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Indefinite-Metric Quantum Field Theory of General Relativity. VIII

——*Commutators Involving* b_ρ——

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The expression for the equal-time commutator between b_ρ and $\dot{b}_\sigma$, which was derived previously from the consistency requirement of the BRS transformation, is proved on the basis of canonical commutation relations and field equations. It is found that the commutators involving b_ρ exhibit "tensor-like" behavior quite analogously to general covariance.

The manifestly covariant quantum field theory is formulated for the coupled Einstein-Maxwell system.

§ 1. Introduction

The manifestly covariant canonical theory of quantum gravity has been formulated in quite a satisfactory way in a series of papers.[1)~7),*)] When matter fields are of integer spins only, the gravitational part is described by the metric tensor $g_{\mu\nu}$, an auxiliary boson field b_ρ and a pair of the Faddeev-Popov (FP) ghosts c^ρ and $\bar{c}_\rho$. If there are half-odd-spin fields, then the gravitational part is described by the vierbein $h_{\mu a}$ instead of $g_{\mu\nu}(=h_{\mu a}h_\nu{}^a)$, a new auxiliary antisymmetric boson field s_{ab} and the corresponding FP ghosts t_{ab} and $\bar{t}_{ab}$ in addition to b_ρ, c^ρ and $\bar{c}_\rho$. An important result found in V and VI is that in the latter case, though $h_{\mu a}$, but not $g_{\mu\nu}$, is regarded as a fundamental field, *all the field equations*[**)] *and the equal-time commutation relations for the "old" fields* $g_{\mu\nu}$, b_ρ, c^ρ *and* $\bar{c}_\rho$ *are the same as those in the former case.*

In II, we calculated the equal-time commutators between Heisenberg fields and their first time derivatives from canonical commutation relations. Since b_ρ's are *not* canonical variables, it is not an easy task to prove the important formula[***)]

$$[b_\rho, \dot{b}_\sigma'] = i\kappa(\tilde{g}^{00})^{-1}(\partial_\rho b_\sigma + \partial_\sigma b_\rho)\cdot\delta^3. \tag{1·1}$$

This commutation relation was derived from the consistency between the classical

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*) The second, fifth and sixth papers are referred to as II, V and VI, respectively.

**) Of course, the energy-momentum tensor $T_{\mu\nu}$ of the Einstein equation is modified by the introduction of new fields.

***) The notation is explained at the end of this section.

and quantum definitions of the BRS transformation, but such reasoning cannot be regarded as a proof because it is based on something other than canonical commutation relations and field equations. Solving the Einstein equation with respect to $\dot{b}_\sigma$, we found some necessary and sufficient conditions for the validity of $(1\cdot1)$, but their validity was not checked because the calculation to be done was extremely cumbersome.

In the present paper, we complete the proof of $(1\cdot1)$. The essential idea for avoiding brute-force labor is to employ the vierbein formalism developed in V and VI. The Ricci tensor $R_{\mu\nu}$ is usually expressed in terms of the affine connection $\Gamma_{\sigma\tau}{}^{\lambda}$, but it can also be expressed elegantly in terms of the spin affine connection $\Gamma_\lambda{}^{ab}$, as is familiar in the supergravity theory. The point is that the expression for the commutator $[\Gamma_\lambda{}^{ab}, b_\rho']$ is very simple in contrast with that for $[\Gamma_{\sigma\tau}{}^{\lambda}, b_\rho']$. Owing to this observation, we can very easily prove that the conditions for the validity of $(1\cdot1)$ are indeed satisfied.

In addition, we can proceed further. We can calculate $[R_{\mu\nu}, b_\rho']$, which is found to be of the same form as that of $[g_{\mu\nu}, b_\rho']$. Furthermore, many quantities are found to have the commutator with b_ρ of the same form. Then we are naturally led to introducing a new notion of the *tensor-like commutation relation*. Its physical meaning is unknown, but it is quite analogous to general covariance, despite the fact that it is a notion completely at the operator level.

In § 2, the proof of $(1\cdot1)$ is presented. In § 3, tensor-like commutation relations are discussed. In § 4, we formulate the manifestly covariant quantum gravielectrodynamics for providing additional examples of tensor-like commutation relations. The Appendix is devoted to the calculation of $[T_{\mathrm{LL}\,\mu\nu}, b_\rho']$.

Throughout the present paper, we employ the notation introduced previously. For example,

$$\partial_\mu X\cdot Y\equiv(\partial_\mu X)Y\,,$$

$$\dot{X}\equiv\partial_0 X\,,$$

$$[X, Y']\equiv[X(x), Y(x')] \quad \text{at } x^0=x'^0,$$

$$\delta^3\equiv\sum_{k=1}^{3}\delta(x_k-x_k')\,,$$

$$h\equiv(-\det g_{\mu\nu})^{1/2}=-\det h_{\mu a}\,,$$

$$\tilde{g}^{\mu\nu}\equiv hg^{\mu\nu},$$

and κ denotes the gravitational constant.

§ 2. Proof of $(1\cdot1)$

In this section, we prove the commutation relation $(1\cdot1)$ on the basis of canonical commutation relations and field equations. For simplicity, we here assume that matter field is a scalar field ϕ alone; extension to more general cases is presented in later sections.

In our formalism, the Einstein equation is written as

$$R_{\mu\nu}-E_{\mu\nu}+\kappa(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)=0\,, \tag{2·1}$$

where*)

$$E_{\mu\nu}\equiv\partial_\mu b_\nu+\partial_\nu b_\mu-i\kappa(\partial_\mu\bar{c}_\sigma\cdot\partial_\nu c^\sigma+\partial_\nu\bar{c}_\sigma\cdot\partial_\mu c^\sigma)\,, \tag{2·2}$$

$$T_{\mu\nu}=T_{S\,\mu\nu}\equiv\partial_\mu\phi\cdot\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}(g^{\sigma\tau}\partial_\sigma\phi\cdot\partial_\tau\phi-m^2\phi^2) \tag{2·3}$$

and $T\equiv T^\mu{}_\mu$. Furthermore, $g_{\mu\nu}$ satisfies the de Donder condition

$$\partial_\mu\tilde{g}^{\mu\nu}=0\,. \tag{2·4}$$

Now, in order to prove (1·1), we must solve (2·1) with respect to $\dot{b}_\rho$ after eliminating $\ddot{g}_{jk}$. This procedure was explicitly carried out in II. It was found there that the necessary and sufficient conditions for the validity of (1·1) are as follows:

$$[2R^0{}_0-R,\,b_0{}']=0\,, \tag{2·5}$$

$$[2R^0{}_0-R,\,b_j{}']=-2i\kappa(\tilde{g}^{00})^{-1}R^0{}_j\delta^3, \tag{2·6}$$

$$[R^0{}_k,\,b_0{}']=i\kappa(\tilde{g}^{00})^{-1}R^0{}_k\delta^3, \tag{2·7}$$

$$[R^0{}_k,\,b_j{}']=0\,, \tag{2·8}$$

where $R\equiv R^\mu{}_\mu$. They are summarized as

$$[R^0{}_\nu-\tfrac{1}{2}\delta^0{}_\nu R,\,b_\rho{}']=i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\rho R^0{}_\nu-\delta^0{}_\nu R^0{}_\rho)\,\delta^3. \tag{2·9}$$

In the following, we prove (2·9).

As is well known (and can be easily confirmed), the Riemannian tensor $R_{\mu\nu\alpha\beta}$ can be expressed in terms of the spin affine connection $\Gamma_\lambda{}^{ab}$ in an elegant way:

$$R_{\mu\nu\alpha\beta}=h_{\alpha a}h_{\beta b}\hat{R}_{\mu\nu}{}^{ab} \tag{2·10}$$

with

$$\hat{R}_{\mu\nu}{}^{ab}\equiv\partial_\mu\Gamma_\nu{}^{ab}-\partial_\nu\Gamma_\mu{}^{ab}+\Gamma_\mu{}^{ac}\Gamma_\nu{}^b{}_c-\Gamma_\nu{}^{ac}\Gamma_\mu{}^b{}_c\,, \tag{2·11}$$

which is antisymmetric both in $a\leftrightarrow b$ and in $\mu\leftrightarrow\nu$. Hence

$$R^\mu{}_\nu=h^\mu{}_a h^\lambda{}_b\hat{R}_{\nu\lambda}{}^{ab}\,. \tag{2·12}$$

Since the commutation relations $[g_{\mu\nu},\,b_\rho{}']$ and $[\dot{g}_{\mu\nu},\,b_\rho{}']$ are reproduced from $[h_{\mu a},\,b_\rho{}']$ and $[\dot{h}_{\mu a},\,b_\rho{}']$ as shown in VI, we may calculate $[R^\mu{}_\nu,\,b_\rho{}']$ in the

*) Throughout the present paper, $T_{\mu\nu}$ denotes the energy-momentum tensor but not the tensor density. If one wishes to introduce a cosmological term, suppose that $T_{\mu\nu}$ includes a term proportional to $g_{\mu\nu}$.

vierbein formalism. We employ the following commutation relations established in VI:

$$[h_{\mu a}, b_\rho'] = -i\kappa (\tilde{g}^{00})^{-1} h_{\rho a} \delta^0{}_\mu \delta^3, \tag{2·13}$$

$$[h^{\mu a}, b_\rho'] = i\kappa (\tilde{g}^{00})^{-1} h^{0a} \delta^\mu{}_\rho \delta^3, \tag{2·14}$$

$$[\Gamma_\mu{}^{ab}, b_\rho'] = -i\kappa (\tilde{g}^{00})^{-1} \delta^0{}_\mu \Gamma_\rho{}^{ab} \delta^3. \tag{2·15}$$

The simplicity of (2·15) is noteworthy. From it we see that $\Gamma_k{}^{ab}$ *commutes with* b_ρ.

Owing to the antisymmetry of $\hat{R}_{\nu\lambda}{}^{ab}$ in $a \leftrightarrow b$, we can write

$$R^0{}_k = h^0{}_a h^l{}_b \hat{R}_{kl}{}^{ab}\,. \tag{2·16}$$

Since (2·11) and (2·15) imply

$$[\hat{R}_{kl}{}^{ab}, b_\rho'] = 0\,, \tag{2·17}$$

from (2·14) we immediately obtain the $\nu = k$ case of (2·9). Next, since

$$\begin{aligned} R &= h^\mu{}_a h^\lambda{}_b \hat{R}_{\mu\lambda}{}^{ab} \\ &= 2h^0{}_a h^l{}_b \hat{R}_{0l}{}^{ab} + h^k{}_a h^l{}_b \hat{R}_{kl}{}^{ab}\,, \end{aligned} \tag{2·18}$$

we have

$$2R^0{}_0 - R = -h^k{}_a h^l{}_b \hat{R}_{kl}{}^{ab}\,. \tag{2·19}$$

Hence, again owing to (2·17) together with (2·14) and (2·16), we obtain the $\nu = 0$ case of (2·9). This completes the proof.

§ 3. Tensor-like commutation relations

We continue our analysis concerning the commutators involving b_ρ. Since it is easy to show that

$$[E_{kl}, b_\rho'] = 0\,, \tag{3·1}$$

$$[T_{S\,kl}, b_\rho'] = 0\,, \tag{3·2}$$

(2·1) implies

$$[R_{kl}, b_\rho'] = 0\,. \tag{3·3}$$

With the aid of the identities

$$R_{0k} = (g^{00})^{-1} (R^0{}_k - g^{0l} R_{kl})\,, \tag{3·4}$$

$$R_{00} = (g^{00})^{-1} (2R^0{}_0 - R + g^{jk} R_{jk}) \tag{3·5}$$

together with

$$[g^{\mu\nu}, b_\rho'] = i\kappa (\tilde{g}^{00})^{-1} (\delta^\nu{}_\rho g^{\mu 0} + \delta^\mu{}_\rho g^{0\nu})\, \delta^3, \tag{3·6}$$

we can rewrite (2·9) and (3·3) into

$$[R_{\mu\nu}, b_\rho'] = -i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu R_{\rho\nu} + \delta^0{}_\nu R_{\mu\rho})\delta^3. \tag{3·7}$$

It is remarkable that (3·7) is of the same form as

$$[g_{\mu\nu}, b_\rho'] = -i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu g_{\rho\nu} + \delta^0{}_\nu g_{\mu\rho})\delta^3. \tag{3·8}$$

In general, if an operator $X_{\mu\nu}$, which may not necessarily be symmetric in $\mu\leftrightarrow\nu$, satisfies

$$[X_{\mu\nu}, b_\rho'] = -i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu X_{\rho\nu} + \delta^0{}_\nu X_{\mu\rho})\delta^3, \tag{3·9}$$

then with the aid of (3·6) we see that

$$[X^\mu{}_\nu, b_\rho'] = i\kappa(\tilde{g}^{00})^{-1}(\delta^\mu{}_\rho X^0{}_\nu - \delta^0{}_\nu X^\mu{}_\rho)\delta^3, \tag{3·10}$$

$$[X^{\mu\nu}, b_\rho'] = i\kappa(\tilde{g}^{00})^{-1}(\delta^\mu{}_\rho X^{0\nu} + \delta^\nu{}_\rho X^{\mu 0})\delta^3. \tag{3·11}$$

In particular, $X^\mu{}_\mu$ commutes with b_ρ.

It is easy to see that the operators $\partial_\mu b_\nu + \partial_\nu b_\mu$, $\partial_\mu \bar{c}_\sigma \cdot \partial_\nu c^\sigma$, $T_{S\mu\nu}$ and $g_{\mu\nu}T_S$, which appear in (2·1), all satisfy (3·9). On the other hand, as remarked in the Appendix of VI, many quantities such as $h_{\mu a}$, $\Gamma_\mu{}^{ab}$, $\partial_\mu c^\sigma$, $\partial_\mu \bar{c}_\sigma$, $\partial_\mu s_{ab}$, $\partial_\mu t_{ab}$, $\partial_\mu \bar{t}_{ab}$, $\partial_\mu \phi$ and $\partial_\mu \psi$ (ψ denotes the Dirac field) satisfy the commutation relation

$$[X_\mu, b_\rho'] = -i\kappa(\tilde{g}^{00})^{-1}\delta^0{}_\mu X_\rho \delta^3. \tag{3·12}$$

Furthermore, b_σ, c^σ, $\bar{c}_\sigma$, s_{ab}, t_{ab}, $\bar{t}_{ab}$, ϕ and ψ commute with b_ρ. Accordingly, it is quite natural to introduce a new notion of the *tensor-like commutation relation*, that is, we generally consider the following commutation relation:

$$[X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}, b_\rho'] = i\kappa(\tilde{g}^{00})^{-1}\Big(\sum_{i=1}^{k}\delta^{\mu_i}{}_\rho X^{\mu_1\cdots 0\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - \sum_{j=1}^{l}\delta^0{}_{\nu_j} X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\rho\cdots\nu_l}\Big)\delta^3. \tag{3·13}$$

It is "tensor-like" because *(3·13) is consistent with taking tensor product and with contracting upper and lower indices.* For instance, if X_μ and Y_μ satisfy (3·12) [i.e., (3·13) with $k=0$ and $l=1$], then $X_\mu Y_\nu$ satisfies (3·9) [i.e., (3·13) with $k=0$ and $l=2$] and $g^{\mu\nu}X_\mu Y_\nu$ commutes with b_ρ [i.e., (3·13) with $k=l=0$]. In this sense, our tensor-like commutation relation is quite analogous to general covariance, but it should be noted that the former has no classical counterpart in contrast with the BRS transformation. We conjecture that *any quantity which is a tensor under the general coordinate transformation at the classical level satisfies (3·13) with the same tensor character,* though the converse is not true. The tensor-like property is quite useful in practical calculations.

For a tensor density, of course, a term

$$-\delta^0{}_\rho X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \tag{3·14}$$

should be added in the parentheses of (3·13). Therefore, it cancels the contribution from a contravariant zeroth component; for example, if $X^\mu{}_\nu$ is a mixed tensor density in the sense stated above, $X^0{}_\nu$ has the commutator with b_ρ just like a covariant vector (cf. the Appendix of VI).

In § 2, we have restricted our consideration to the case in which matter field is a scalar field alone. We now extend our analysis to more general cases. In order to guarantee the validity of (1·1) and (3·7), it remains to show that $T_{\mu\nu}$ satisfies (3·9).

The energy-momentum tensor, $T_{\mathrm{D}\,\mu\nu}$, for the Dirac field (see V) is given by

$$T_{\mathrm{D}\,\mu\nu}=\tfrac{1}{4}\,i\bar{\psi}\,[\gamma_\mu(\partial_\nu-\Gamma_\nu)-(\overleftarrow{\partial}_\mu+\Gamma_\mu)\gamma_\nu]\psi+(\mu\leftrightarrow\nu)\,, \tag{3·15}$$

where

$$\bar{\psi}\equiv\psi^\dagger\mathring{\gamma}_0,\qquad \gamma_\mu\equiv h_\mu{}^a\mathring{\gamma}_a,\qquad \Gamma_\mu\equiv\tfrac{1}{8}[\mathring{\gamma}_a,\mathring{\gamma}_b]\,\Gamma_\mu{}^{ab}, \tag{3·16}$$

$\mathring{\gamma}_a$ being the γ-matrices in the flat space-time. Since ψ commutes with b_ρ and since γ_μ, Γ_μ and $\partial_\mu\psi$ satisfy (3·12), we immediately see that $T_{\mathrm{D}\mu\nu}$ satisfies (3·9) owing to the tensor-like property.

In the vierbein formalism presented in V, $T_{\mu\nu}$ contains an additional contribution arising from the local-Lorentz gauge fixing term and the corresponding FP-ghost one, that is, we must discuss $T_{\mathrm{LL}\,\mu\nu}$, which is a sum of

$$T_{\mathrm{LLGF}\,\mu\nu}=-g_{\mu\nu}g^{\sigma\tau}\Gamma_\sigma{}^{ab}\partial_\tau s_{ab}+\Gamma_\mu{}^{ab}\partial_\nu s_{ab}+\Gamma_\nu{}^{ab}\partial_\mu s_{ab}+f_{\mu\nu} \tag{3·17}$$

and

$$\begin{aligned}T_{\mathrm{LLFP}\,\mu\nu}=&\,ig_{\mu\nu}g^{\sigma\tau}\partial_\sigma\bar{t}_{ab}\cdot(\partial_\tau t^{ab}+t^a{}_c\Gamma_\tau{}^{cb}-t^b{}_c\Gamma_\tau{}^{ca})\\&-i[\partial_\mu\bar{t}_{ab}\cdot(\partial_\nu t^{ab}+t^a{}_c\Gamma_\nu{}^{cb}-t^b{}_c\Gamma_\nu{}^{ca})+(\mu\leftrightarrow\nu)]+\hat{f}_{\mu\nu}\,,\end{aligned} \tag{3·18}$$

where

$$\begin{aligned}f_{\mu\nu}\equiv&-g^{\sigma\tau}g_{\lambda\nu}h_{\mu c}(\partial\Gamma_\tau{}^{ab}/\partial h_{\lambda c})\,\partial_\sigma s_{ab}\\&+h^{-1}h_{\mu c}g_{\lambda\nu}\partial_\alpha\{\tilde{g}^{\sigma\tau}[\partial\Gamma_\tau{}^{ab}/\partial(\partial_\alpha h_{\lambda c})]\,\partial_\sigma s_{ab}\}\end{aligned} \tag{3·19}$$

and $\hat{f}_{\mu\nu}$ is obtained from (3·19) by replacing $\partial_\sigma s_{ab}$ by $-i[\partial_\sigma\bar{t}_{cb}\cdot t^c{}_a-(a\leftrightarrow b)]$,*) In the Appendix we prove that both $T_{\mathrm{LLGF}\,\mu\nu}$ and $T_{\mathrm{LLFP}\,\mu\nu}$ satisfy (3·9) separately. Thus the validity of (1·1) and (3·7) has been established also in the coupled Einstein-Dirac system.

§ 4. Quantum gravi-electrodynamics

In this section, we formulate the manifestly covariant quantum field theory of the coupled Einstein-Maxwell system, and then check the validity of the conjecture

*) Here we have antisymmetrized it for later convenience; this is possible because $\Gamma_\tau{}^{ab}$ is antisymmetric in $a\leftrightarrow b$.

proposed in § 3 concerning the commutators involving b_ρ in this theory. It is important that (1·1), (3·7), etc. are valid independently of matter fields.

The Lagrangian density for the electromagnetic field A_μ is given by

$$\mathcal{L}_{\mathrm{EM}} \equiv -\tfrac{1}{4} h g^{\mu\sigma} g^{\nu\tau} F_{\mu\nu} F_{\sigma\tau} - \tilde{g}^{\mu\nu} A_\nu \partial_\mu B + \tfrac{1}{2} \alpha h B^2 - i\tilde{g}^{\mu\nu} \partial_\mu \bar{C} \cdot \partial_\nu C \,, \tag{4·1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, B is an auxiliary scalar field, C and $\bar{C}$ are the FP ghosts of the electromagnetic field and α denotes a gauge parameter. The field equations related to the electromagnetic field are

$$\partial_\mu (h F^{\mu\nu}) - \tilde{g}^{\mu\nu} \partial_\mu B = 0 \,, \tag{4·2}$$

$$g^{\mu\nu} \partial_\mu A_\nu + \alpha B = 0 \,, \tag{4·3}$$

$$g^{\mu\nu} \partial_\mu \partial_\nu C = 0 \,, \tag{4·4}$$

$$g^{\mu\nu} \partial_\mu \partial_\nu \bar{C} = 0 \,, \tag{4·5}$$

where use has been made of (2·4). From (4·2) it follows that

$$g^{\mu\nu} \partial_\mu \partial_\nu B = 0 \,. \tag{4·6}$$

Note that B no longer satisfies a free-field equation.

In making canonical quantization, we replace $\mathcal{L}_{\mathrm{EM}}$ by $\tilde{\mathcal{L}}_{\mathrm{EM}}$, because we wish to regard A_0 rather than B as a canonical variable. Here $\tilde{\mathcal{L}}_{\mathrm{EM}}$ is obtained from $\mathcal{L}_{\mathrm{EM}}$ by replacing the second term of (4·1) by

$$+\partial_\mu (\tilde{g}^{\mu\nu} A_\nu) \cdot B \,. \tag{4·7}$$

It should be noted that (4·7) contains time derivatives of $g_{\mu\nu}$.

The canonical conjugates are defined by

$$\pi_A{}^\nu \equiv \partial \tilde{\mathcal{L}}_{\mathrm{EM}} / \partial \dot{A}_\nu = -h F^{0\nu} + \tilde{g}^{0\nu} B \,, \tag{4·8}$$

$$\pi_C \equiv \partial \tilde{\mathcal{L}}_{\mathrm{EM}} / \partial \dot{C} = -i \tilde{g}^{0\mu} \partial_\mu \bar{C} \,, \tag{4·9}$$

$$\pi_{\bar{C}} \equiv \partial \tilde{\mathcal{L}}_{\mathrm{EM}} / \partial \dot{\bar{C}} = +i \tilde{g}^{0\nu} \partial_\nu C \,. \tag{4·10}$$

The non-vanishing canonical (anti-) commutation relations are

$$[A_\mu, \pi_A{}^{\nu\prime}] = i \delta^\nu{}_\mu \delta^3, \tag{4·11}$$

$$\{C, \pi_C{}'\} = \{\bar{C}, \pi_{\bar{C}}{}'\} = i\delta^3. \tag{4·12}$$

Since $B = (\tilde{g}^{00})^{-1} \pi_A{}^0$, canonical commutation relations yield

$$[B, B'] = 0 \,, \tag{4·13}$$

$$[A_\mu, B'] = i (\tilde{g}^{00})^{-1} \delta^0{}_\mu \delta^3. \tag{4·14}$$

From the $\nu = k$ part of (4·11), we can show that

$$[A_\mu, \dot{A}_l{}'] = -i (\tilde{g}^{00})^{-1} g_{\mu l} \delta^3. \tag{4·15}$$

Hence from (4·3) we have

$$[A_\mu, \dot{A}_0'] = -i(\tilde{g}^{00})^{-1}[g_{\mu 0} - (1-\alpha)\delta^0{}_\mu (g^{00})^{-1}]\delta^3. \tag{4·16}$$

It is straightforward to see

$$[F_{\mu\nu}, B'] = 0 \tag{4·17}$$

from $[\pi_A{}^k, \pi_A{}^{0\prime}] = 0$ together with (4·13) and (4·14).

In order to calculate the commutators involving $\dot{B}$, using (2·4) we rewrite the $\nu=0$ component of (4·2) as

$$\begin{aligned} g^{0\tau}\partial_\tau B &= g^{\mu\sigma} g^{0\tau}\partial_\mu F_{\sigma\tau} + g^{\mu\sigma}\partial_\mu g^{0\tau}\cdot F_{\sigma\tau} \\ &= g^{k\sigma} g^{0\tau}\partial_k F_{\sigma\tau} - g^{0\tau}\partial_\mu g_{\tau\lambda}\cdot F^{\mu\lambda}. \end{aligned} \tag{4·18}$$

Since the right-hand side contains no $\ddot{A}_\nu$, we can calculate commutation relations of $\dot{B}$. For example,

$$\begin{aligned} g^{00}[\dot{B}, A_\mu'] = &-g^{0k}[\partial_k B, A_\mu'] + g^{k\sigma} g^{0\tau}[\partial_k F_{\sigma\tau}, A_\mu'] \\ &- g^{0\tau}\partial_\nu g_{\tau\lambda}\cdot [F^{\nu\lambda}, A_\mu']. \end{aligned} \tag{4·19}$$

With the aid of (2·4), we obtain

$$[\dot{B}, A_\mu'] = i\partial_\mu(\tilde{g}^{00})^{-1}\cdot\delta^3 + i[2\delta^0{}_\mu (g^{00})^{-1} g^{0k} - \delta^k{}_\mu](\delta^3)_k, \tag{4·20}$$

where

$$(\delta^3)_k \equiv \partial_k[(\tilde{g}^{00})^{-1}\delta^3]. \tag{4·21}$$

It is easy to show that

$$[\dot{B}, B'] = [\dot{B}, C'] = [\dot{B}, \bar{C}'] = 0. \tag{4·22}$$

The electromagnetic BRS charge Q_B is defined by

$$Q_B \equiv \int d^3x \tilde{g}^{0\nu}(B\partial_\nu C - \partial_\nu B\cdot C). \tag{4·23}$$

By using the commutation relations obtained above, it can be shown that Q_B is indeed the generator of the electromagnetic BRS transformation $\boldsymbol{\delta}_{\mathrm{EM}}$:

$$i[Q_B, X]_\mp = \boldsymbol{\delta}_{\mathrm{EM}}(X), \tag{4·24}$$

where

$$\boldsymbol{\delta}_{\mathrm{EM}}(A_\mu) = \partial_\mu C, \tag{4·25}$$

$$\boldsymbol{\delta}_{\mathrm{EM}}(\bar{C}) = iB, \tag{4·26}$$

and the $\boldsymbol{\delta}_{\mathrm{EM}}$ of any other field vanishes. The electromagnetic subsidiary condition is, of course, defined by

$$Q_B|\text{phys}\rangle=0\,. \tag{4·27}$$

Having formulated the theory proper to the electromagnetic field, we discuss the relation between the electromagnetic part and the gravitational one. First, we note that the canonical conjugate, $\pi^{\mu\nu}$, of $g_{\mu\nu}$ is modified due to the presence of $\dot{g}_{\mu\nu}$ in (4·7). The modification to be made is just to replace b_ρ by $b_\rho+\kappa A_\rho B$ in $\pi^{\mu\nu}$. Hence no change occurs about $[g_{\mu\nu}, \dot{g}_{\sigma\tau}']$ and $[g_{\mu\nu}, b_\rho']$.

Now, we investigate the commutators involving b_ρ. From $[A_\mu, \pi^{0\nu\prime}]=0$ and $[\pi_A{}^0, \pi^{0\nu\prime}]=0$, we have

$$[A_\mu, b_\rho']=-i\kappa(\tilde{g}^{00})^{-1}\delta^0{}_\mu A_\rho\delta^3, \tag{4·28}$$

$$[B, b_\rho']=0\,, \tag{4·29}$$

respectively, where use has been made of (4·14) and $[\tilde{g}^{00}, b_\rho']=i\kappa\delta^0{}_\rho\delta^3$. Likewise, $[\pi_A{}^k, \pi^{0\nu\prime}]=0$ together with (4·11) yields

$$[F_{0l}, b_\rho']=-i\kappa(\tilde{g}^{00})^{-1}F_{\rho l}\delta^3, \tag{4·30}$$

whence

$$[F_{\mu\nu}, b_\rho']=-i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu F_{\rho\nu}+\delta^0{}_\nu F_{\mu\rho})\delta^3. \tag{4·31}$$

It is easy to confirm that $[b_\rho, b_\sigma']=0$ (see II) remains unchanged by using (4·28), (4·29) and (4·14).

The electromagnetic energy-momentum tensor $T_{\mathrm{EM}\,\mu\nu}$ is defined by

$$\begin{aligned} T_{\mathrm{EM}\,\mu\nu}&\equiv 2h^{-1}\partial\mathcal{L}_{\mathrm{EM}}/\partial g^{\mu\nu} \\ &=-g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau}-(A_\nu\partial_\mu B+A_\mu\partial_\nu B) \\ &\quad -i(\partial_\mu\bar{C}\cdot\partial_\nu C+\partial_\nu\bar{C}\cdot\partial_\mu C)-g_{\mu\nu}h^{-1}\mathcal{L}_{\mathrm{EM}}\,. \end{aligned} \tag{4·32}$$

It is evident from the tensor-like property that $T_{\mathrm{EM}\,\mu\nu}$ satisfies (3·9) if the expected tensor-like commutation relation hold for each of A_μ, $F_{\mu\nu}$, B, $\partial_\mu B$, $\partial_\mu C$ and $\partial_\mu\bar{C}$. The first three are established already by (4·28), (4·31) and (4·29). The last two are obviously all right. Accordingly, it remains only to calculate $[\dot{B}, b_\rho']$. From (4·18) we have

$$\begin{aligned} g^{00}[\dot{B}, b_\rho']=&-[g^{0\tau}, b_\rho']\partial_\tau B+[g^{k\sigma}g^{0\tau}\partial_k F_{\sigma\tau}, b_\rho'] \\ &-[g^{0\tau}\partial_\mu g_{\tau\lambda}\cdot F^{\mu\lambda}, b_\rho']. \end{aligned} \tag{4·33}$$

For calculating the last term, it is convenient to make use of (A·5). After some calculation, in which (4·31) and (4·18) are made use of, we find

$$[\dot{B}, b_\rho']=-i\kappa(\tilde{g}^{00})^{-1}\partial_\rho B\cdot\delta^3, \tag{4·34}$$

that is, $\partial_\mu B$ satisfies (3·12). Thus $T_{\mathrm{EM}\,\mu\nu}$ has the tensor-like commutation relation.

Appendix

——Commutator between $T_{\mathrm{LL}\,\mu\nu}$ and b_ρ——

We first consider $[T_{\mathrm{LLGF}\,\mu\nu}, b_\rho']$. It is evident from the tensor-like property that each of the first three terms of (3·17) satisfies (3·9). Hence we have only to calculate $[f_{\mu\nu}, b_\rho']$. The explicit expression for $f_{\mu\nu}$ is calculated from (3·19):

$$\begin{aligned} f_{\mu\nu} = & [(h_\mu{}^b g^{\sigma\tau}\partial_\tau h_\sigma{}^a + h_\mu{}^c h^{\sigma a} h^{\tau b}\partial_\tau h_{\sigma c})\partial_\nu s_{ab} + (\mu\leftrightarrow\nu)] \\ & + F_{\mu\nu}{}^{\sigma ab}\partial_\sigma s_{ab} - [h_\mu{}^a h^{\sigma b}\partial_\nu\partial_\sigma s_{ab} + (\mu\leftrightarrow\nu)] \\ & + h_\mu{}^a h_\nu{}^b g^{\sigma\tau}\partial_\sigma\partial_\tau s_{ab} \end{aligned} \tag{A·1}$$

with

$$\begin{aligned} F_{\mu\nu}{}^{\sigma ab} \equiv & -h_\mu{}^a h_\nu{}^c g^{\sigma\tau} h^{ab}\partial_\tau h_{ac} + h_\nu{}^b g^{\sigma\tau}\partial_\tau h_\mu{}^a \\ & - h_\mu{}^b h^{\tau a} g^{\sigma\lambda}\partial_\tau g_{\lambda\nu} - h_\nu{}^b h^{\tau a} g^{\sigma\lambda}\partial_\mu g_{\lambda\tau}\,. \end{aligned} \tag{A·2}$$

We know all necessary commutation relations (see II and VI) except for $[\ddot{s}_{ab}, b_\rho']$, but the calculation is rather cumbersome.

First, by direct computation, we find

$$[g^{\sigma\tau}\partial_\tau h_{\sigma a}, b_\rho'] = 0\,, \tag{A·3}$$

from which the first term $h_\mu{}^b g^{\sigma\tau}\partial_\tau h_\sigma{}^a\partial_\nu s_{ab}$ is seen to satisfy (3·9). More generally, we obtain

$$\begin{aligned} & [h^{\sigma a}h^{\tau b}\partial_\tau h_{\sigma c}, b_\rho'] \\ & = i\kappa h_{\rho c}[2(g^{00})^{-1}h^{0a}h^{0b}g^{0k} - h^{ka}h^{0b} - h^{0a}h^{kb}](\delta^3)_k\,, \end{aligned} \tag{A·4}$$

where $(\delta^3)_k$ is defined by (4·21). Since (A·4) is symmetric in $a\leftrightarrow b$, its contribution vanishes when multiplied by $\partial_\nu s_{ab}$.*) Hence the second term of (A·1) also satisfies (3·9).

Next, we calculate $[F_{\mu\nu}{}^{\sigma ab}\partial_\sigma s_{ab}, b_\rho']$. Since this calculation is complicated, we omit its details. The calculation of the first term of (A·2) is simplified by making use of (A·4). For calculating third and fourth terms of (A·2), the following formula is useful:

$$\begin{aligned} & [g^{\sigma\lambda}\partial_\tau g_{\lambda\nu}, b_\rho'] \\ & = i\kappa(\tilde{g}^{00})^{-1}[\delta^\sigma{}_\rho g^{0\lambda}\partial_\tau g_{\lambda\nu} - \delta^0{}_\nu g^{\sigma\lambda}\partial_\tau g_{\lambda\rho} - \delta^0{}_\tau g^{\sigma\lambda}\partial_\rho g_{\lambda\nu}]\cdot\delta^3 \\ & \quad + i\kappa[-\delta^k{}_\tau(g^{\sigma 0}g_{\nu\rho} + \delta^0{}_\nu\delta^\sigma{}_\rho) - \delta^0{}_\tau(g^{\sigma k}g_{\nu\rho} + \delta^k{}_\nu\delta^\sigma{}_\rho) \\ & \quad + 2\delta^0{}_\tau(g^{00})^{-1}g^{0k}(g^{\sigma 0}g_{\nu\rho} + \delta^0{}_\nu\delta^\sigma{}_\rho)](\delta^3)_k\,. \end{aligned} \tag{A·5}$$

After somewhat tedious calculation, we find

*) Note that $h^{\sigma a}h^{\tau b}\partial_\tau h_{\sigma c} - (a\leftrightarrow b)$ is a scalar under the general coordinate transformation.

$$[F_{\mu\nu}{}^{\sigma ab}\partial_\sigma s_{ab}, b_\rho']$$
$$= -i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu F_{\rho\nu}{}^{\sigma ab}\partial_\sigma s_{ab} + \delta^0{}_\nu F_{\mu\rho}{}^{\sigma ab}\partial_\sigma s_{ab})\cdot\delta^3$$
$$+ i\kappa[-2(g^{00})^{-1}g^{0k}h^{0a}h_\nu{}^b\delta^0{}_\mu + h_\nu{}^b h^{ka}\delta^0{}_\mu$$
$$+ h_\nu{}^b h^{0a}\delta^k{}_\mu + (\mu\leftrightarrow\nu)](\delta^3)_{,k}\partial_\rho s_{ab}\,. \tag{A·6}$$

Since the remainder of (A·1) contains $\ddot{s}_{ab}$, we must make use of the field equation of s_{ab} presented in V:

$$g^{\mu\nu}(\partial_\mu\partial_\nu s_{ab} + \partial_\mu s_{ac}\cdot\Gamma_\nu{}^c{}_b - \partial_\mu s_{bc}\cdot\Gamma_\nu{}^c{}_a)$$
$$-i[g^{\mu\nu}\partial_\mu\bar{t}_{ac}(\partial_\nu t^c{}_b + t^c{}_d\Gamma_\nu{}^d{}_b - t_{bd}\Gamma_\nu{}^{dc}) - (a\leftrightarrow b)] = 0\,. \tag{A·7}$$

It is evident from the tensor-like property that each term of (A·7) except for the first one commutes with b_ρ. Hence

$$[g^{\mu\nu}\partial_\mu\partial_\nu s_{ab}, b_\rho'] = 0\,. \tag{A·8}$$

Accordingly, the last term of (A·1) satisfies (3·9). Furthermore, from (A·8) we have

$$[\ddot{s}_{ab}, b_\rho'] = 2i\kappa(\tilde{g}^{00})^{-1}[-\partial_\rho\dot{s}_{ab}\cdot\delta^3 + \tilde{g}^{0k}\partial_\rho s_{ab}\cdot(\delta^3)_{,k}]\,. \tag{A·9}$$

Then it is straightforward to show that

$$[h_\nu{}^a h^{\sigma b}\partial_\mu\partial_\sigma s_{ab}, b_\rho']$$
$$= -i\kappa(\tilde{g}^{00})^{-1}(\delta^0{}_\mu h_\nu{}^a h^{\sigma b}\partial_\rho\partial_\sigma s_{ab} + \delta^0{}_\nu h_\rho{}^a h^{\sigma b}\partial_\mu\partial_\sigma s_{ab})\cdot\delta^3$$
$$+ i\kappa h_\nu{}^a[2(g^{00})^{-1}g^{0k}h^{0b}\delta^0{}_\mu - h^{kb}\delta^0{}_\mu - h^{0b}\delta^k{}_\mu]\partial_\rho s_{ab}\cdot(\delta^3)_{,k}\,. \tag{A·10}$$

Comparing (A·10) with (A·6), we find that the second line of (A·1) satisfies (3·9). Thus $T_{\mathrm{LLGF}\,\mu\nu}$ satisfies (3·9).

It is now very easy to see that $T_{\mathrm{LLFP}\,\mu\nu}$ also satisfies (3·9), because $\hat{f}_{\mu\nu}$ is very similar to $f_{\mu\nu}$. Owing to the existence of ∂_α in (3·19), the replacement of $\partial_\sigma s_{ab}$ by

$$-i[\partial_\sigma\bar{t}_{cb}\cdot t^c{}_a - (a\leftrightarrow b)] \tag{A·11}$$

yields extra terms

$$i\{h_\nu{}^a h^{\alpha b}[\partial_\mu\bar{t}_{cb}\cdot\partial_\alpha t^c{}_a - (a\leftrightarrow b)] + (\mu\leftrightarrow\nu)\}$$
$$-ih_\mu{}^a h_\nu{}^b g^{\sigma\alpha}[\partial_\sigma\bar{t}_{cb}\cdot\partial_\alpha t^c{}_a - (a\leftrightarrow b)]\,, \tag{A·12}$$

which have no corresponding one in $T_{\mathrm{LLGF}\mu\nu}$, but each term of (A·12) evidently satisfies (3·9). Since it is straightforward to prove

$$[g^{\mu\nu}\partial_\mu\partial_\nu\bar{t}_{ab}, b_\rho'] = 0 \tag{A·13}$$

from the field equation

$$g^{\mu\nu}(\partial_\mu\partial_\nu\bar{t}_{ab}+\partial_\mu\bar{t}_{ac}\cdot\Gamma_\nu{}^c{}_b-\partial_\mu\bar{t}_{bc}\cdot\Gamma_\nu{}^c{}_a)=0\,, \tag{A·14}$$

everything goes well in parallel with the analysis of $T_{\mathrm{LLGF}\mu\nu}$.

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Note Added: On the validity of our conjecture stated in §3, we have obtained the following additional supports:
i) The covariant derivative of A_μ satisfies the tensor-like commutation relation.
ii) Let $X^{\mu\cdots}{}_{\nu\cdots}$ be a tensor at the classical level. If both $X^{\mu\cdots}{}_{\nu\cdots}$ and $\varDelta_\lambda X^{\mu\cdots}{}_{\nu\cdots}$ satisfy the tensor-like commutation relation, $[X^{\mu\cdots}{}_{\nu\cdots}, \dot{b}_\rho{}']$ can be calculated by means of the Leibniz rule. Then $[X^{\mu\cdots}{}_{\nu\cdots}, Q_b]$ can be calculated if $X^{\mu\cdots}{}_{\nu\cdots}$ is a canonical field, and we find that it equals $-i\delta^*(X^{\mu\cdots}{}_{\nu\cdots})$. Thus the tensor-like commutation relation is consistent with BRS transformation.