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Indefinite-Metric Quantum Field Theory of General Relativity. V

——*Vierbein Formalism*——

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The indefinite-metric quantum field theory of general relativity is extended to the coupled system of the gravitational field and a Dirac field on the basis of the vierbein formalism. The six extra degrees of freedom involved in vierbein are made unobservable by introducing an extra subsidiary condition $Q_s|\text{phys}\rangle=0$, where Q_s denotes a new BRS charge corresponding to the local Lorentz invariance. It is shown that a manifestly covariant, unitary, canonical theory can be constructed consistently on the basis of the vierbein formalism.

§ 1. Introduction

In a series of papers,[1)~4)] we have successfully developed the indefinite-metric quantum field theory of gravity. In the first paper,[1)] we emphasized the importance of the following four fundamental principles: Lagrangian and canonical formalism, manifest covariance, indefinite-metric Hilbert space with subsidiary conditions (so as to make the physical S-matrix unitary), and asymptotic completeness. The present author believes that the true fundamental theory describing Nature should satisfy the above four principles, unless space-time itself needs to be quantized.

The existence of Dirac fields in Nature is undoubtedly true. In order to conform to our standpoint, therefore, we must extend our indefinite-metric quantum field theory of general relativity to the coupled system of the gravitational field and a Dirac field. This problem is highly non-trivial, because the generally-covariant formulation of a Dirac field cannot be made in terms of the metric tensor $g_{\mu\nu}$ alone. As is well known, the Dirac theory is most conveniently formulated in terms of vierbein.[5)] The vierbein $h_{\mu a}$ $(a=0, 1, 2, 3)$ involves six extra degrees of freedom, which are nothing but the freedom of choosing the directions of the four axes labeled as $a=0, 1, 2, 3$ at each space-time point. Since the transformation between two choices of four axes is a Lorentz transformation, it is usually called the local Lorentz (LL) transformation, though this name is somewhat misleading, because it is not a coordinate transformation.

It is known that the generally-covariant Dirac Lagrangian density is invariant under the LL transformation. Accordingly, the new situation encountered here is quite similar to the Yang-Mills theory; we have a *local* internal symmetry, which is a Lorentz group. It is natural, therefore, to introduce the BRS transformation corresponding to it and set up a Kugo-Ojima subsidiary condition.[6]

In the present paper, we show that the extension of our canonical formalism[1)~3)] to the vierbein case is carried out consistently. In § 2, we review the generally-covariant formulation of a Dirac field in terms of vierbein. In § 3, after defining the LL-BRS transformation, we introduce the LL-gauge-fixing Lagrangian density and the LL-FP-ghost one. Then a new system of field equations is obtained. In § 4, we discuss the LL-FP-ghost current $J_t{}^\mu$, the LL-BRS current $J_s{}^\mu$, the FP-ghost current $J_c{}^\mu$, the BRS current $J_b{}^\mu$ and the Poincaré generators P_μ and $M_{\mu\nu}$. It is shown that the expressions for Q_c, Q_b and P_μ remain unchanged. In § 5, we introduce the asymptotic fields and show the unitarity of the physical S-matrix in the Heisenberg picture. Discussion is made on our choice of the LL-gauge-fixing term in the final section.

The analysis of commutation relations will be presented in a succeeding paper.

§ 2. Vierbein and a Dirac field

We denote the vierbein by $h_{\mu a}$, which satisfy

$$\eta^{ab}h_{\mu a}h_{\nu b}=g_{\mu\nu}\,, \tag{2·1}$$

$$g^{\mu\nu}h_{\mu a}h_{\nu b}=\eta_{ab}\,, \tag{2·2}$$

where $g_{\mu\nu}$ and η_{ab} are the gravitational field and the Minkowski metric $(+---)$, respectively. Greek indices and Latin ones are raised by $g^{\mu\nu}$ and by η^{ab}, respectively. Let $h\equiv-\det h_{\mu a}$, then $h=\sqrt{-g}$, where $g\equiv\det g_{\mu\nu}$. For any derivation ∂, from (2·2) we have

$$\partial h^{\mu a}=-h^{\mu b}h^{\nu a}\partial h_{\nu b}\,, \tag{2·3}$$

$$\partial h=hh^{\mu a}\partial h_{\mu a}\,. \tag{2·4}$$

Expressing the affine connection $\Gamma_{\mu\nu}{}^\lambda$ in terms of vierbein, we see*)

$$\Gamma_{\mu\nu}{}^\lambda M^\mu{}_\lambda=h^{\lambda a}\partial_\nu h_{\mu a}\cdot M^\mu{}_\lambda \tag{2·5}$$

for any $M^{\mu\nu}=M^{\nu\mu}$.

The generally-covariant Dirac γ-matrices are defined by

$$\gamma^\mu\equiv h^{\mu a}\mathring{\gamma}_a\,, \tag{2·6}$$

so that $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, where $\mathring{\gamma}_a$'s denote the usual γ-matrices in the flat space-time. The spin affine connection is defined by

$$\Gamma_\mu\equiv\tfrac{1}{2}\,\mathring{\sigma}_{ab}\Gamma_\mu{}^{ab}, \tag{2·7}$$

*) A middle dot indicates that the preceding differential operator does not act beyond it.

where $\mathring{\sigma}_{ab} \equiv (\mathring{\gamma}_a\mathring{\gamma}_b - \mathring{\gamma}_b\mathring{\gamma}_a)/4$ and

$$\begin{aligned}\Gamma_\mu{}^{ab} &\equiv (\partial_\mu h_\nu{}^a - \Gamma_{\mu\nu}{}^\rho h_\rho{}^a)\, h^{\nu b} \\ &= \tfrac{1}{2}\{[h^{\nu b}\partial_\mu h_\nu{}^a - h^{\nu b}\partial_\nu h_\mu{}^a - h^{\nu a}h^{\lambda b}h_\mu{}^c\partial_\lambda h_{\nu c}] - (a \leftrightarrow b)\} \\ &= -\Gamma_\mu{}^{ba}. \end{aligned} \tag{2·8}$$

Then the covariant derivative of γ^ν vanishes:

$$\nabla_\mu \gamma^\nu \equiv \partial_\mu \gamma^\nu + \Gamma_{\mu\lambda}{}^\nu \gamma^\lambda - [\Gamma_\mu, \gamma^\nu] = 0\,. \tag{2·9}$$

The Dirac field ψ satisfies[5)]

$$(\gamma^\mu \nabla_\mu + im)\,\psi = 0\,, \tag{2·10}$$

where m stands for a mass and $\nabla_\mu \psi \equiv (\partial_\mu - \Gamma_\mu)\,\psi$. The conjugate field $\bar{\psi}$ is defined by $\bar{\psi} \equiv \psi^\dagger \mathring{\gamma}_0$. Since $h^{\mu a}$ is hermitian, (2·6) implies $\mathring{\gamma}_0(\gamma^\mu)^\dagger \mathring{\gamma}_0 = \gamma^\mu$. Hence (2·10) implies

$$\bar{\psi}\,(\overleftarrow{\nabla}_\mu \gamma^\mu - im) = 0 \tag{2·11}$$

with $\bar{\psi}\overleftarrow{\nabla}_\mu \equiv \bar{\psi}\,(\overleftarrow{\partial}_\mu + \Gamma_\mu)$. The Dirac equations (2·10) and (2·11) follow from the following Dirac Lagrangian density $\mathcal{L}_\mathrm{D}$ as is seen by using (2·9):

$$\mathcal{L}_\mathrm{D} \equiv \frac{1}{2} ih\bar{\psi}\,(\gamma^\mu \vec{\partial}_\mu - \overleftarrow{\partial}_\mu \gamma^\mu - \{\gamma^\mu, \Gamma_\mu\})\,\psi - mh\bar{\psi}\psi\,. \tag{2·12}$$ *)

It can be shown that $\mathcal{L}_\mathrm{D}$ is invariant under both the general coordinate transformation and the LL one.

In the Dirac theory, the canonical energy-momentum tensor density,**)

$$\begin{aligned} T_\mathrm{D}^{\mathrm{can}\,\mu}{}_\lambda &\equiv [\partial\mathcal{L}_\mathrm{D}/\partial(\partial_\mu\psi)]\,\partial_\lambda\psi - \partial_\lambda\bar{\psi}\,[\partial\mathcal{L}_\mathrm{D}/\partial(\partial_\mu\bar{\psi})] - \delta^\mu{}_\lambda \mathcal{L}_\mathrm{D} \\ &= \frac{1}{2} ih\bar{\psi}\,(\gamma^\mu \vec{\partial}_\lambda - \overleftarrow{\partial}_\lambda \gamma^\mu)\,\psi - \delta^\mu{}_\lambda \mathcal{L}_\mathrm{D}\,, \end{aligned} \tag{2·13}$$

is different from the gravitational-source energy-momentum tensor density $T_\mathrm{D}{}^\mu{}_\lambda$ defined by

$$\begin{aligned} T_\mathrm{D}{}^{\mu\nu} &\equiv -h^\nu{}_a\left(\frac{\partial\mathcal{L}_\mathrm{D}}{\partial h_{\mu a}} - \partial_\rho \frac{\partial\mathcal{L}_\mathrm{D}}{\partial(\partial_\rho h_{\mu a})}\right) \\ &= -g^{\mu\nu}\mathcal{L}_\mathrm{D} + \frac{1}{2} ih\bar{\psi}\,(\gamma^\mu \vec{\nabla}^\nu - \overleftarrow{\nabla}^\nu \gamma^\mu)\,\psi \\ &\quad + \frac{1}{16} ih\bar{\psi}\,(\{\gamma^\mu, [\gamma^\nu, \gamma^\lambda]\}\vec{\nabla}_\lambda + \overleftarrow{\nabla}_\lambda \{\gamma^\mu, [\gamma^\nu, \gamma^\lambda]\})\,\psi\,. \end{aligned} \tag{2·14}$$

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*) The differential operator $\overleftarrow{\partial}_\mu$ acts on $\bar{\psi}$ but not on h.

**) In the present paper, we always consider energy-momentum *tensor densities* instead of tensors.

This expression for $T_D{}^{\mu\nu}$ looks non-symmetric in its appearance under $\mu\leftrightarrow\nu$, but with the aid of the Dirac equations, it can be rewritten into a symmetric form:

$$T_D{}^{\mu\nu}=\frac{1}{4}i\hbar\bar{\psi}(\gamma^\mu\vec{\nabla}^\nu+\gamma^\nu\vec{\nabla}^\mu-\overleftarrow{\nabla}^\nu\gamma^\mu-\overleftarrow{\nabla}^\mu\gamma^\nu)\psi\,. \quad (2\cdot15)$$

This fact is a consequence of the LL invariance of $\mathcal{L}_D$.

Finally, we review some important consequences of the invariant variation theory (the second Noether theorem)[7] for later convenience.

Let Λ be any scalar density depending on some fields Φ_A. Then the invariance of $\int_\Omega d^4x\Lambda$ under the general coordinate transformation implies that the following three identities hold:

$$\sum_A\{\partial_\mu[\delta_{\Phi_A}(\Lambda)[\Phi_A]^\mu{}_\lambda]+\delta_{\Phi_A}(\Lambda)\partial_\lambda\Phi_A\}=0\,, \quad (2\cdot16)$$

$$\sum_A[\partial\Lambda/\partial(\partial_\mu\Phi_A)]\partial_\lambda\Phi_A-\delta^\mu{}_\lambda\Lambda-\sum_A\delta_{\Phi_A}(\Lambda)[\Phi_A]^\mu{}_\lambda-\partial_\nu K^{\mu\nu}{}_\lambda=0\,, \quad (2\cdot17)$$

$$K^{\mu\nu}{}_\lambda=-K^{\nu\mu}{}_\lambda \quad (2\cdot18)$$

with

$$K^{\mu\nu}{}_\lambda\equiv-\sum_A[\partial\Lambda/\partial(\partial_\mu\Phi_A)][\Phi_A]^\nu{}_\lambda\,, \quad (2\cdot19)$$

where δ_{Φ_A} denotes the Euler derivative with respect to Φ_A and $[\Phi_A]^\mu{}_\lambda$ is the transformation matrix of Φ_A under the infinitesimal general coordinate transformation (i.e., the infinitesimal change of Φ_A is written as $[\Phi_A]^\mu{}_\lambda\partial_\mu\varepsilon^\lambda$).

Since

$$[h_{\nu a}]^\mu{}_\lambda=-\delta^\mu{}_\nu h_{\lambda a}\,, \quad (2\cdot20)$$

$$[\psi]^\mu{}_\lambda=[\bar{\psi}]^\mu{}_\lambda=0\,, \quad (2\cdot21)$$

for $\Lambda=\mathcal{L}_D$ (2·16) becomes

$$\partial_\mu[\delta_{h_{\nu a}}(\mathcal{L}_D)(-\delta^\mu{}_\nu h_{\lambda a})]+\delta_{h_{\nu a}}(\mathcal{L}_D)\partial_\lambda h_{\nu a}+\delta_\psi(\mathcal{L}_D)\partial_\lambda\psi-\partial_\lambda\bar{\psi}\cdot\delta_{\bar{\psi}}(\mathcal{L}_D)=0\,. \quad (2\cdot22)$$

The last two terms vanish because of the field equations for ψ and $\bar{\psi}$, namely, (2·10) and (2·11). Therefore (2·22) reduces to

$$\partial_\mu T_D{}^\mu{}_\lambda-h^{\nu a}\partial_\lambda h_{\mu a}\cdot T_D{}^\mu{}_\nu=0\,, \quad (2\cdot23)$$

that is,

$$\nabla_\mu T_D{}^\mu{}_\lambda=0 \quad (2\cdot24)$$

because of $T_D{}^{\mu\nu}=T_D{}^{\nu\mu}$ and (2·5).

The remaining identities (2·17) and (2·18) become

$$T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_{\lambda}+\partial_\lambda h_{\nu a}\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial(\partial_\mu h_{\nu a})}-T_{\mathrm{D}}{}^{\mu}{}_{\lambda}=\partial_\nu\left[h_{\lambda a}\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial(\partial_\mu h_{\nu a})}\right], \tag{2·25}$$

$$h_{\lambda a}\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial(\partial_\mu h_{\nu a})}=-h_{\lambda a}\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial(\partial_\nu h_{\mu a})}, \tag{2·26}$$

respectively. They are important in § 4.

§ 3. Lagrangian and field equations

We first introduce the LL-BRS transformation, which is denoted by $\boldsymbol{\delta}_{\mathrm{LL}}$, while we denote by $\boldsymbol{\delta}$ the BRS transformation corresponding to the general coordinate transformation as before.[1),*)] Since the LL transformations form a Lorentz group, we have[8)]

$$\boldsymbol{\delta}_{\mathrm{LL}}(h_{\mu a})=-t_a{}^b h_{\mu b}, \tag{3·1}$$

$$\boldsymbol{\delta}_{\mathrm{LL}}(\psi)=-\tfrac{1}{2}t^{ab}\mathring{\sigma}_{ab}\psi, \qquad \boldsymbol{\delta}_{\mathrm{LL}}(\bar\psi)=+\tfrac{1}{2}t^{ab}\bar\psi\mathring{\sigma}_{ab}, \tag{3·2}$$

$$\boldsymbol{\delta}_{\mathrm{LL}}(t_{ab})=-t_a{}^c t_{cb}, \tag{3·3}$$

where t_{ab} is one of the LL-FP ghosts. We also introduce another LL-FP ghost $\bar t_{ab}$ and an auxiliary boson field s_{ab}. As usual,[6),1)] we assume that

$$\boldsymbol{\delta}_{\mathrm{LL}}(\bar t_{ab})=is_{ab}, \tag{3·4}$$

$$\boldsymbol{\delta}_{\mathrm{LL}}(s_{ab})=0. \tag{3·5}$$

The three fields s_{ab}, t_{ab} and $\bar t_{ab}$ are all hermitian and antisymmetric under $a\leftrightarrow b$. Furthermore, they are assumed to be BRS-invariant ($\boldsymbol{\delta}(s_{ab})=0$, etc.). The LL-FP ghosts t_{ab} and $\bar t_{ab}$ are fermion fields. They may be commutative or anticommutative with the FP ghost c^ρ and $\bar c_\rho$ and with the Dirac fields ψ and $\bar\psi$. But, for definiteness, we assume that all fermion fields are mutually anticommutative (at the classical level). Hence $\{\boldsymbol{\delta},\boldsymbol{\delta}_{\mathrm{LL}}\}=0$. In contrast with the BRS transformation $\boldsymbol{\delta}$, the LL-BRS transformation $\boldsymbol{\delta}_{\mathrm{LL}}$ commutes with ∂_μ because the LL invariance is an internal symmetry.

All the "old" fields $g_{\mu\nu}$, b_ρ, c^ρ and $\bar c_\rho$, which have been considered previously,[1)~4)] are LL-BRS-invariant. Indeed, we can immediately confirm that

$$\boldsymbol{\delta}_{\mathrm{LL}}(g_{\mu\nu})=0 \tag{3·6}$$

from (3·1) and (2·1). Hence $\boldsymbol{\delta}_{\mathrm{LL}}(h)=0$. We can also easily show that

$$\boldsymbol{\delta}_{\mathrm{LL}}(\Gamma_\mu{}^{ab})=-t^a{}_c\Gamma_\mu{}^{cb}+t^b{}_c\Gamma_\mu{}^{ca}-\partial_\mu t^{ab}, \tag{3·7}$$

$$\boldsymbol{\delta}_{\mathrm{LL}}(\mathcal{L}_{\mathrm{D}})=0. \tag{3·8}$$

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*) $\boldsymbol{\delta}(h_{\mu a})=\kappa\partial_\mu c^\lambda\cdot h_{\lambda a}$, $\boldsymbol{\delta}(\psi)=\boldsymbol{\delta}(\bar\psi)=0$.

Thus both the old Lagrangian density,*) $\tilde{\mathcal{L}}$, considered previously[1] and the Dirac one $\mathcal{L}_D$ are LL-BRS-invariant.

Now, we introduce the gauge-fixing term for the LL transformation. It must meet the following requirements:

1. It should be non-invariant under the LL-BRS transformation.
2. It should depend only on s_{ab} and $h_{\mu c}$.
3. When multiplied by h^{-1}, it should be BRS-invariant.
4. The number of ∂_0 involved in any term of it should not exceed two.
5. It should involve the six degrees of freedom of $\dot{h}_{\mu a}$ independent of $\dot{g}_{\sigma\tau}$.

Then the following choice of the LL-gauge-fixing Lagrangian density is the simplest possible and most natural one:

$$\mathcal{L}_{\rm LLGF} = hg^{\mu\nu}\Gamma_\nu{}^{ab}\partial_\mu s_{ab}\,. \tag{3·9}$$

The corresponding LL-FP-ghost term $\mathcal{L}_{\rm LLFP}$ is determined by

$$\mathcal{L}_{\rm LL} \equiv \mathcal{L}_{\rm LLGF} + \mathcal{L}_{\rm LLFP} = -i\boldsymbol{\delta}_{\rm LL}(hg^{\mu\nu}\Gamma_\nu{}^{ab}\partial_\mu \bar{t}_{ab}), \tag{3·10}$$

that is,

$$\mathcal{L}_{\rm LLFP} \equiv -ihg^{\mu\nu}\partial_\mu \bar{t}_{ab}\cdot(t^a{}_c\Gamma_\nu{}^{cb} - t^b{}_c\Gamma_\nu{}^{ca} + \partial_\nu t^{ab})\,. \tag{3·11}$$

The total Lagrangian density,

$$\mathcal{L}_{\rm tot} \equiv \tilde{\mathcal{L}} + \mathcal{L}_{\rm D} + \mathcal{L}_{\rm LL}\,, \tag{3·12}$$

is invariant under the LL-BRS transformation, and the total action is BRS-invariant.

The field equations which follow from (3·12) are as follows. The Einstein equation[1] becomes

$$h(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R - B^{\mu\nu}) = -\kappa(T_{\rm D}{}^{\mu\nu} + T_{\rm LL}{}^{\mu\nu})\,, \tag{3·13}$$

where $R^{\mu\nu}$ is the Ricci tensor ($R \equiv R^\mu{}_\mu$) and

$$B^{\mu\nu} \equiv (g^{\mu\sigma}g^{\nu\tau} - \tfrac{1}{2}g^{\mu\nu}g^{\sigma\tau})\,[(\partial_\sigma b_\tau - i\kappa\partial_\sigma\bar{c}_\rho\cdot\partial_\tau c^\rho) + (\sigma\leftrightarrow\tau)]\,, \tag{3·14}$$

$$T_{\rm LL}{}^{\mu\nu} \equiv -h^\nu{}_a\left(\frac{\partial\mathcal{L}_{\rm LL}}{\partial h_{\mu a}} - \partial_\rho\frac{\partial\mathcal{L}_{\rm LL}}{\partial(\partial_\rho h_{\mu a})}\right). \tag{3·15}$$

The following field equations[1] remain unchanged:

$$\partial_\mu(hg^{\mu\nu}) = 0\,, \tag{3·16}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu c^\rho = 0\,, \tag{3·17}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho = 0\,. \tag{3·18}$$

*) We consider the Landau-gauge case alone and, for simplicity, omit any matter field other than the Dirac field.

New field equations are

$$g^{\mu\nu}\partial_\mu \Gamma_\nu{}^{ab}=0\,, \tag{3·19}$$

$$g^{\mu\nu}(\partial_\mu t^a{}_c\cdot\Gamma_\nu{}^{cb}-\partial_\mu t^b{}_c\cdot\Gamma_\nu{}^{ca}+\partial_\mu\partial_\nu t^{ab})=0\,, \tag{3·20}$$

$$g^{\mu\nu}(\partial_\mu \bar{t}^a{}_c\cdot\Gamma_\nu{}^{cb}-\partial_\mu \bar{t}^b{}_c\cdot\Gamma_\nu{}^{ca}+\partial_\mu\partial_\nu \bar{t}^{ab})=0 \tag{3·21}$$

together with the Dirac equations (2·10) and (2·11).

From (3·7), we see that (3·20) coincides with the $\boldsymbol{\delta}_{\rm LL}$ of (3·19). Likewise, the $\boldsymbol{\delta}_{\rm LL}$ of (3·21) is

$$ig^{\mu\nu}(\partial_\mu s^a{}_c\cdot\Gamma_\nu{}^{cb}-\partial_\mu s^b{}_c\cdot\Gamma_\nu{}^{ca}+\partial_\mu\partial_\nu s^{ab})$$
$$+[g^{\mu\nu}\partial_\mu\bar{t}^a{}_c\cdot(t^c{}_d\Gamma_\nu{}^{db}-t^b{}_d\Gamma_\nu{}^{dc}+\partial_\nu t^{cb})-(a\leftrightarrow b)]=0\,. \tag{3·22}$$

It is quite instructive to see that (3·22) coincides with the antisymmetric part of the field equation (3·13).

Since $T_{\rm D}{}^{\mu\nu}$ is symmetric, the antisymmetric part of (3·13) is

$$T_{\rm LL}{}^{\mu\nu}-T_{\rm LL}{}^{\nu\mu}=0\,. \tag{3·23}$$

Here

$$T_{\rm LL}{}^{\mu\nu}=T_{\rm LLGF}{}^{\mu\nu}+T_{\rm LLFP}{}^{\mu\nu} \tag{3·24}$$

with

$$h_\mu{}^b h_\nu{}^a T_{\rm LLGF}{}^{\mu\nu}\equiv -h_\mu{}^b\left(\frac{\partial\mathcal{L}_{\rm LLGF}}{\partial h_{\mu a}}-\partial_\rho\frac{\partial\mathcal{L}_{\rm LLGF}}{\partial(\partial_\rho h_{\mu a})}\right)$$
$$=-\eta^{ab}\mathcal{L}_{\rm LLGF}+h(h^{\sigma a}h^{\nu b}+h^{\sigma b}h^{\nu a})\Gamma_\nu{}^{cd}\partial_\sigma s_{cd}$$
$$-hg^{\sigma\nu}h_\mu{}^b\frac{\partial\Gamma_\nu{}^{cd}}{\partial h_{\mu a}}\partial_\sigma s_{cd}+h_\mu{}^b\partial_\rho\left(hg^{\sigma\nu}\frac{\partial\Gamma_\nu{}^{cd}}{\partial(\partial_\rho h_{\mu a})}\partial_\sigma s_{cd}\right) \tag{3·25}$$

and

$$h_\mu{}^b h_\nu{}^a T_{\rm LLFP}{}^{\mu\nu}\equiv -h_\mu{}^b\left(\frac{\partial\mathcal{L}_{\rm LLFP}}{\partial h_{\mu a}}-\partial_\rho\frac{\partial\mathcal{L}_{\rm LLFP}}{\partial(\partial_\rho h_{\mu a})}\right)$$
$$=-\eta^{ab}\mathcal{L}_{\rm LLFP}-ih(h^{\sigma a}h^{\nu b}+h^{\sigma b}h^{\nu a})\partial_\sigma\bar{t}_{de}\cdot(t^d{}_c\Gamma_\nu{}^{ce}-t^e{}_c\Gamma_\nu{}^{cd}++\partial_\nu t^{de})$$
$$+2ihg^{\sigma\nu}\partial_\sigma\bar{t}_{de}\cdot t^d{}_c h_\mu{}^b\partial\Gamma_\nu{}^{ce}/\partial h_{\mu a}$$
$$-2ih_\mu{}^b\partial_\rho[hg^{\sigma\nu}\partial_\sigma\bar{t}_{de}\cdot t^d{}_c\partial\Gamma_\nu{}^{ce}/\partial(\partial_\rho h_{\mu a})]\,. \tag{3·26}$$

The explicit expressions

$$2\partial\Gamma_\nu{}^{cd}/\partial h_{\mu a}=[-h^{\rho a}h^{\mu d}\partial_\nu h_\rho{}^c+h^{\rho a}h^{\mu d}\partial_\rho h_\nu{}^c$$
$$+h^{\rho a}h^{\mu c}h^{\lambda d}h_\nu{}^e\partial_\lambda h_{\rho e}+h^{\rho c}h^{\lambda a}h^{\mu d}h_\nu{}^e\partial_\lambda h_{\rho e}$$
$$-\delta^\mu{}_\nu h^{\rho c}h^{\lambda d}\partial_\lambda h_\rho{}^a]-(c\leftrightarrow d)\,, \tag{3·27}$$

$$2\partial\Gamma_\nu{}^{cd}/\partial(\partial_\rho h_{\mu a}) = [\delta^\rho{}_\nu \eta^{ac} h^{\mu d} - \delta^\mu{}_\nu \eta^{ac} h^{\rho d} - h^{\mu c} h^{\rho d} h_\nu{}^a] - (c \leftrightarrow d) \tag{3·28}$$

yield the following useful identities:

$$h_\mu{}^b \frac{\partial\Gamma_\nu{}^{cd}}{\partial(\partial_\rho h_{\mu a})} - (a \leftrightarrow b) = \delta^\rho{}_\nu \eta^{ac} \eta^{bd} - (a \leftrightarrow b), \tag{3·29}$$

$$h_\mu{}^b \left(\frac{\partial\Gamma_\nu{}^{cd}}{\partial h_{\mu a}} - \partial_\rho \frac{\partial\Gamma_\nu{}^{cd}}{\partial(\partial_\rho h_{\mu a})} \right) - (a \leftrightarrow b) = (\eta^{ad}\Gamma_\nu{}^{cb} - \eta^{ac}\Gamma_\nu{}^{db}) - (a \leftrightarrow b). \tag{3·30}$$

With the aid of them together with (3·16), we can show that

$$h_\mu{}^b h_\nu{}^a T_{\mathrm{LLGF}}{}^{\mu\nu} - (a \leftrightarrow b) = 2hg^{\sigma\nu}(\partial_\sigma s^a{}_c \cdot \Gamma_\nu{}^{cb} - \partial_\sigma s^b{}_c \cdot \Gamma_\nu{}^{ca} + \partial_\nu \partial_\sigma s^{ab}), \tag{3·31}$$

$$h_\mu{}^b h_\nu{}^a T_{\mathrm{LLFP}}{}^{\mu\nu} - (a \leftrightarrow b) = -2ih[g^{\sigma\nu}\partial_\sigma \bar{t}^a{}_c \cdot (t^c{}_d \Gamma_\nu{}^{db} - t^b{}_d \Gamma_\nu{}^{dc} + \partial_\nu t^{cb}) - (a \leftrightarrow b)]. \tag{3·32}$$

Thus (3·22) is equivalent to (3·23).

Finally, we show that the covariant derivative of $T_{\mathrm{LL}}{}^\mu{}_\lambda$ vanishes. Applying the same reasoning as the one at the end of §2 to $\Lambda = \mathcal{L}_{\mathrm{LL}}$, we see that (2·16) yields

$$\partial_\mu T_{\mathrm{LL}}{}^\mu{}_\lambda - h^{\nu a}\partial_\lambda h_{\mu a} \cdot T_{\mathrm{LL}}{}^\mu{}_\lambda = 0. \tag{3·33}$$

Because of (3·23), the antisymmetric part of $T_{\mathrm{LL}}{}^{\mu\nu}$ vanishes. Hence (3·33) implies

$$\nabla_\mu T_{\mathrm{LL}}{}^\mu{}_\lambda = 0. \tag{3·34}$$

Therefore, the covariant derivative of (3·13) yields[1]

$$g^{\mu\nu}\partial_\mu \partial_\nu b_\rho = 0. \tag{3·35}$$

We note that (3·35) is a direct consequence of the BRS invariance, because it is essentially the $\boldsymbol{\delta}$ of (3·18).

§4. Conserved quantities

In our theory, there are many conserved currents in addition to the Poincaré generators. First, the Dirac current $J_{\mathrm{D}}{}^\mu$ is given by

$$J_{\mathrm{D}}{}^\mu \equiv h\bar{\psi}\gamma^\mu\psi, \tag{4·1}$$

which plays no essential role in our formalism.

Next, the LL-FP-ghost current $J_{\mathrm{t}}{}^\mu$ is defined by

$$J_{\mathrm{t}}{}^{\mu} \equiv [\partial \mathcal{L}_{\mathrm{LLFP}} / \partial(\partial_\mu t_{ab})] t_{ab} + \bar{t}_{ab} [\partial \mathcal{L}_{\mathrm{LLFP}} / \partial(\partial_\mu \bar{t}_{ab})]$$
$$= ihg^{\mu\nu}(\bar{t}_{ab}\partial_\nu t^{ab} - \partial_\nu \bar{t}_{ab} \cdot t^{ab}) + 2ihg^{\mu\nu}\Gamma_\nu{}^{cb}\bar{t}_{ab}t^a{}_c. \tag{4·2}$$

It is easy to confirm $\partial_\mu J_{\mathrm{t}}{}^\mu = 0$ by means of (3·20) and (3·21).

The LL-BRS current $J_{\mathrm{s}}{}^\mu$ is defined by

$$J_{\mathrm{s}}{}^\mu \equiv \sum_A \boldsymbol{\delta}_{\mathrm{LL}}(\Phi_A)[\partial(\partial_\mu \Phi_A) \diagdown \partial \mathcal{L}_{\mathrm{tot}}], \tag{4·3}$$

where the summation goes over all fields. If Λ is a quantity whose dependence on $h_{\mu a}$ is only through $g_{\sigma\tau}$, then we see

$$\boldsymbol{\delta}_{\mathrm{LL}}(h_{\nu a}) \frac{\partial \Lambda}{\partial(\partial_\mu h_{\nu a})} = -t_{ab} h_\nu{}^b h_\sigma{}^a \left(\frac{\partial \Lambda}{\partial(\partial_\mu g_{\nu\sigma})} + \frac{\partial \Lambda}{\partial(\partial_\mu g_{\sigma\nu})} \right) = 0 \tag{4·4}$$

owing to the antisymmetry of t_{ab}. Hence $J_{\mathrm{s}}{}^\mu$ receives no contribution from $\tilde{\mathcal{L}}$. After some calculation, we find

$$J_{\mathrm{s}}{}^\mu = hg^{\mu\nu}(s_{ab}\partial_\nu t^{ab} - \partial_\nu s_{ab} \cdot t^{ab} + 2\Gamma_\nu{}^{cb} s_{ab} t^a{}_c)$$
$$+ ihg^{\mu\nu}\partial_\nu \bar{t}_{ab} \cdot t^{bc} t^a{}_c$$
$$= -\boldsymbol{\delta}_{\mathrm{LL}}(J_{\mathrm{t}}{}^\mu). \tag{4·5}$$

Of course, one can directly confirm $\partial_\mu J_{\mathrm{s}}{}^\mu = 0$ by using (3·20) ~ (3·22).

Since $\mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{LL}}$ is independent of the FP ghosts c^ρ and $\bar{c}_\rho$, the FP-ghost current $J_{\mathrm{c}}{}^\mu$ remains unchanged.

One naturally expects also that the BRS current $J_{\mathrm{b}}{}^\mu$ remains unchanged because we still have (3·17) and (3·35). But the validity of this statement is non-trivial, because the BRS Noether current receives non-vanishing contributions from $\mathcal{L}_{\mathrm{D}}$ and $\mathcal{L}_{\mathrm{LL}}$. Indeed, the additional contribution [see (4·1) of Ref. 2)] is

$$-\kappa(\partial_\nu c^\lambda \cdot h_{\lambda a} + c^\lambda \partial_\lambda h_{\nu a}) \partial(\mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{LL}}) / \partial(\partial_\mu h_{\nu a}) - \kappa c^\lambda (T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_\lambda + T_{\mathrm{LL}}^{\mathrm{can}\,\mu}{}_\lambda), \tag{4·6}$$

where $T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_\lambda$ is defined by (2·13) and

$$T_{\mathrm{LL}}^{\mathrm{can}\,\mu}{}_\lambda \equiv [\partial \mathcal{L}_{\mathrm{LL}} / \partial(\partial_\mu s_{ab})] \partial_\lambda s_{ab} + [\partial \mathcal{L}_{\mathrm{LL}} / \partial(\partial_\mu t_{ab})] \partial_\lambda t_{ab}$$
$$+ [\partial \mathcal{L}_{\mathrm{LL}} / \partial(\partial_\mu \bar{t}_{ab})] \partial_\lambda \bar{t}_{ab} - \delta^\mu{}_\lambda \mathcal{L}_{\mathrm{LL}}. \tag{4·7}$$

In order to simplify the expression for the BRS Noether current, we made use of the Einstein equation.[2] The Einstein equation is now (3·13), which contains additional terms $\kappa(T_{\mathrm{D}}{}^{\mu\nu} + T_{\mathrm{LL}}{}^{\mu\nu})$. Therefore the total change of the expression for the BRS Noether current is the sum of (4·6) and

$$\kappa c^\lambda (T_{\mathrm{D}}{}^\mu{}_\lambda + T_{\mathrm{LL}}{}^\mu{}_\lambda). \tag{4·8}$$

Because of its dependence on c^λ, it can be written as a total divergence of an antisymmetric tensor density if and only if

$$T_{\mathrm{D}}{}^{\mu}{}_{\lambda}+T_{\mathrm{LL}}{}^{\mu}{}_{\lambda}=T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_{\lambda}+T_{\mathrm{LL}}^{\mathrm{can}\,\mu}{}_{\lambda}+\partial_{\lambda}h_{\nu a}\frac{\partial(\mathcal{L}_{\mathrm{D}}+\mathcal{L}_{\mathrm{LL}})}{\partial(\partial_{\mu}h_{\nu a})}-\partial_{\nu}K^{\mu\nu}{}_{\lambda} \tag{4·9}$$

and

$$K^{\mu\nu}{}_{\lambda}=-K^{\nu\mu}{}_{\lambda}, \tag{4·10}$$

where

$$K^{\mu\nu}{}_{\lambda}\equiv h_{\lambda a}\frac{\partial(\mathcal{L}_{\mathrm{D}}+\mathcal{L}_{\mathrm{LL}})}{\partial(\partial_{\mu}h_{\nu a})}. \tag{4·11}$$

As explained at the end of § 2, *they are indeed identities.* More precisely, (4·9) and (4·10) hold, without using any field equation, for each contribution from $\mathcal{L}_{\mathrm{D}}$, $\mathcal{L}_{\mathrm{LLGF}}$ and $\mathcal{L}_{\mathrm{LLFP}}$ separately. Their validity can also be directly confirmed by means of the formulae

$$\frac{\partial\Gamma_{\sigma}{}^{cd}}{\partial(\partial_{\mu}h_{\nu a})}=-\frac{\partial\Gamma_{\sigma}{}^{cd}}{\partial(\partial_{\nu}h_{\mu a})}, \tag{4·12}$$

$$h_{\lambda a}\frac{\partial\Gamma_{\sigma}{}^{cd}}{\partial h_{\mu a}}+(\partial_{\nu}h_{\lambda a}-\partial_{\lambda}h_{\nu a})\frac{\partial\Gamma_{\sigma}{}^{cd}}{\partial(\partial_{\nu}h_{\mu a})}=\delta^{\mu}{}_{\sigma}\Gamma_{\lambda}{}^{cd}. \tag{4·13}$$

Thus the expression for the BRS current[1] $J_{\mathrm{b}}{}^{\mu}$ remains unchanged.

Quite a similar mechanism takes place also for the (total) canonical energy-momentum tensor density[3]

$$\mathcal{T}^{\mu}{}_{\lambda}\equiv\sum_{A}[\partial\mathcal{L}_{\mathrm{tot}}/\partial(\partial_{\mu}\Phi_{A})]\partial_{\lambda}\Phi_{A}-\delta^{\mu}{}_{\lambda}\mathcal{L}_{\mathrm{tot}}. \tag{4·14}$$

Its effective additional contribution is

$$\frac{\partial(\mathcal{L}_{\mathrm{D}}+\mathcal{L}_{\mathrm{LL}})}{\partial(\partial_{\mu}h_{\nu a})}\partial_{\lambda}h_{\nu a}+(T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_{\lambda}+T_{\mathrm{D}}^{\mathrm{can}\,\mu}{}_{\lambda})-(T_{\mathrm{D}}{}^{\mu}{}_{\lambda}+T_{\mathrm{LL}}{}^{\mu}{}_{\lambda}), \tag{4·15}$$

which is precisely equal to $\partial_{\nu}K^{\mu\nu}{}_{\lambda}$, as is shown above. Thus the expression for the translation generator,[3]

$$P_{\mu}=\kappa^{-1}\int d^{3}x\, hg^{0\sigma}\partial_{\sigma}b_{\mu}, \tag{4·16}$$

remains unchanged.

Finally, the canonical angular-momentum tensor density is defined by[3]

$$\mathcal{M}^{\lambda}{}_{\mu\nu}\equiv x_{\mu}\mathcal{T}^{\lambda}{}_{\nu}-x_{\nu}\mathcal{T}^{\lambda}{}_{\mu}+\mathcal{S}^{\lambda}{}_{\mu\nu}, \tag{4·17}$$

where $\mathcal{S}^{\lambda}{}_{\mu\nu}$ stands for the spin angular momentum. Under the *true* Lorentz transformation, ψ should transform like a spinor. Correspondingly, the vierbein $h_{\mu a}$ should transform not like a vector but like a tensor. In general, any Latin index should *not* be distinguished from a Greek one under the Lorentz transformation.

Hence we must be very careful about raising and lowering indices.

Since

$$(\eta_{\mu\rho}h_{\nu a}-\eta_{\nu\rho}h_{\mu a}+\eta_{\mu a}h_{\rho\nu}-\eta_{\nu a}h_{\rho\mu})\,\partial\tilde{\mathcal{L}}/\partial(\partial_\lambda h_{\rho a})$$
$$=(\eta_{\mu\rho}g_{\nu\sigma}-\eta_{\nu\rho}g_{\mu\sigma})\,[\partial\tilde{\mathcal{L}}/\partial(\partial_\lambda g_{\rho\sigma})+\partial\tilde{\mathcal{L}}/\partial(\partial_\lambda g_{\sigma\rho})] \tag{4·18}$$

owing to the cancellation of the last two terms, the additional contribution to $\mathcal{S}^{\lambda}{}_{\mu\nu}$ is

$$(\eta_{\mu\rho}h_{\nu a}-\eta_{\nu\rho}h_{\mu a}+\eta_{\mu a}h_{\rho\nu}-\eta_{\nu a}h_{\rho\mu})\,\partial(\mathcal{L}_{\rm D}+\mathcal{L}_{\rm LL})/\partial(\partial_\lambda h_{\rho a})$$
$$+[\partial\mathcal{L}_{\rm D}/\partial(\partial_\lambda\psi)]\,\mathring{\sigma}_{\mu\nu}\psi+\bar{\psi}\mathring{\sigma}_{\mu\nu}[\partial\mathcal{L}_{\rm D}/\partial(\partial_\lambda\bar{\psi})]+\mathcal{S}_{\rm LL}{}^{\lambda}{}_{\mu\nu} \tag{4·19}$$

with

$$\mathcal{S}_{\rm LL}{}^{\lambda}{}_{\mu\nu}\equiv(\eta_{\mu a}s_{\nu b}-\eta_{\nu a}s_{\mu b}+\eta_{\mu b}s_{a\nu}-\eta_{\nu b}s_{a\mu})\,\partial\mathcal{L}_{\rm LLGF}/\partial(\partial_\lambda s_{ab})$$
$$-(\eta_{\mu a}t_{\nu b}-\eta_{\nu a}t_{\mu b}+\eta_{\mu b}t_{a\nu}-\eta_{\nu b}t_{a\mu})\,\partial\mathcal{L}_{\rm LLFP}/\partial(\partial_\lambda t_{ab})$$
$$-(\eta_{\mu a}\bar{t}_{\nu b}-\eta_{\nu a}\bar{t}_{\mu b}+\eta_{\mu b}\bar{t}_{a\nu}-\eta_{\nu b}\bar{t}_{a\mu})\,\partial\mathcal{L}_{\rm LLFP}/\partial(\partial_\lambda\bar{t}_{ab})\,. \tag{4·20}$$

On the other hand, the orbital angular momentum $x_\mu\mathcal{T}^{\lambda}{}_{\nu}-x_\nu\mathcal{T}^{\lambda}{}_{\mu}$ acquires an additional contribution

$$x_\mu\partial_\rho K^{\lambda\rho}{}_{\nu}-x_\nu\partial_\rho K^{\lambda\rho}{}_{\mu}. \tag{4·21}$$

That is, apart from a total divergence of a quantity antisymmetric under $\mu\leftrightarrow\nu$, $\mathcal{M}^{\lambda}{}_{\mu\nu}$ receives the contribution

$$-\eta_{\mu\rho}K^{\lambda\rho}{}_{\nu}+\eta_{\nu\rho}K^{\lambda\rho}{}_{\mu}=(-\eta_{\mu\rho}h_{\nu a}+\eta_{\nu\rho}h_{\mu a})\frac{\partial(\mathcal{L}_{\rm D}+\mathcal{L}_{\rm LL})}{\partial(\partial_\lambda h_{\rho a})}\,, \tag{4·22}$$

which exactly cancels the first two terms of (4·19). Furthermore, by direct calculation, we find

$$(\eta_{\mu a}h_{\rho\nu}-\eta_{\nu a}h_{\rho\mu})\,\partial\mathcal{L}_{\rm D}/\partial(\partial_\lambda h_{\rho a})$$
$$=-\frac{1}{2}ih\bar{\psi}\{\gamma^{\lambda},\mathring{\sigma}_{\mu\nu}\}\psi$$
$$=-[\partial\mathcal{L}_{\rm D}/\partial(\partial_\lambda\psi)]\,\mathring{\sigma}_{\mu\nu}\psi-\bar{\psi}\mathring{\sigma}_{\mu\nu}[\partial\mathcal{L}_{\rm D}/\partial(\partial_\lambda\bar{\psi})]. \tag{4·23}$$

(This identity is a consequence of the LL invariance of $\mathcal{L}_{\rm D}$.) Thus no contribution from $\mathcal{L}_{\rm D}$ remains.*)

After all, the Lorentz generator[3)] $M_{\mu\nu}$ acquires an extra contribution

$$M_{{\rm LL}\,\mu\nu}\equiv\int d^3x[(\eta_{\mu a}h_{\rho\nu}-\eta_{\nu a}h_{\rho\mu})\,\partial\mathcal{L}_{\rm LL}/\partial\dot{h}_{\rho a}+\mathcal{S}_{\rm LL}{}^{0}{}_{\mu\nu}]$$
$$=2\int d^3x\;hg^{0\sigma}L_{\sigma\mu\nu}\,, \tag{4·24}$$

*) In the flat space-time, this corresponds to the well-known fact that the angular momentum tensor contains no extra spin term when expressed in terms of the symmetric energy-momentum tensor.

where

$$L_\sigma{}^{ab} \equiv \partial_\sigma s^{ab} - \Gamma_\sigma{}^{ca} s^b{}_c + \Gamma_\sigma{}^{cb} s^a{}_c$$
$$- i[\bar{t}^a{}_c(\partial_\sigma t^{cb} + \Gamma_\sigma{}^{db} t^c{}_d - \Gamma_\sigma{}^{dc} t^b{}_d) - (a \leftrightarrow b)]. \tag{4·25}$$

Since

$$\partial_\lambda (h g^{\lambda\sigma} L_{\sigma\mu\nu}) = 0 \tag{4·26}$$

owing to (3·16), (3·19), (3·20) and (3·22), we see that $M_{\mathrm{LL}\,\mu\nu}$ is a conserved quantity.

§ 5. Asymptotic fields

Since it is inadequate to eliminate $\dot{s}_{ab}$ from our Lagrangian by integrating by parts, *the six components of* s_{ab} *must be regarded as canonical variables,* that is, s_{ab} is *not* a Lagrange multiplier field. For $\mathcal{L}_{\mathrm{D}}$, we should eliminate $\partial_\mu \bar{\psi}$ by integrating by parts. Thus the canonical variables are $h_{\mu a}$, c^ρ, $\bar{c}_\rho$, s_{ab}, t_{ab}, $\bar{t}_{ab}$ and ψ. Canonical quantization can be carried out consistently. Detailed analysis will be presented in a succeeding paper. We shall show there that all commutation relations concerning the old fields ($g_{\mu\nu} = h_{\mu a} h_\nu{}^a$, b_ρ, c^ρ, $\bar{c}_\rho$)[2] are precisely reproduced.

The physical states are defined by the subsidiary conditions

$$Q_{\mathrm{b}} |\mathrm{phys}\rangle = 0\,, \quad Q_{\mathrm{s}} |\mathrm{phys}\rangle = 0\,, \tag{5·1}$$

where, of course, both $Q_{\mathrm{b}} \equiv \int d^3x J_{\mathrm{b}}{}^0$ and $Q_{\mathrm{s}} \equiv \int d^3x J_{\mathrm{s}}{}^0$ are conserved.

In order to show the positive semi-definiteness of the norm of the physical-state subspace, we investigate the asymptotic fields under the postulate of asymptotic completeness.

We introduce the asymptotic fields by*)

$$\begin{aligned}
&(h_{\mu a} - \eta_{\mu a})/2\sqrt{\kappa} \to \chi_{\mu a} + \text{Källén term}\,, \\
&b_\rho/\sqrt{\kappa} \to \beta_\rho, \quad c^\rho \to \gamma^\rho, \quad \bar{c}_\rho \to \bar{\gamma}_\rho\,, \\
&\sqrt{\kappa}\, s_{ab} \to \sigma_{ab}, \quad t_{ab} \to \tau_{ab}, \quad \bar{t}_{ab} \to \bar{\tau}_{ab}\,, \\
&\psi \to \psi^{\mathrm{asym}}
\end{aligned} \tag{5·2}$$

as $x^0 \to -\infty$ (or $x^0 \to +\infty$). It is convenient to set

$$\varphi_{ab} \equiv \chi_{ab} + \chi_{ba} = \varphi_{ba}\,, \tag{5·3}$$

$$\epsilon_{ab} \equiv \chi_{ab} - \chi_{ba} = -\epsilon_{ba}\,, \tag{5·4}$$

so that $\varphi_{\mu\nu}$ coincides with the asymptotic field of $g_{\mu\nu}$.

As before,[1),4)] we assume that the properties of the asymptotic fields are gov-

*) Here we omit Z-factors and neglect the problem of ultraviolet divergence.

erned by the linearized Lagrangian density of $\mathcal{L}_{\text{tot}}$ except for the renormalization of the parameters involved. The asymptotic Lagrangian density, $\mathcal{L}^{\text{asym}}$, corresponding to $\mathcal{L}$ remains unchanged. The total asymptotic-field Lagrangian density is given by

$$\mathcal{L}_{\text{tot}}^{\text{asym}} \equiv \tilde{\mathcal{L}}^{\text{asym}} - \partial^c \epsilon^{ab} \cdot \partial_c \sigma_{ab} - 2\partial^b \varphi^{ac} \cdot \partial_c \sigma_{ab} - i\partial_c \bar{\tau}_{ab} \cdot \partial^c \tau^{ab} + \bar{\psi}^{\text{asym}} [\tfrac{1}{2} i \mathring{\gamma}^a (\vec{\partial}_a - \overleftarrow{\partial}_a) - m] \psi^{\text{asym}}, \tag{5·5}$$

from which we have

$$\Box \epsilon^{ab} + \partial^b \partial_c \varphi^{ac} - \partial^a \partial_c \varphi^{bc} = 0\,, \tag{5·6}$$

$$\Box \sigma_{ab} = 0, \quad \Box \tau_{ab} = 0, \quad \Box \bar{\tau}_{ab} = 0\,. \tag{5·7}$$

By using the linearized De Donder condition[1)]

$$\partial_c \varphi^{ac} = \tfrac{1}{2} \partial^a \varphi^c{}_c, \tag{5·8}$$

(5·6) is simplified into

$$\Box \epsilon_{ab} = 0\,. \tag{5·9}$$

Owing to the third term of (5·5), the field equation for φ_{ab} is modified into

$$\Box \varphi_{ab} = \partial_a \beta_b + \partial_b \beta_a + \partial_b \partial^c \sigma_{ac} + \partial_a \partial^c \sigma_{bc}\,. \tag{5·10}$$

It is straightforward to analyze the canonical commutation relations for the asymptotic fields. We find that the four-dimensional commutation relations between the old fields[1)] remain unchanged. Those which involve the new fields are found to be

$$[\varphi_{ab}(x), \epsilon_{cd}(y)] = [\varphi_{ab}(x), \sigma_{cd}(y)] = 0\,, \tag{5·11}$$

$$[\epsilon_{ab}(x), \sigma_{cd}(y)] = -\tfrac{1}{2} i (\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) D(x-y)\,, \tag{5·12}$$

$$[\epsilon_{ab}(x), \epsilon_{cd}(y)] = [\sigma_{ab}(x), \sigma_{cd}(y)] = 0\,, \tag{5·13}$$

$$\{\tau_{ab}(x), \bar{\tau}_{cd}(y)\} = -\tfrac{1}{2} (\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) D(x-y)\,, \tag{5·14}$$

$$\{\tau_{ab}(x), \tau_{cd}(y)\} = \{\bar{\tau}_{ab}(x), \bar{\tau}_{cd}(y)\} = 0\,, \tag{5·15}$$

etc.

The expressions for Q_s and Q_t in terms of the asymptotic fields are, up to a multiplicative constant,

$$Q_s = \int d^3x \, \sigma_{ab} (\vec{\partial}_0 - \overleftarrow{\partial}_0) \tau^{ab}, \tag{5·16}$$

$$Q_t = i \int d^3x \, \bar{\tau}_{ab} (\vec{\partial}_0 - \overleftarrow{\partial}_0) \tau^{ab}. \tag{5·17}$$

Hence

$$[\epsilon_{ab}, Q_s] = i\tau_{ab}, \qquad [\sigma_{ab}, Q_s] = 0\,, \tag{5·18}$$

$$\{\tau_{ab}, Q_s\} = 0, \qquad \{\bar{\tau}_{ab}, Q_s\} = \sigma_{ab}\,. \tag{5·19}$$

Then applying the Kugo-Ojima theorem,[6] we see that the physical-state subspace is positive semidefinite.

Thus the physical S-matrix is unitary.

§ 6. Discussion

In the present paper, we have established that the quantum field theory of the coupled Einstein-Dirac system can be consistently formulated in the framework of the manifestly-covariant canonical formalism.

We make some remarks on the choice of the LL-gauge-fixing term. In the path-integral formalism, one can introduce almost any kind of the gauge-fixing term, though then gauge theories always suffer from the difficulty caused by the Gribov ambiguity.[9] On the contrary, in the covariant canonical formalism, to which the Gribov ambiguity is totally irrelevant, the choice of the gauge-fixing term is quite restrictive. In our theory, our choice (3·9) is practically unique under the conditions stated in § 3. Simpler-looking choices,

$$\mathcal{L}'_{\rm LLGF} \equiv hh^{\lambda a}\partial_\lambda h^{\mu b}\partial_\mu s_{ab} \tag{6·1}$$

and

$$\mathcal{L}''_{\rm LLGF} \equiv \partial_\lambda (hh^{\lambda a})\, h^{\mu b}\partial_\mu s_{ab}\,, \tag{6·2}$$

which are mutually equivalent, satisfy the first four conditions but not the last one. With (6·1) or (6·2), all canonical conjugates of $h_{\mu a}$ are not independent. Of course, (3·9) is not unique in the mathematical sense, for instance, we may add $\mathcal{L}'_{\rm LLGF}$ and/or $hs_{ab}s^{ab}$ to $\mathcal{L}_{\rm LLGF}$. But such modifications are not interesting. We shall see in a succeeding paper that the LL-gauge-fixing term (3·9) yields quite natural equal-time commutation relations between Heisenberg fields.

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