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Indefinite-Metric Quantum Field Theory of General Relativity. VII

——*Supplementary Remarks*——

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The following remarks are made concerning the manifestly-covariant canonical formalism of quantum gravity.

1. It is proved that all non-matter fields must be exactly massless.
2. An argument is made for supporting the absence of a tripole ghost.
3. The graviton components are taken out from the gravitational asymptotic field without referring to a special Lorentz frame.

§ 1. Introduction

In the present paper, we make some miscellaneous remarks on our manifestly-covariant canonical formalism of quantum gravity developed in a series of papers,[1)~6)] which are referred to as I, ···, VI, consecutively.

In § 2, we prove that all non-matter fields appearing in our formalism must be exactly massless, as is expected. We show that each of them is the Goldstone field corresponding to a certain spontaneously broken symmetry.

In § 3, we present some justification of our prescription of constructing the asymptotic-field Lagrangian. Though the invariance under the general coordinate transformation is broken by the presence of the gauge-fixing term, we find that there remains its "remnant", which induces some additional restriction on the form of the asymptotic-field Lagrangian.

In § 4, we show that the graviton components can be taken out from the gravitational asymptotic field in the position space, namely, without restricting ourselves to a special Lorentz frame. Of course, our method also applies to quantum electrodynamics for taking out the transverse photons.

§ 2. Exact masslessness of all non-matter fields

In this section, we prove that all fields of our quantum-gravity theory, of course excluding matter fields, are exactly massless. This fact means that the two-point Green's functions have a massless pole despite the relevance of the infrared problem. Our result is thus an important support to the postulate of asymptotic completeness in quantum gravity.

The Lagrangian density proposed in I is written as*)

$$\tilde{\mathcal{L}}=\tilde{\mathcal{L}}_{\rm E}+\kappa^{-1}\partial_\mu\tilde{g}^{\mu\nu}\cdot b_\nu+i\tilde{g}^{\mu\nu}\partial_\mu\bar{c}_\rho\cdot\partial_\nu c^\rho+\mathcal{L}_M, \tag{2·1}$$

where $\tilde{\mathcal{L}}_{\rm E}$ denotes the Einstein Lagrangian density; κ is the gravitational constant and $\tilde{g}^{\mu\nu}\equiv\sqrt{-g}g^{\mu\nu}$ with $g\equiv\det g_{\mu\nu}$; b_ν is an auxiliary boson field, c^ρ and $\bar{c}_\rho$ being the Faddeev-Popov (FP) ghosts; $\mathcal{L}_M$ denotes the Lagrangian density of some integer-spin fields. The action $\int d^4x\tilde{\mathcal{L}}$ is invariant under the Poincaré group, under the BRS transformation and under the FP-ghost scaling transformation; the corresponding conserved quantities are the Poincaré generators P_μ and $M_{\mu\nu}$, the BRS charge Q_b and the FP-ghost charge Q_c, respectively.

Now, we note that the action is invariant also under the following transformations:

(a) general linear group $GL(4)$;

(b) $b_\rho\to b_\rho+$const, others invariant;

(c) $c^\rho\to c^\rho+$const, others invariant;

(d) $\bar{c}_\rho\to\bar{c}_\rho+$const, others invariant.

Here the last two transformations are formal in the sense that they can be finite transformations only at the classical level.**)

The expression for the generator $\hat{M}^\mu{}_\nu$ of $GL(4)$ was presented in III explicitly:[8),***)]

$$\hat{M}^\mu{}_\nu=\kappa^{-1}\int d^3x\tilde{g}^{0\sigma}[x^\mu\partial_\sigma b_\nu-\delta^\mu{}_\sigma b_\nu+i\kappa(\bar{c}_\nu\partial_\sigma c^\mu-\partial_\sigma\bar{c}_\nu\cdot c^\mu)]. \tag{2·2}$$

The commutation relations between $\hat{M}^\mu{}_\nu$ and the field operators are as follows:

$$[g_{\rho\sigma},\hat{M}^\mu{}_\nu]=ix^\mu\partial_\nu g_{\rho\sigma}+i\delta^\mu{}_\rho g_{\nu\sigma}+i\delta^\mu{}_\sigma g_{\rho\nu}, \tag{2·3}$$

$$[b_\rho,\hat{M}^\mu{}_\nu]=ix^\mu\partial_\nu b_\rho+i\delta^\mu{}_\rho b_\nu, \tag{2·4}$$

$$[c^\rho,\hat{M}^\mu{}_\nu]=ix^\mu\partial_\nu c^\rho-i\delta^\rho{}_\nu c^\mu, \tag{2·5}$$

$$[\bar{c}_\rho,\hat{M}^\mu{}_\nu]=ix^\mu\partial_\nu\bar{c}_\rho+i\delta^\mu{}_\rho\bar{c}_\nu. \tag{2·6}$$

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*) We regard the non-Landau-gauge formalism[7)] as unsatisfactory because it contains an unnatural term $\alpha\sqrt{-g}\eta^{\mu\nu}b_\mu b_\nu$.

**) The use of Grassmann numbers at the quantum level is meaningless. For example, introducing a Grassmann number θ is equivalent to considering a direct sum of two state-vector spaces $\mathcal{V}$ (the part independent of θ) and $\mathcal{V}_\theta$ (the part proportional to θ) such that $\mathcal{V}$ contains the complete information of the theory and $\mathcal{V}_\theta$ has no effect on $\mathcal{V}$; thus $\mathcal{V}_\theta$ is totally redundant.

***) Here, since the three-dimensional integral is divergent at infinity, the integration should be carried out after calculating commutators, as is usually understood.

Since the Lorentz group is a subgroup of $GL(4)$, the Lorentz generator $M_{\mu\nu}$ is expressible in terms of $\widehat{M}^{\mu}{}_{\nu}$:

$$M_{\mu\nu}=\eta_{\mu\lambda}\widehat{M}^{\lambda}{}_{\nu}-\eta_{\nu\lambda}\widehat{M}^{\lambda}{}_{\mu}\,, \tag{2·7}$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric.

Corresponding to the remaining three symmetries, we have the following conserved currents:

$$j_{\rm b}{}^{\mu\nu}\equiv\kappa^{-1}\tilde{g}^{\mu\nu}, \tag{2·8}$$

$$j_{\rm c}{}^{\mu}{}_{\nu}\equiv\tilde{g}^{\mu\sigma}\partial_{\sigma}\bar{c}_{\nu}\,, \tag{2·9}$$

$$j_{\bar{\rm c}}{}^{\mu\nu}\equiv\tilde{g}^{\mu\sigma}\partial_{\sigma}c^{\nu}. \tag{2·10}$$

They are indeed conserved with respect to the index μ owing to the field equations

$$\partial_{\mu}\tilde{g}^{\mu\nu}=0\,, \tag{2·11}$$

$$\partial_{\mu}(\tilde{g}^{\mu\nu}\partial_{\nu}\bar{c}_{\rho})=0\,, \tag{2·12}$$

$$\partial_{\mu}(\tilde{g}^{\mu\nu}\partial_{\nu}c^{\rho})=0\,, \tag{2·13}$$

which directly follow from (2·1) (as given in I). We can therefore define the corresponding conserved charges formally:

$$q_{\rm b}{}^{\nu}\equiv\kappa^{-1}\int d^{3}x\tilde{g}^{0\nu}, \tag{2·14}$$

$$q_{\rm c}{}^{\nu}\equiv\int d^{3}x\tilde{g}^{0\sigma}\partial_{\sigma}\bar{c}_{\nu}\,, \tag{2·15}$$

$$q_{\bar{\rm c}}{}^{\nu}\equiv\int d^{3}x\tilde{g}^{0\sigma}\partial_{\sigma}c^{\nu}. \tag{2·16}$$

It is interesting to note that these charges and the translation generator (see III),

$$P_{\nu}\equiv\kappa^{-1}\int d^{3}x\tilde{g}^{0\sigma}\partial_{\sigma}b_{\nu}\,, \tag{2·17}$$

form a "quartet" in the sense that P_{ν} and $q_{\bar{\rm c}}{}^{\nu}$ are the BRS transforms*) of $-i\kappa^{-1}q_{{\rm c}\nu}$ and $-q_{\rm b}{}^{\nu}$, respectively.

By using the (anti-)commutation relations presented in II, it is easily shown that

$$[b_{\rho},\,q^{{\rm b}\nu}]=-i\delta^{\nu}{}_{\rho}\,, \tag{2·18}$$

$$\{c^{\rho},\,q_{{\rm c}\nu}\}=\delta^{\rho}{}_{\nu}\,, \tag{2·19}$$

*) Here, of course, surface integrals should be discarded.

$$\{\bar{c}_\rho, q_{\bar{c}}{}^\nu\} = -\delta^\nu{}_\rho \tag{2·20}$$

and that all other (anti-) commutators vanish. Thus $q_b{}^\nu$, $q_{c\nu}$ and $q_{\bar{c}}{}^\nu$ are indeed the generators of the transformations (b), (c) and (d), respectively.

Now, as pointed out previously,[8] the $GL(4)$ invariance is spontaneously broken up to the Lorentz group if space-time is asymptotically flat, that is, if

$$\langle 0|g_{\mu\nu}|0\rangle = \eta_{\mu\nu}\,, \tag{2·21}$$

$$\langle 0|b_\rho|0\rangle = 0,\ \langle 0|c^\rho|0\rangle = 0,\ \langle 0|\bar{c}_\rho|0\rangle = 0\,, \tag{2·22}$$

where $|0\rangle$ denotes the true vacuum. This fact implies that $g_{\mu\nu}$ is the Goldstone field, and therefore gravitons must be exactly massless owing to the Goldstone theorem. Likewise, but independently of the asymptotic flatness, (2·18) ~ (2·20) imply that the symmetries represented by $q_b{}^\nu$, $q_{c\nu}$ and $q_{\bar{c}}{}^\nu$ are spontaneously broken. Accordingly, we see that b_ρ, c^ρ and $\bar{c}_\rho$ must also be exactly massless.

The above consideration can be extended to the case of the vierbein formalism developed in V and VI. In this case, the gravitational field $g_{\mu\nu}$ was replaced by the vierbein $h_{\mu a}$ ($h_{\mu a}h_\nu{}^a = g_{\mu\nu}$), and we introduced three antisymmetric fields s_{ab}, t_{ab} and $\bar{t}_{ab}$ so as to break the local Lorentz invariance. Our Lagrangian density is given by

$$\mathcal{L}_{\rm tot} = \tilde{\mathcal{L}} + \mathcal{L}_{\rm D} + \mathcal{L}_{\rm LL}\,, \tag{2·23}$$

where $\mathcal{L}_{\rm D}$ denotes the generally covariant Dirac Lagrangian density and

$$\mathcal{L}_{\rm LL} \equiv \tilde{g}^{\mu\nu}\Gamma_\nu{}^{ab}\partial_\mu s_{ab} - i\tilde{g}^{\mu\nu}\partial_\mu \bar{t}_{ab}\cdot(\partial_\nu t^{ab} + \Gamma_\nu{}^{cb}t^a{}_c - \Gamma_\nu{}^{ca}t^b{}_c)\,, \tag{2·24}$$

$\Gamma_\nu{}^{ab}$ being the spin affine connection.

As shown in V, the expression (2·2) for the generator $\hat{M}^\mu{}_\nu$ remains unchanged, but the relation (2·7) must be replaced by

$$M_{\mu\nu} = \eta_{\mu\lambda}\hat{M}^\lambda{}_\nu - \eta_{\nu\lambda}\hat{M}^\lambda{}_\mu + M_{{\rm LL}\mu\nu}\,, \tag{2·25}$$

where $M_{{\rm LL}\mu\nu}$ is a new conserved quantity defined in [V, (4·24)]. As shown in VI, we have

$$[h_{\rho a}, \hat{M}^\mu{}_\nu] = ix^\mu\partial_\nu h_{\rho a} + i\delta^\mu{}_\rho h_{\nu a}\,, \tag{2·26}$$

$$[h_{\rho a}, M_{{\rm LL}\mu\nu}] = i(\eta_{\mu a}h_{\rho\nu} - \eta_{\nu a}h_{\rho\mu})\,. \tag{2·27}$$

Hence, if space-time is asymptotically flat, that is, if

$$\langle 0|h_{\mu a}|0\rangle = \eta_{\mu a}\,, \tag{2·28}$$

then both invariances under $\hat{M}^\mu{}_\nu$ and $M_{{\rm LL}\mu\nu}$ are spontaneously broken in such a way that the invariance under $M_{\mu\nu}$ remains unbroken. Since the invariance under $\hat{M}^\mu{}_\nu$ is totally broken in the present case, all 16 components of $h_{\mu a}$ must be exactly massless.

Finally, we note that there are three new conserved currents

$$j_{\rm s}{}^{\mu ab} \equiv \tilde{g}^{\mu\nu} \Gamma_\nu{}^{ab}, \tag{2·29}$$

$$j_{\rm t}{}^{\mu ab} \equiv \tilde{g}^{\mu\nu} (\partial_\nu \bar{t}^{ab} + \Gamma_\nu{}^{cb} \bar{t}^a{}_c - \Gamma_\nu{}^{ca} \bar{t}^b{}_c), \tag{2·30}$$

$$j_{\bar{\rm t}}{}^{\mu ab} \equiv \tilde{g}^{\mu\nu} (\partial_\nu t^{ab} + \Gamma_\nu{}^{cb} t^a{}_c - \Gamma_\nu{}^{ca} t^b{}_c). \tag{2·31}$$

Let $q_{\rm s}{}^{ab}$, $q_{\rm t}{}^{ab}$ and $q_{\bar{\rm t}}{}^{ab}$ be the corresponding conserved charges. Then they and $M_{\rm LL}{}^{ab} \equiv \eta^{\mu a}\eta^{\nu b} M_{{\rm LL}\mu\nu}$ form a quartet in the sense that $M_{\rm LL}{}^{ab}$ and $q_{\bar{\rm t}}{}^{ab}$ are the local-Lorentz-BRS transforms of $-2iq_{\rm t}{}^{ab}$ and $-q_{\rm s}{}^{ab}$, respectively.

By using the results of VI, we can easily show that

$$[s_{cd}, q_{\rm s}{}^{ab}] = \tfrac{1}{2} i (\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c), \tag{2·32}$$

$$[s_{cd}, q_{\rm t}{}^{ab}] = \tfrac{1}{2} i [(\delta^b{}_d \bar{t}^a{}_c - \delta^a{}_d \bar{t}^b{}_c) - (c \leftrightarrow d)], \tag{2·33}$$

$$\{t_{cd}, q_{\rm t}{}^{ab}\} = -\tfrac{1}{2} (\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c), \tag{2·34}$$

$$\{\bar{t}_{cd}, q_{\bar{\rm t}}{}^{ab}\} = \tfrac{1}{2} (\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c), \tag{2·35}$$

and that all other (anti-)commutators vanish. Thus we see that the symmetries represented by $q_{\rm s}{}^{ab}$, $q_{\rm t}{}^{ab}$ and $q_{\bar{\rm t}}{}^{ab}$ are all spontaneously broken, whence s_{cd}, t_{cd} and $\bar{t}_{cd}$ are exactly massless owing to the Goldstone theorem.

§ 3. Asymptotic-field Lagrangian

In order to determine the field equations and the equal-time commutation relations for the asymptotic fields, in I (and later also in V) we made the following assumption: The (renormalized) asymptotic fields are governed by the asymptotic-field Lagrangian, which has the same *form* as the quadratic part*) of the exact Lagrangian. Subsequently, Kugo and Ojima[9] criticized this assumption and showed that the positive semi-definiteness of the physical subspace could be proved even under weaker assumptions. They emphasized that by means of the Ward-Takahashi identities one could not exclude the possible existence of a tripole ghost, which was not encountered in I. As pointed out in II, however, our assumption seems to be natural in the sense that if it is not the case then one cannot prove the equivalence between Dyson's S-matrix in the interaction picture and the S-matrix $\{\langle \text{out}|\text{in}\rangle\}$ in the Heisenberg picture. Therefore, from the aesthetic point of view, we still believe that our assumption is basically correct. The purpose of this section is to provide a strong support to this belief.

There is no objection to assuming that the asymptotic-field Lagrangian $\mathcal{L}^{\rm asym}$ exists and is quadratic in asymptotic fields, and of course, the corresponding action should be Poincaré invariant. Furthermore, it is quite natural to assume that the

*) Of course, quadratic with respect to the fields having a vanishing vacuum expectation value.

canonical degrees of freedom remain unchanged for the asymptotic fields which uniquely have the corresponding Heisenberg field. In particular, the number of ∂_0 should not exceed two in each term of $\mathcal{L}^{\mathrm{asym}}$. Finally, if the exact action is invariant under certain transformations, $\mathcal{L}^{\mathrm{asym}}$ should be invariant, up to a total divergence, under their linearized versions if and only if they are not broken by the representation adopted.

For example, consider the Landau-gauge quantum electrodynamics. For simplicity of notation, let A_μ and B be the asymptotic fields of the electromagnetic field and of the auxiliary scalar field, respectively. From the general principles only, the possible expression for $\mathcal{L}^{\mathrm{asym}}$, apart from the FP-ghost term and the charged-field part, is

$$a_1(-1/4)F^{\mu\nu}F_{\mu\nu}+a_2(\partial_\mu A^\mu)^2+a_3A^\mu A_\mu+a_4B\partial_\mu A^\mu+a_5B^2, \tag{3·1}$$

where $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$ and the a_j's are real numbers. But, since quantum electrodynamics is a gauge theory, it is invariant under the BRS transformation. This invariance implies that $a_2=a_3=0$. Furthermore, the Landau-gauge Lagrangian is invariant under the constant translation of the auxiliary scalar field ($B\to B+$const). Though this symmetry is spontaneously broken by definition, it behaves like an *unbroken* symmetry because the manner of breaking is independent of the representation adopted. This symmetry implies that $a_5=0$. Finally, assuming*) $a_1>0$ on the basis of perturbation theory, we can set $a_1=a_4=1$ by redefining A_μ and B. We thus have

$$\mathcal{L}^{\mathrm{asym}}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+B\partial_\mu A^\mu, \tag{3·2}$$

which is nothing but the quadratic part of the exact Lagrangian density apart from the parts of other fields.

Now, we return to quantum gravity. First, we consider the case (2·1). Let $\varphi_{\mu\nu}$, β_ρ, γ^ρ and $\bar{\gamma}_\rho$ be the asymptotic fields of $g_{\mu\nu}$, b_ρ, c^ρ and $\bar{c}_\rho$, respectively, as in I. The BRS transforms of the asymptotic fields are as follows:

$$\boldsymbol{\delta}(\varphi_{\mu\nu})=\tfrac{1}{2}\sqrt{\kappa}(\partial_\mu\gamma_\nu+\partial_\nu\gamma_\mu), \tag{3·3}$$

$$\boldsymbol{\delta}(\beta_\rho)=0,\ \boldsymbol{\delta}(\gamma^\rho)=0,\ \boldsymbol{\delta}(\bar{\gamma}_\rho)=i\sqrt{\kappa}\beta_\rho. \tag{3·4}$$

From the general principles and the invariance under the BRS transformation together with the invariance under the FP-ghost scaling transformation, the possible expression for $\mathcal{L}^{\mathrm{asym}}$, apart from the non-gravitational terms, is

$$a_1\mathcal{L}_{\mathrm{E}}^{\mathrm{lin}}+a_2[-(2\partial_\mu\varphi^{\mu\nu}-\partial^\nu\varphi)\beta_\nu+i\partial^\mu\bar{\gamma}_\nu\cdot\partial_\mu\gamma^\nu]$$
$$+a_3(\partial^\nu\varphi\cdot\beta_\nu-i\partial^\nu\bar{\gamma}_\nu\cdot\partial_\mu\gamma^\mu)+a_4\beta_\nu\beta^\nu \tag{3·5}$$

up to a total divergence, where $\varphi\equiv\varphi^\mu{}_\mu$ and $\mathcal{L}_{\mathrm{E}}^{\mathrm{lin}}$ denotes the Lagrangian of the

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*) Nobody can *prove* the positivity of the norm of transverse photons.

linearized Einstein theory.

Since the symmetry $b_\rho \to b_\rho + \text{const}$ behaves like an unbroken one just as in quantum electrodynamics, $\mathcal{L}^{\text{asym}}$ should be invariant under $\beta_\rho \to \beta_\rho + \text{const}$, whence $a_4 = 0$. If we assume $a_1 > 0$ (and $a_2 \neq 0$) as in quantum electrodynamics, then we can set $a_1 = a_2 = 1$ by redefining the asymptotic fields. Thus the extra term is the third term, its coefficient a_3 being completely arbitrary. The possible existence of such a term was noted previously by Kimura[10] apart from the FP-ghost contribution. Since the existence of this term implies the presence of a tripole ghost, we must find the reason why it does not appear. What is the essential difference between the second term and the third one?

Let the λ_ν's be four c-number functions satisfying

$$\Box \lambda_\nu = 0\,, \tag{3·6}$$

and consider the transformation defined by

$$\varphi_{\mu\nu} \to \varphi_{\mu\nu} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu\,, \tag{3·7}$$

the other asymptotic fields being invariant. We call this transformation the λ transformation. Both the first term and the second one of (3·5) are invariant under the λ transformation, but the third term is *not*. Thus, if the λ transformation is a linearized version of a transformation under which (2·1) is invariant up to a total divergence, then we can exclude the third term.

Fortunately, such a transformation does exist. Though the invariance under the general coordinate transformation is explicitly broken by the gauge-fixing term, there still remains a "remnant" of it.*) This transformation, which we call the Λ transformation, is defined as follows.

Let the Λ^μ's be four infinitesimal functions satisfying a condition determined later. Every field is transformed in the same way as under the general coordinate transformation whose infinitesimal transformation functions are the Λ^μ's. Here, we regard b_ρ as a covariant vector but c^σ and $\bar{c}_\rho$ as *scalars* but not as vectors. For example,

$$\delta_\Lambda (g^{\mu\nu}) = \partial_\lambda \Lambda^\mu \cdot g^{\lambda\nu} + \partial_\lambda \Lambda^\nu \cdot g^{\mu\lambda} - \Lambda^\lambda \partial_\lambda g^{\mu\nu}, \tag{3·8}$$

$$\delta_\Lambda (\sqrt{-g}) = -\partial_\lambda (\Lambda^\lambda \sqrt{-g})\,, \tag{3·9}$$

$$\delta_\Lambda (c^\sigma) = -\Lambda^\lambda \partial_\lambda c^\sigma, \tag{3·10}$$

where δ_Λ stands for the Lie derivative of the Λ transformation. Then it is evident that (2·1) is invariant, up to a total divergence, under the Λ transformation if $\partial_\mu \tilde{g}^{\mu\nu}$ transforms like a contravariant vector density. Hence we require

$$\delta_\Lambda (\partial_\mu \tilde{g}^{\mu\nu}) = \partial_\lambda \Lambda^\nu \cdot \partial_\mu \tilde{g}^{\mu\lambda} - \partial_\lambda (\Lambda^\lambda \partial_\mu \tilde{g}^{\mu\nu})\,, \tag{3·11}$$

*) As is well known, there is such invariance also in quantum electrodynamics.

that is,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Lambda^\lambda=0\,. \tag{3·12}$$

We can explicitly construct the conserved current corresponding to the Λ transformation:

$$J_\Lambda{}^\mu\equiv\kappa^{-1}\tilde{g}^{\mu\nu}(b_\rho\partial_\nu\Lambda^\rho-\partial_\nu b_\rho\cdot\Lambda^\rho)\,. \tag{3·13}$$

Accordingly, the generator of the Λ transformation is given by

$$Q_\Lambda\equiv\kappa^{-1}\int d^3x\tilde{g}^{0\nu}(b_\rho\partial_\nu\Lambda^\rho-\partial_\nu b_\rho\cdot\Lambda^\rho)\,. \tag{3·14}$$

Since it is possible to assume that Λ^λ and $\dot{\Lambda}^\lambda$ are c-numbers at a particular time, it is easy to calculate the commutator between Q_Λ and an arbitrary field X. We then find

$$[X, Q_\Lambda]=i\delta_\Lambda(X)\,. \tag{3·15}$$

Now, it is obvious from the above construction that the linearized form of the Λ transformation is nothing but the λ transformation. We can thus infer that the third term of (3·5) is to be absent.

§ 4. Separation of the graviton components

In the analysis of the asymptotic fields presented in I, we separated the graviton components from the dipole ghosts in order to prove the positive semi-definiteness of the physical subspace. To do this, we made use of a special Lorentz frame $p_1=p_2=0$ in the momentum space. Strictly speaking, however, such a procedure is justified only in the one-particle sector, because, for example, two quanta generally move in different directions. We should therefore make the separation of the graviton components in an arbitrary Lorentz frame. Furthermore, it is better to work out this procedure in the position space because then it serves as the zeroth approximation in the case of the background curved space-time discussed in IV.

The asymptotic fields $\varphi_{\mu\nu}$ and β_ρ satisfy the following equations and the four-dimensional commutation relations:[1),10),*)]

$$\Box\varphi_{\mu\nu}=\partial_\mu\beta_\nu+\partial_\nu\beta_\mu\,, \tag{4·1}$$

$$\partial^\mu\varphi_{\mu\nu}-\tfrac{1}{2}\,\partial_\nu\varphi=0; \tag{4·2}$$

$$\begin{aligned}[\varphi_{\mu\nu}(x)\,,\,\varphi_{\rho\sigma}(y)]&=\tfrac{1}{2}\,i\,(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})\,D(x-y)\\&\quad-\tfrac{1}{2}\,i\,(\eta_{\mu\sigma}\partial_\nu\partial_\rho+\eta_{\nu\rho}\partial_\mu\partial_\sigma+\eta_{\mu\rho}\partial_\nu\partial_\sigma+\eta_{\nu\sigma}\partial_\mu\partial_\rho)\,E(x-y)\,,\end{aligned} \tag{4·3}$$

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*) On the right-hand sides of commutation relations, differentiation should always be understood as the one with respect to x.

$$[\varphi_{\mu\nu}(x),\beta_\rho(y)]=\tfrac{1}{2}i(\eta_{\mu\rho}\partial_\nu+\eta_{\nu\rho}\partial_\mu)D(x-y),\tag{4·4}$$

$$[\beta_\rho(x),\beta_\sigma(y)]=0,\tag{4·5}$$

where $\Box E(z)=D(z)$.

As the independent components, we adopt $\{\varphi_{11},\varphi_{12},\varphi_{00},\varphi_{01},\varphi_{02},\varphi_{33}\}$, that is, we eliminate $\{\varphi_{22},\varphi_{03},\varphi_{13},\varphi_{23}\}$ by means of (4·2). It is possible to solve (4·2) with respect to the latter four components because the determinant formed from their coefficients is

$$\tfrac{1}{2}\partial_3^{\,2}(\partial_0^{\,2}-\partial_1^{\,2}+\partial_2^{\,2}+\partial_3^{\,2}),\tag{4·6}$$

which is non-vanishing generically*) on the mass shell.

Now, the crucial characteristic of the graviton components is that they commute with $\beta_\rho(y)$. From the explicit expression for $\eta_{\mu\rho}\partial_\nu+\eta_{\nu\rho}\partial_\mu$, it is elementary to see that $\varphi^{(11)}$ and $\varphi^{(12)}$ commute with β_ρ, where

$$\varphi^{(11)}\equiv\partial_0^{\,2}\varphi_{11}-2\partial_0\partial_1\varphi_{01}+\partial_1^{\,2}\varphi_{00},\tag{4·7}$$

$$\varphi^{(12)}\equiv\partial_0^{\,2}\varphi_{12}-\partial_0\partial_2\varphi_{01}-\partial_0\partial_1\varphi_{02}+\partial_1\partial_2\varphi_{00}.\tag{4·8}$$

Then, from (4·1) we have

$$\Box\varphi^{(11)}=0,\quad \Box\varphi^{(12)}=0,\tag{4·9}$$

and (4·3) yields

$$[\varphi^{(11)}(x),\varphi^{(11)}(y)]=\tfrac{1}{2}i(\partial_0^{\,2}-\partial_1^{\,2})^2D(x-y),\tag{4·10}$$

$$[\varphi^{(11)}(x),\varphi^{(12)}(y)]=-\tfrac{1}{2}i(\partial_0^{\,2}-\partial_1^{\,2})\partial_1\partial_2D(x-y),\tag{4·11}$$

$$[\varphi^{(12)}(x),\varphi^{(12)}(y)]=\tfrac{1}{2}i(\partial_0^{\,2}-\partial_1^{\,2})(\partial_0^{\,2}-\partial_2^{\,2})D(x-y).\tag{4·12}$$

Hence, if we set

$$\partial_0\varphi_1\equiv\partial_2\varphi^{(11)}-\partial_1\varphi^{(12)},\tag{4·13}$$

$$\varphi_2\equiv\partial_1\varphi^{(11)}+\partial_2\varphi^{(12)},\tag{4·14}$$

that is,

$$\varphi_1=\partial_0\partial_2\varphi_{11}-\partial_0\partial_1\varphi_{12}-\partial_1\partial_2\varphi_{01}+\partial_1^{\,2}\varphi_{02},\tag{4·15}$$

$$\begin{aligned}\varphi_2=&\,\partial_0^{\,2}\partial_1\varphi_{11}+\partial_0^{\,2}\partial_2\varphi_{12}-\partial_0(2\partial_1^{\,2}+\partial_2^{\,2})\varphi_{01}\\&-\partial_0\partial_1\partial_2\varphi_{02}+\partial_1(\partial_1^{\,2}+\partial_2^{\,2})\varphi_{00},\end{aligned}\tag{4·16}$$

then we obtain

$$[\varphi_1(x),\varphi_1(y)]=\tfrac{1}{2}i(\partial_1^{\,2}+\partial_2^{\,2})(\partial_2^{\,2}+\partial_3^{\,2})D(x-y),\tag{4·17}$$

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*) That is, the exceptional case is negligible if we consider wave packets.

$$[\varphi_1(x), \varphi_2(y)] = 0, \tag{4·18}$$

$$[\varphi_2(x), \varphi_2(y)] = -\tfrac{1}{2} i (\partial_1{}^2+\partial_2{}^2)(\partial_2{}^2+\partial_3{}^2)\partial_3{}^2 D(x-y), \tag{4·19}$$

where use has been made of $\Box D=0$. We can identify φ_1 and φ_2 as the graviton components. The positive-frequency parts of (4·17) and (4·19) are indeed positive definite generically on the mass shell in the momentum space.

Next, we construct four dipole ghosts $\chi^{(\sigma)}$ ($\sigma=0, 1, 2, 3$) in such a way that

$$[\varphi_k(x), \chi^{(\sigma)}(y)] = 0. \quad (k=1, 2) \tag{4·20}$$

For example, (4·20) is satisfied by

$$\chi^{(0)} \equiv \varphi_{11} - \varphi_{00}, \tag{4·21}$$

$$\chi^{(1)} \equiv \partial_1\varphi_{12} - \partial_0\varphi_{02}, \tag{4·22}$$

$$\chi^{(2)} \equiv \partial_1\varphi_{11} - \partial_1\varphi_{33} + \partial_2\varphi_{12} - \partial_0\varphi_{01}, \tag{4·23}$$

$$\chi^{(3)} \equiv \partial_0\varphi_{11} + \partial_0\varphi_{33} - \partial_1\varphi_{01} - \partial_2\varphi_{02}. \tag{4·24}$$

Furthermore, we can diagonalize them in the sense that

$$[\chi_\sigma(x), \beta_\rho(y)] = 0 \qquad \text{for } \sigma \neq \rho \tag{4·25}$$

by setting

$$\chi_0 \equiv \tfrac{1}{2}\partial_0(\partial_0{}^2 - 3\partial_1{}^2 - \partial_2{}^2)\chi^{(0)} + \partial_0\partial_1\chi^{(2)} + \partial_1{}^2\chi^{(3)}, \tag{4·26}$$

$$\chi_1 \equiv -\tfrac{1}{2}\partial_1(\partial_0{}^2 + \partial_1{}^2 + \partial_2{}^2)\chi^{(0)} + \partial_0{}^2\chi^{(2)} + \partial_0\partial_1\chi^{(3)}, \tag{4·27}$$

$$\chi_2 \equiv -\tfrac{1}{2}\partial_2\chi^{(0)} + \chi^{(1)}, \tag{4·28}$$

$$\chi_3 \equiv -\partial_1{}^2\chi^{(0)} - \partial_2\chi^{(1)} + \partial_1\chi^{(2)} + \partial_0\chi^{(3)}. \tag{4·29}$$

In this way, we obtain the formulae more reasonable than [I, (5·19) ~ (5·21)].

In order to understand what we have done in the above, it is instructive to apply the above method to quantum electrodynamics for taking out the transverse-photon components.

In quantum electrodynamics, the asymptotic fields A_μ and B satisfy

$$\Box A_\mu - (1-\alpha)\partial_\mu B = 0, \tag{4·30}$$

$$\partial_\mu A^\mu + \alpha B = 0, \tag{4·31}$$

$$[A_\mu(x), A_\nu(y)] = -i\eta_{\mu\nu}D(x-y) + i(1-\alpha)\partial_\mu\partial_\nu E(x-y), \tag{4·32}$$

$$[A_\mu(x), B(y)] = -i\partial_\mu D(x-y). \tag{4·33}$$

As the independent components, we take A_0, A_1 and A_2; then from (4·31) A_3 is expressed as

$$\partial_3 A_3 = \alpha B + \partial_0 A_0 - \partial_1 A_1 - \partial_2 A_2 \,. \tag{4·34}$$

Corresponding to (4·7) and (4·8), we set

$$F_{0k} \equiv \partial_0 A_k - \partial_k A_0 \,, \quad (k=1, 2) \tag{4·35}$$

which commutes with B. Then (4·32) implies that

$$[F_{0k}(x), F_{0l}(y)] = i(\eta_{kl}\partial_0^2 + \partial_k \partial_l) D(x-y) \,. \tag{4·36}$$

Next, corresponding to (4·13) and (4·14), we set

$$\partial_0 F_{12} \equiv \partial_2 F_{01} - \partial_1 F_{02} \,, \tag{4·37}$$

$$G \equiv \partial_1 F_{01} + \partial_2 F_{02} \,, \tag{4·38}$$

that is,

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 \,, \tag{4·39}$$

$$G = \partial_3(\partial_3 A_0 - \partial_0 A_3) + \partial_0 B \,. \tag{4·40}$$

Then

$$[F_{12}(x), F_{12}(y)] = -i(\partial_1^2 + \partial_2^2) D(x-y) \,, \tag{4·41}$$

$$[F_{12}(x), G(y)] = 0 \,, \tag{4·42}$$

$$[G(x), G(y)] = i(\partial_1^2 + \partial_2^2)\partial_3^2 D(x-y) \,. \tag{4·43}$$

Thus we essentially take F_{12} and F_{30} as the transverse photons.

Finally, the longitudinal photon is defined by

$$X \equiv \partial_0 A_0 - \partial_1 A_1 - \partial_2 A_2 = \partial_3 A_3 - \alpha B \,, \tag{4·44}$$

so that X commutes with both F_{12} and G.

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Note added in proof: The proof of unitarity can be made without explicitly constructing asymptotic fields. See the author's other paper in this issue.