Progress of Theoretical Physics, Vol. 49, No. 2, February 1973

Indefinite-Metric Quantum Theory of Genuine and Higgs-Type Massive Vector Fields

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(Received August 25, 1972)

On the basis of the indefinite-metric vector field theory proposed previously, Johnson's proposition that the physical mass of a vector field tends to zero as its bare mass goes to zero, is shown to be valid if the vector field couples with a charged scalar field in the minimal interaction. In the case of the theory of spontaneously broken gauge invariance, the reason why the vector field acquires a non-zero mass in spite of the above theorem is clarified. The theory of a vector field which is massive owing to the spontaneous breakdown of gauge invariance is consistently formulated in the framework of the indefinite-metric quantum field theory. In this formalism, both renormalizability and the unitarity of the physical S-matrix are self-evident.

§ 1. Introduction

Recently, the present author[1),*)] has proposed an indefinite-metric theroy[2)] of a massive vector field such that as its mass goes to zero the theory smoothly tends to the Landau-gauge quantum electrodynamics.[3),4)] As one of important consequences of this theory, we can reasonably show the validity of Johnson's proposition[5),**)] that if the bare mass of the vector field U_μ goes to zero, its physical mass must also tend to zero, provided that there are no other massless physical particles, under the assumption that the current j_μ is conserved and does not explicitly depend on U_μ.

On the other hand, in connection with Weinberg's theory of leptons,[6)] much attention has been paid to the spontaneous breakdown of gauge invariance in the massless vector field theories. Several years ago, Higgs and others[7)] noted that if gauge invariance of the theory is spontaneously broken, the massless vector field acquires a non-zero mass, but then Goldstone bosons do not appear in the Coulomb gauge because we do not have manifest covariance, which is necessary for the proof of the Goldstone theorem.[8)] If one reconsiders this situation in a covariant gauge, in which we have to introduce indefinite metric, Goldstone bosons appear but they become unphysical. This interesting phenomenon is now called the Higgs phenomenon. Recently, 't Hooft[9)] has applied it to the Yang-Mills

*) Unfortunately, the publication of this paper was much delayed.

**) Johnson's reasoning was based on the conventional massive vector field theory whose massless limit is non-existent.

field in order to construct a renormalizable theory of massive charged vector fields.*) B. W. Lee[10] has made a detailed study of the Higgs phenomenon in the theory of a neutral vector field which couples with a charged scalar field (Higgs model). All these recent investigations are made in the Feynman functional-integral formalism.

The purpose of the present paper is to analyze the Higgs phenomenon on the basis of our indefinite-metric theory of a vector field. We first extend our proof of Johnson's proposition to the case in which the neutral vector field U_μ couples with a charged scalar field (§ 2). Then we encounter an apparent dilemma between Johnson's proposition and the Higgs phenomenon. In order to resolve this paradox, we investigate a solvable model, which is essentially the zeroth approximation to the Higgs model (§ 3). The reason for the dilemma is found to be a special character of j_μ in the case of the spontaneously broken gauge theory. Finally, the full Higgs model is studied (§ 4). We clarify why Goldstone bosons become unphysical in such a way that the unitarity of the physical S-matrix is not violated. The main results obtained by B. W. Lee[10] are reproduced in quite a transparent way.

§ 2. Genuine massive vector field

In this section, we consider a neutral vector field U_μ, which couples with a charged scalar field ϕ. We assume that the interaction between them is the so-called minimal interaction. The Lagrangian density $\mathcal{L}$ of the system is given by

$$\mathcal{L}=-\tfrac{1}{4}(\partial^\mu U^\nu-\partial^\nu U^\mu)(\partial_\mu U_\nu-\partial_\nu U_\mu)+\tfrac{1}{2}m^2U^\mu U_\mu+B\partial^\mu U_\mu+\mathcal{L}_\phi \tag{2·1}$$

with

$$\mathcal{L}_\phi\equiv(\partial^\mu+igU^\mu)\phi^\dagger(\partial_\mu-igU_\mu)\phi+F(\phi^\dagger\phi). \tag{2·2}$$

Here, B is an auxiliary scalar field having negative norm; m and g denote the bare mass of U_μ and the bare coupling constant, respectively; F is a quadratic real polynomial; a dagger stands for hermitian conjugation and the Minkowski metric employed is $(1, -1, -1, -1)$.

The field equations for U_μ and B are

$$\partial^\mu U_\mu=0\,, \tag{2·3}$$

$$(\Box+m^2)U_\mu-\partial_\mu B=j_\mu \tag{2·4}$$

with $\Box\equiv\partial^\mu\partial_\mu$. Here the current

$$\begin{aligned} j_\mu&\equiv-\delta\mathcal{L}_\phi/\delta U^\mu\\ &=-ig[\phi^\dagger\partial_\mu\phi-(\partial_\mu\phi^\dagger)\phi]-2g^2\phi^\dagger\phi U_\mu \end{aligned} \tag{2·5}$$

*) We note, however, that this theory is not a genuine Lagrangian field theory because one has to introduce *ad hoc* Feynman's fictitious quanta. On the other hand, Weinberg's idea is based on the Lagrangian formalism.

is conserved:

$$\partial^\mu j_\mu = 0\,, \tag{2·6}$$

as is easily confirmed, but it explicitly depends on U_μ. From (2·4), (2·3) and (2·6), we have

$$\Box B = 0\,. \tag{2·7}$$

The canonical conjugates of $U_l(l=1, 2, 3)$, U_0, ϕ and $\phi^\dagger$ are

$$\begin{aligned}
\Pi_l &\equiv \delta\mathcal{L}/\delta\dot{U}_l = \dot{U}_l - \partial_l U_0\,,\\
\Pi_0 &\equiv \delta\mathcal{L}/\delta\dot{U}_0 = B\,,\\
\pi &\equiv \delta\mathcal{L}/\delta\dot{\phi} = \dot{\phi}^\dagger + igU_0\phi^\dagger,\\
\pi^\dagger &\equiv \delta\mathcal{L}/\delta\dot{\phi}^\dagger = \dot{\phi} - igU_0\phi\,,
\end{aligned} \tag{2·8}$$

respectively, where a dot stands for differentiation with respect to time. The equal-time commutators for canonical variables are

$$\begin{aligned}
&[U_\mu, \Pi_\nu] = i\delta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[\phi, \pi] = [\phi^\dagger, \pi^\dagger] = i\delta(\boldsymbol{x}-\boldsymbol{y}),
\end{aligned} \tag{2·9}$$

and vanishing commutators for all other combinations. In terms of field variables, the equal-time commutators are rewritten as*)

$$\begin{aligned}
&[U_k(x), \dot{U}_l(y)]_{x_0=y_0} = i\delta_{kl}\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[U_0(x), B(y)]_{x_0=y_0} = i\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[\dot{U}_k(x), B(y)]_{x_0=y_0} = i\partial_k{}^x\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[B(x), \dot{\phi}(y)]_{x_0=y_0} = g\phi(y)\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[B(x), \dot{\phi}^\dagger(y)]_{x_0=y_0} = -g\phi^\dagger(y)\delta(\boldsymbol{x}-\boldsymbol{y}),\\
&[\phi(x), \dot{\phi}^\dagger(y)]_{x_0=y_0} = [\phi^\dagger(x), \dot{\phi}(y)]_{x_0=y_0} = i\delta(\boldsymbol{x}-\boldsymbol{y}),
\end{aligned} \tag{2·10}$$

and vanishing commutators for all other combinations of U_k, U_0, $\dot{U}_l$, B, ϕ, $\dot{\phi}$, $\phi^\dagger$ and $\dot{\phi}^\dagger$. In order to calculate the equal-time commutators involving $\dot{U}_0$ or $\dot{B}$, we have to make use of

$$\begin{aligned}
\dot{U}_0 &= \sum_k \partial_k U_k\,,\\
\dot{B} &= \sum_k \partial_k \dot{U}_k - (\Delta - m^2)U_0 - j_0\,,
\end{aligned} \tag{2·11}$$

the relations which follow from (2·3) and (2·4). Since $\mathcal{L}_\phi$ is of the minimal interaction, we have

*) $\partial_\mu{}^x \equiv \partial/\partial x_\mu$.

$$j_0=-\frac{\delta\mathcal{L}_\phi}{\delta U_0}=\frac{\delta\mathcal{L}_\phi}{\delta\dot{\phi}}ig\phi-ig\phi^\dagger\frac{\delta\mathcal{L}_\phi}{\delta\dot{\phi}^\dagger}$$
$$=ig(\pi\phi-\phi^\dagger\pi^\dagger). \tag{2·12}$$

Hence the canonical commutation relations imply that

$$[U_\mu(x),j_0(y)]_{x_0=y_0}=0\,,$$
$$[\dot{U}_l(x),j_0(y)]_{x_0=y_0}=0\,,$$
$$[B(x),j_0(y)]_{x_0=y_0}=0\,. \tag{2·13}$$

In order to find four-dimensional commutation relations involving B, we re-write (2·7) as[4)]

$$B(x)=-\int dz[\dot{D}(x-z)B(z)+D(x-z)\dot{B}(z)], \tag{2·14}$$

where z_0 is a free parameter. Setting $z_0=y_0$ in (2·14), with the aid of (2·11), (2·10) and (2·13), it is straightforward to obtain

$$[B(x),\,U_\mu(y)]=i\partial_\mu{}^xD(x-y), \tag{2·15}$$
$$[B(x),B(y)]=-im^2D(x-y) \tag{2·16}$$

and $[B(x),j_0(y)]=0$. Because of manifest covariance, therefore, we have

$$[B(x),j_\mu(y)]=0\,. \tag{2·17}$$

Since $B(x)$ satisfies a free-field equation (2·7), we can consistently define*) its positive frequency part $B^{(+)}(x)$. The constraint for the physical states is

$$B^{(+)}(x)|\text{phys}\rangle=0\,. \tag{2·18}$$

From (2·17) we see that $j_\mu(y)|\text{phys}\rangle$ is also a physical state.

Let $|\Omega\rangle$ be the true vacuum. From manifest covariance, local commutativity and the Lorentz condition (2·3), we have a spectral representation:

$$\langle\Omega|[U_\mu(x),\,U_\nu(y)]|\Omega\rangle$$
$$=-i\int_0^\infty ds\rho(s)\,(g_{\mu\nu}+s^{-1}\partial_\mu{}^x\partial_\nu{}^x)\,\Delta(x-y,\,s)+im^{-2}h\partial_\mu{}^x\partial_\nu{}^xD(x-y). \tag{2·19}$$

The parameter h is determined as follows. By making use of (2·4), (2·16) and (2·17), we obtain

$$(\Box^x+m^2)\,(\Box^y+m^2)\langle\Omega|[U_\mu(x),\,U_\nu(y)]|\Omega\rangle$$
$$=\langle\Omega|[j_\mu(x),j_\nu(y)]|\Omega\rangle+im^2\partial_\mu{}^x\partial_\nu{}^xD(x-y). \tag{2·20}$$

Because of (2·6), we should have

*) To define $B^{(+)}(x)$, replace D by $D^{(+)}$ in (2·14).

$$\langle\Omega|[j_\mu(x), j_\nu(y)]|\Omega\rangle = -i\int_a^\infty ds\tilde{\rho}(s)\,(g_{\mu\nu}+s^{-1}\partial_\mu{}^x\partial_\nu{}^x)\,\Delta(x-y, s) \qquad (2\cdot21)$$

with $a>0$, provided that no massless *physical* particles are present. On substituting (2·19) and (2·21) in (2·20), we find

$$h=1\,,$$

$$(s-m^2)^2\rho(s)=\tilde{\rho}(s). \qquad (2\cdot22)$$

From (2·19) together with (2·22) and the first commutator in (2·10), we find Johnson's formulas[5)]

$$\int_b^\infty ds\rho(s)=1\,,$$

$$\int_b^\infty ds\rho(s)/s=m^{-2}, \qquad (2\cdot23)$$

where $b\equiv\min(a, m^2)$. From (2·23), we conclude that the physical mass $m_{\rm phys}$ of U_μ, which is a point spectrum of $\rho(s)$, must tend to zero as $m\to0$.

The above reasoning is applicable to any theory in which U_μ couples with its source in the minimal interaction.

§ 3. Boulware-Gilbert model

The Lagrangian density of the Higgs model[7)] is essentially the same as (2·1) with $m\to0$, though we here adopt the Landau-gauge formulation. All field equations and canonical commutation relations remain unchanged. The only difference consists in the non-vanishing vacuum expectation value of ϕ:

$$\langle\Omega|\phi(x)|\Omega\rangle=v/\sqrt{2}\neq0\,, \qquad (3\cdot1)$$

which was not used in the proof, presented in § 2, of Johnson's proposition that $m\to0$ implies $m_{\rm phys}\to0$. Nevertheless, it is known that U_μ acquires a non-zero physical mass ($m_{\rm phys}\neq0$) at $m=0$ in the Higgs model.

As usual, we set

$$\sqrt{2}\phi(x)=v+\psi(x)+i\chi(x), \qquad (3\cdot2)$$

where $v^*=v$, $\psi^\dagger=\psi$ and $\chi^\dagger=\chi$, so that

$$\langle\Omega|\psi(x)|\Omega\rangle=\langle\Omega|\chi(x)|\Omega\rangle=0\,. \qquad (3\cdot3)$$

On substituting (3·2) in (2·2), we have

$$\mathcal{L}_\phi=\tfrac{1}{2}M^2U^\mu U_\mu+\tfrac{1}{2}\partial^\mu\psi\partial_\mu\psi+\tfrac{1}{2}\partial^\mu\chi\partial_\mu\chi-MU^\mu\partial_\mu\chi+\tfrac{1}{2}g^2U^\mu U_\mu(\psi^2+\chi^2)$$
$$+gMU^\mu U_\mu\psi+gU^\mu(\chi\partial_\mu\psi-\psi\partial_\mu\chi)+F(\tfrac{1}{2}(v+\psi)^2+\tfrac{1}{2}\chi^2), \qquad (3\cdot4)$$

where

$$M\equiv gv\,. \qquad (3\cdot5)$$

As the zeroth approximation to the Higgs model, we consider the case in which $g\to 0$ (and $F\to 0$) but M is kept finite. Since ψ then decouples from the rest, we have an effective Lagrangian density

$$\mathcal{L}_0\equiv -\tfrac{1}{4}(\partial^\mu U^\nu-\partial^\nu U^\mu)(\partial_\mu U_\nu-\partial_\nu U_\mu)+\tfrac{1}{2}(m^2+M^2)U^\mu U_\mu$$
$$+B\partial^\mu U_\mu - MU^\mu\partial_\mu\chi+\tfrac{1}{2}\partial^\mu\chi\partial_\mu\chi\,. \qquad (3\cdot 6)$$

This is essentially the model considered by Boulware and Gilbert[11] as an example of a gauge-invariant massive vector field. Since this model is exactly solvable, in this section we analyze it in detail in order to see why the proof of $m_{\rm phys}\to 0$ as $m\to 0$ does not apply.

The field equations are (2·3) and (2·4) together with

$$j_\mu\equiv M\partial_\mu\chi - M^2 U_\mu \qquad (3\cdot 7)$$

and

$$\Box\chi=0\,. \qquad (3\cdot 8)$$

We may rewrite (2·4) with (3·7) as

$$(\Box+m^2+M^2)U_\mu-\partial_\mu(B+M\chi)=0\,. \qquad (3\cdot 9)$$

From (3·9), (2·3) and (3·8), we have

$$\Box B=0\,, \qquad (3\cdot 10)$$

whence

$$\Box(\Box+m^2+M^2)U_\mu=0\,. \qquad (3\cdot 11)$$

The field equations are thus the same as those in the free-field case having a mass squared m^2+M^2 and an auxiliary field $B+M\chi$. But the constraint is still (2·18).

The equal-time commutators involving χ and/or $\dot\chi$ are as follows:

$$[U_\mu,\chi]=[U_\mu,\dot\chi]=0\,,$$
$$[\dot U_l,\chi]=[\dot U_l,\dot\chi]=0\,,$$
$$[B,\chi]=[\chi,\chi]=[\dot\chi,\dot\chi]=0\,,$$
$$[B,\dot\chi]=-iM\delta(\boldsymbol{x}-\boldsymbol{y}),$$
$$[\chi,\dot\chi]=i\delta(\boldsymbol{x}-\boldsymbol{y})\,. \qquad (3\cdot 12)$$

Hence, by using (2·14), (2·11) and (3·7), it is easy to confirm (2·16) and (2·17).*) For completeness, we here write all four-dimensional commutation relations between fields:

$$[U_\mu(x),U_\nu(y)]=-i[g_{\mu\nu}+(m^2+M^2)^{-1}\partial_\mu{}^x\partial_\nu{}^x]\Delta(x-y,m^2+M^2)$$
$$+i(m^2+M^2)^{-1}\partial_\mu{}^x\partial_\nu{}^x D(x-y),$$

*) A contrary statement was erroneously made in Ref. 1).

$$[U_\mu(x), B(y)] = -i\partial_\mu{}^x D(x-y),$$
$$[B(x), B(y)] = -im^2 D(x-y),$$
$$[U_\mu(x), \chi(y)] = 0,$$
$$[B(x), \chi(y)] = -iMD(x-y),$$
$$[\chi(x), \chi(y)] = iD(x-y). \qquad (3\cdot13)$$

Therefore, from (3·7) we have

$$[j_\mu(x), j_\nu(y)] = M^2\partial_\mu{}^x\partial_\nu{}^y[\chi(x), \chi(y)] + M^4[U_\mu(x), U_\nu(y)]$$
$$= -iM^4[g_{\mu\nu} + (m^2+M^2)^{-1}\partial_\mu{}^x\partial_\nu{}^x]\Delta(x-y, m^2+M^2)$$
$$-im^2M^2(m^2+M^2)^{-1}\partial_\mu{}^x\partial_\nu{}^x D(x-y). \qquad (3\cdot14)$$

The remarkable point of the Boulware-Gilbert model is the existence of the term proportional to $\partial_\mu{}^x\partial_\nu{}^x D(x-y)$ in the current-current commutator. This fact, which *contradicts* (2·21), is due to the presence of massless physical particles. Indeed, let

$$\tilde{\chi}(x) \equiv -M(m^2+M^2)^{-1}[m^2\chi(x) - MB(x)], \qquad (3\cdot15)$$

since

$$\Box\tilde{\chi}(x) = 0, \qquad (3\cdot16)$$
$$[\tilde{\chi}(x), B(y)] = 0, \qquad (3\cdot17)$$
$$[\tilde{\chi}(x), \tilde{\chi}(y)] = im^2M^2(m^2+M^2)^{-1}D(x-y), \qquad (3\cdot18)$$

$\tilde{\chi}(x)$ is massless, physical (i.e., $B^{(+)}(x)[\tilde{\chi}(y)|\Omega\rangle] = 0$) and of positive norm.*) The intermediate states consisting of a $\tilde{\chi}$ particle gives a non-zero contribution to $\langle\Omega|[j_\mu, j_\nu]|\Omega\rangle$.

It is important to note that though $\tilde{\chi}(x)$ is physical, as $m\to 0$ it tends to $B(x)$ so that its norm tends to zero; therefore the $\tilde{\chi}$ particles become *unobservable.* As remarked previously,[1),2)] $B(x)$ is unphysical for $m\neq 0$, but it becomes physical for $m=0$ because it then commutes with $B(y)$. For $m=0$, the massless *unphysical* field is

$$X(x) \equiv \chi(x) + \tfrac{1}{2}M^{-1}B(x); \qquad (3\cdot19)$$

both $B(x)$ and $X(x)$ are of zero norm, but $[B(x), X(y)]$ is non-vanishing.

§ 4. Higgs-type massive vector field

In § 3, we have seen that the reason why the physical mass of U_μ can be non-zero as $m\to 0$ in the theory of spontaneously broken gauge invariance is the existence of *massless physical particles,* which are *not* identical with Goldstone

*) The normalization of $\tilde{\chi}(x)$ is chosen so as to account for the massless spectrum of $[j_\mu, j_\nu]$.

bosons. The crucial point is that as $m\to 0$ those massless physical particles become unobservable just like the quanta of the Coulomb interaction. Having understood the mechanism of yielding a non-zero physical mass, in this section we study the Higgs model by setting $m=0$ from the beginning. We rewrite U_μ as A_μ in order to stress $m=0$, and we consider the general covariant gauge by adding $\frac{1}{2}\alpha B^2$ to $\mathcal{L}$ for the convenience of the comparison with B. W. Lee's work.[10),*)]

The field equations are

$$\partial^\mu A_\mu + \alpha B = 0\,, \tag{4·1}$$

$$(\Box + M^2) A_\mu - (1-\alpha)\partial_\mu B - M\partial_\mu \chi = J_\mu \tag{4·2}$$

with

$$\begin{aligned} J_\mu &\equiv j_\mu + M^2 A_\mu - M\partial_\mu \chi \\ &= -g[gA_\mu(\psi^2+\chi^2) + 2MA_\mu\psi + \chi\partial_\mu\psi - \psi\partial_\mu\chi]. \end{aligned} \tag{4·3}$$

Of course, $\partial^\mu j_\mu = 0$ but $\partial^\mu J_\mu \neq 0$. We still have**)

$$\Box B = 0\,, \tag{4·4}$$

but χ no longer satisfies a free-field equation. The constraint (2·18) remains unchanged.

The equal-time commutators (2·10) remain valid if U_μ and ϕ are replaced by A_μ and by $(1/\sqrt{2})(v+\psi+i\chi)$, respectively. Hence we have four-dimensional commutation relations

$$[B(x), A_\mu(y)] = i\partial_\mu{}^x D(x-y), \tag{4·5}$$

$$[B(x), B(y)] = 0\,, \tag{4·6}$$

$$[B(x), \chi(y)] = -i[M + g\psi(y)]D(x-y), \tag{4·7}$$

$$[B(x), J_\mu(y)] = -igM\partial_\mu{}^y[\psi(y)D(x-y)]. \tag{4·8}$$

From (4·7) and (4·8), we have

$$\langle\Omega|[B(x), \chi(y)]|\Omega\rangle = -iMD(x-y), \tag{4·9}$$

$$\langle\Omega|[B(x), J_\mu(y)]|\Omega\rangle = 0\,, \tag{4·10}$$

respectively. The non-vanishing of (4·9) is the important characteristic of the spontaneously broken gauge theory. From (4·9) together with (4·1), we must have

$$\langle\Omega|[A_\mu(x), \chi(y)]|\Omega\rangle = i\alpha M\partial_\mu{}^x E(x-y), \tag{4·11}$$

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*) In his treatment, the Landau-gauge case is ill-defined in contrast with our formalism. For example, the proper self-energy part of A_μ is singular at $\alpha=0$ in his formalism.

**) If $m\neq 0$ and $\alpha\neq 0$, B becomes massive; every $D(x-y)$ appearing in § 2 then is replaced by $\Delta(x-y, \alpha m^2)$.

because of manifest covariance and the vanishing equal-time commutators, where

$$E(x) \equiv -(\partial/\partial m^2)\Delta(x, m^2)|_{m=0}$$
$$= -(8\pi)^{-1}\varepsilon(x_0)\theta(x^2), \qquad (4\cdot12)$$
$$\Box E(x) = D(x). \qquad (4\cdot13)$$

As seen in § 3, A_μ acquires a non-zero mass at least if g is small. Hence A_μ contains no massless transverse components. From manifest covariance, local commutativity, (4·5) together with (4·1) and the equal-time commutators, we have a spectral representation

$$\langle\Omega|[A_\mu(x), A_\nu(y)]|\Omega\rangle = -i\int_c^\infty ds\rho(s)[(g_{\mu\nu}+s^{-1}\partial_\mu{}^x\partial_\nu{}^x)\Delta(x-y, s)$$
$$-s^{-1}\partial_\mu{}^x\partial_\nu{}^x D(x-y)] - i\alpha\partial_\mu{}^x\partial_\nu{}^x E(x-y) \qquad (4\cdot14)$$

with $c>0$.

Now, we consider the asymptotic fields. Since in-fields and out-fields can be discussed in the same way, for definiteness we consider in-fields alone. In order to avoid gauge complication, we first discuss the Landau-gauge case ($\alpha=0$). Suppose that

$$A_\mu(x)\to A_\mu{}^{\rm in}(x), \qquad B(x)\to B^{\rm in}(x),$$
$$\phi(x)\to\phi^{\rm in}(x), \qquad \chi(x)\to\chi^{\rm in}(x) \qquad (4\cdot15)$$

as $x_0\to-\infty$. Each in-field has to satisfy a free-field equation. Hence (4·5), (4·6), (4·9) and (4·11) with $\alpha=0$ yield

$$[B^{\rm in}(x), A_\mu{}^{\rm in}(y)] = i\partial_\mu{}^x D(x-y), \qquad (4\cdot16)$$
$$[B^{\rm in}(x), B^{\rm in}(y)] = 0, \qquad (4\cdot17)$$
$$[B^{\rm in}(x), \chi^{\rm in}(y)] = -iMD(x-y), \qquad (4\cdot18)$$
$$[A_\mu{}^{\rm in}(x), \chi^{\rm in}(y)] = 0, \qquad (4\cdot19)$$

respectively. From (4·18) we see that $\chi^{\rm in}(y)$ must satisfy the d'Alembert equation, that is, the χ field is massless. This fact represents that χ is the Goldstone field. Hence $\chi^{\rm in}(x)$ has to satisfy

$$[\chi^{\rm in}(x), \chi^{\rm in}(y)] = i\gamma D(x-y), \qquad (4\cdot20)$$

where γ is some real dimensionless constant. The constraint for the physical in-states is

$$[B^{\rm in}(x)]^{(+)}|{\rm phys}\rangle = 0. \qquad (4\cdot21)$$

From (4·18) we see that *Goldstone bosons are unphysical.*

Since

$$[B^{\rm in}(x), V_\mu{}^{\rm in}(y)] = 0, \qquad (4\cdot22)$$

where

$$V_\mu^{\text{in}}(x) \equiv A_\mu^{\text{in}}(x) - M^{-1}\partial_\mu \chi^{\text{in}}(x), \tag{4·23}$$

the physical-state subspace is generated by the hermitian conjugates of*)

$$[V_\mu^{\text{in}}(x)]^{(+)}, \qquad [\psi^{\text{in}}(x)]^{(+)}, \qquad [B^{\text{in}}(x)]^{(+)} \tag{4·24}$$

from the vacuum. Since $V_\mu^{\text{in}}(x)$ should satisfy a Klein-Gordon equation, it cannot contain a massless component. From (4·14) with $\alpha=0$, the massless spectrum of $[A_\mu^{\text{in}}(x), A_\nu^{\text{in}}(y)]$ is $iK\partial_\mu^x\partial_\nu^x D(x-y)$ with

$$K \equiv \int_0^\infty ds\rho(s)/s\,. \tag{4·25}$$

Therefore, using (4·20) and (4·19), we find

$$\gamma = M^2 K\,. \tag{4·26}$$

Thus we should have

$$\langle \Omega | [\chi(x), \chi(y)] | \Omega \rangle = iM^2 KD(x-y) + i\int_{+0}^\infty ds\sigma(s)\Delta(x-y, s)\,. \tag{4·27}$$

For $\alpha \neq 0$, we have to be careful of the invalidity of (4·15), as was noted by Källén[12] in quantum electrodynamics. Indeed, the right-hand side of (4·11) is inconsistent with any free-field equation. The appearance of $E(x-y)$ implies that there should exist dipole-ghost states.[2] As is well known, however, the Gupta-Bleuler theory, which corresponds to $\alpha=1$, involves no dipole ghosts. This dilemma is due to the breakdown of the operator manifest covariance of a non-Landau-gauge theory, as has been pointed out recently.[2] As far as two-point functions are concerned, however, this trouble can be bypassed. Following Lautrup,[4] we define an operator

$$\Lambda(x) \equiv \tfrac{1}{2}\Delta^{-1}[x_0\partial_0 B(x) - \tfrac{1}{2}B(x)], \tag{4·28}$$

where Δ denotes the Laplacian. Though $\Lambda(x)$ is *not* a Lorentz scalar, it satisfies

$$\Box \Lambda(x) = B(x), \tag{4·29}$$

$$[B(x), \Lambda(y)] = [\Lambda(x), \Lambda(y)] = 0\,, \tag{4·30}$$

$$[A_\mu(x), \partial_\nu^y \Lambda(y)] + [\partial_\mu^x \Lambda(x), A_\nu(y)] = i\partial_\mu^x\partial_\nu^x E(x-y)\,. \tag{4·31}$$

From (4·14), (4·30) and (4·31), we see that the vacuum expectation value of the commutator of

$$\hat{A}_\mu(x) \equiv A_\mu(x) + \alpha\partial_\mu \Lambda(x) \tag{4·32}$$

has no α-dependent term, that is, it equals (4·14) without the last term. With

*) V_μ^{in} and ψ^{in} are of positive norm and B^{in} is of zero norm. The zero-norm unphysical field [cf. (3·19)] is

$$X^{\text{in}}(x) \equiv \chi^{\text{in}}(x) + \tfrac{1}{2}\gamma M^{-1} B^{\text{in}}(x)\,.$$

the aid of (4·5) and (4·9), we can show that

$$\langle\Omega|M[A_\mu(x), \Lambda(y)]+[\partial_\mu^x \Lambda(x), \chi(y)]|\Omega\rangle = -iM\partial_\mu^x E(x-y). \tag{4·33}$$

Hence if we define

$$\hat{\chi}(x) \equiv \chi(x) + \alpha M \Lambda(x), \tag{4·34}$$

then we have

$$\langle\Omega|[\hat{A}_\mu(x), \hat{\chi}(y)]|\Omega\rangle = 0. \tag{4·35}$$

Therefore, by defining $A_\mu^{\text{in}}(x)$ and $\chi^{\text{in}}(x)$ as the asymptotic fields of $\hat{A}_\mu(x)$ and $\hat{\chi}(x)$, respectively, the discussion of the in-fields reduces to that in the Landau-gauge case. Since from (4·9)

$$\langle\Omega|[\chi(x), \Lambda(y)]+[\Lambda(x), \chi(y)]|\Omega\rangle = -iME(x-y), \tag{4·36}$$

we have

$$\langle\Omega|[\hat{\chi}(x), \hat{\chi}(y)]|\Omega\rangle = \langle\Omega|[\chi(x), \chi(y)]|\Omega\rangle - i\alpha M^2 E(x-y). \tag{4·37}$$

Since the left-hand side of (4·37) should be identified with (4·27), we finally find

$$\langle\Omega|[\chi(x), \chi(y)]|\Omega\rangle$$
$$= iM^2 KD(x-y) + i\int_{+0}^{\infty} ds\sigma(s)\Delta(x-y, s) + i\alpha M^2 E(x-y) \tag{4·38}$$

in the general covariant gauge. The Green's function counterparts of (4·14), (4·11) and (4·38) were given by B. W. Lee[10] by calculating the proper self-energy parts by means of the Ward-Takahashi identities.

To sum up, we have shown that the neutral vector field theory of spontaneously broken gauge invariance can be consistently formulated in the framework of the indefinite-metric quantum field theory. We can avoid the use of complicated functional-integral technique completely. Our theory is manifestly renormalizable, and the unitarity of the physical S-matrix is self-evident because the constraint (2·18) persists at all time. The Goldstone field χ is massless and unphysical, while the massless B field is physical but unobservable because of its zero norm just like the quanta of the Coulomb interaction.

Extension of our formalism to the non-Abelian gauge field will be formally straightforward,[1] but we then encounter the difficulty that the constraint no longer persists.

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