INDENTATION OF AN INCOMPRESSIBLE INHOMOGENEOUS LAYER BY A RIGID CIRCULAR INDENTER

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Summary

This work deals with an incompressible inhomogeneous layer bonded to a rigid substrate and indented without friction by a rigid circular indenter. The corresponding mixed boundary-value problem of elasticity is reduced to equivalent dual integral equations. It is shown that the pliability function in these equations may be found from a system of nonlinear differential equations and that its behaviour is peculiar when the elastic medium is incompressible. A novel technique taking into account this peculiarity is developed in order to reduce the dual integral equations to Fredholm integral equations of the second kind with symmetric strictly coercive operators. For a homogeneous layer and a flat indenter, the structure of the Fredholm integral equations permits an approximate analytical solution which is very accurate for any layer thickness. For an indenter of three-dimensional profile, leading asymptotic terms of the solution are derived in the case of a thin inhomogeneous layer.

1. Introduction

In this paper we consider the indentation problem of a rigid circular indenter in frictionless contact with an elastic layer occupying in the system of cylindrical coordinates (r, θ, z) the region $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant h$. The lower surface of the layer $z = 0$ is bonded to a rigid foundation. The indenter is pressed, under action of the vertical force $\mathcal{P}$ at the point $\theta = 0$, $r = d$, against the surface $z = h$. A contact zone between the indenter and the solid is assumed to be a circle of radius R. The medium, in the frame of the infinitesimal theory of elasticity, is supposed to be isotropic and incompressible; its shear modulus $G(z)$ is a piecewise smooth function.

The model of an incompressible inhomogeneous layer arises in design of functionally graded rubber covers. This model requires special investigation because the solution of the contact problem of an incompressible layer bonded to a rigid foundation has a peculiar asymptotic behaviour as a layer becomes thin. One might find theories for the thin homogeneous incompressible layer derived by physical arguments and the relevant discussions in the papers by Barber (**1**) and Jaffar (**2**), and the book by Johnson (**3**). The axisymmetric case for the homogeneous incompressible layer was studied by Alexandrov (**4**) with asymptotic methods. The method of this paper is a further development of the author's approach for a compressible layer (**5**). This method is based on the operators which transform kernels of the Hankel transform into the kernels of the Weber–Orr transform. It gives regular equations which are convenient for numerical and analytical studies.

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Pliability functions for an inhomogeneous elastic layer are studied in section 2. They are shown to be solutions of an initial-value problem for a certain system of nonlinear differential equations. The important point is that in a neighbourhood of the origin the behaviour of these functions for the incompressible medium is different from the behaviour in the compressible case.

A novel approach to the contact problem is suggested in sections 3 and 4. It takes into account the peculiarity which originates from incompressibility and reduces the governing dual integral equations to Fredholm integral equations of the second kind with strictly coercive operators. The structure of these Fredholm integral equations enables us to derive the leading terms of the asymptotic expansions for a flat indenter.

A homogeneous incompressible layer indented by a circular flat indenter is studied in section 5, where a very accurate approximate solution is derived for any layer thickness. This result is compared with the asymptotic solutions reported in the literature in order to estimate the ranges of their applicability.

In section 6, we investigate the problem of a rigid circular indenter of three-dimensional profile and derive simple asymptotic formulae in the case of a thin layer.

2. Pliability functions

The boundary conditions of the indentation problem for the layer $0 \leqslant z \leqslant h$, $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$, are

$$u_z(r,\theta,h) = -c - c_1 r\cos\theta + w(r,\theta), \quad 0 \leqslant r \leqslant R, \tag{2.1}$$

$$\sigma_z(r,\theta,h) = 0, \quad R \leqslant r < \infty, \tag{2.2}$$

$$\tau_{rz}(r,\theta,h) = \tau_{\theta z}(r,\theta,h) = 0, \quad 0 < r < \infty, \tag{2.3}$$

$$u_z(r,\theta,0) = u_r(r,\theta,0) = u_\theta(r,\theta,0) = 0, \quad 0 < r < \infty, \tag{2.4}$$

where $-c - c_1 r\cos\theta$ is the indenter displacement and $w(r,\theta)$ is the indenter profile.

Since the present problem is linear and the tangential stress is zero on the boundary $z = h$, the relation connecting the Fourier transform of the normal displacement $u_z(x_1, x_2, h)$ with the Fourier transform of the contact stress $\sigma_z(x_1, x_2, h)$ ((x_1, x_2, z) are Cartesian coordinates) is linear as well. This relation should be found by satisfying the boundary conditions (2.3) and (2.4), and can be taken in the form

$$G_0(h)\overline{u}_z(\xi_1,\xi_2,h) = p^{-1} f(p,h)\overline{\sigma}_z(\xi_1,\xi_2,h), \quad p = \sqrt{\xi_1^2+\xi_2^2}, \tag{2.5}$$

where $G_0(z) = G(z)/(1-\nu(z))$, $\nu(z)$ is Poisson's ratio and $f(p,z)$ is a certain *pliability function*. The dependence of the factor $f(p,h)/p$ on the variable p only is dictated by the requirement that (2.5) must be valid for any strain state and, in particular, for the axisymmetric state. The assumption of incompressibility does not facilitate our further work and we shall seek $f(p,h)$ for an arbitrary ν, $0 < \nu(z) \leqslant 0{\cdot}5$, setting $\nu(z) = 0{\cdot}5$ in final formulae.

We note that the pliability function $f(p,z)$ for a layer with piecewise constant $G(z)$ and $\nu(z)$ can be determined explicitly via certain recurrence relations (**6**).

Because (2.5) holds for arbitrary strain states, we consider plane strain in the (x,z)-plane in order to find $f(p,h)$ for a layer bonded to a rigid foundation. In this case $\overline{u}_z(\xi_1,\xi_2,h) = \overline{u}_z(\xi_1,h)\,\delta(\xi_2)$ and $\overline{\sigma}_z(\xi_1,\xi_2,h) = \overline{\sigma}_z(\xi_1,h)\,\delta(\xi_2)$, where $\delta(\xi_2)$ is the impulse function. Then (2.5) becomes equivalent to the equation

$$G_0(h)\overline{u}_z(\xi_1, h) = p^{-1} f(p, h)\overline{\sigma}_z(\xi_1, h), \quad p = |\xi_1|\,. \tag{2.6}$$

The Fourier transforms of the corresponding equilibrium equations and elastic stress–strain relations,

$$\frac{d\overline{\sigma}_z}{dz} - i\xi_1\overline{\tau}_{xz} = 0, \quad \frac{d\overline{\tau}_{xz}}{dz} - i\xi_1\overline{\sigma}_x = 0, \tag{2.7}$$

$$2G\frac{d\overline{u}_z}{dz} = (1-\nu)\,\overline{\sigma}_z - \nu\overline{\sigma}_x, \quad \frac{d\overline{u}_x}{dz} - i\xi_1\overline{u}_z = \frac{1}{G}\overline{\tau}_{xz}, \tag{2.8}$$

$$-2i\xi_1 G\overline{u}_x = (1-\nu)\,\overline{\sigma}_x - \nu\overline{\sigma}_z, \tag{2.9}$$

should be subjected to the boundary conditions

$$\overline{u}_x(\xi_1, 0) = \overline{u}_z(\xi_1, 0) = 0. \tag{2.10}$$

We shall seek a solution of the above equations for plane strain under the conditions (2.10) in the form

$$G_0(z)\,|\xi_1|\,\overline{u}_z(\xi_1, z) = f(|\xi_1|, z)\overline{\sigma}_z(\xi_1, z) - (i\xi_1/|\xi_1|)s(|\xi_1|, z)\overline{\tau}_{xz}(\xi_1, z), \tag{2.11}$$

$$G_0(z)\,|\xi_1|\,\overline{u}_x(\xi_1, z) = (i\xi_1/|\xi_1|)r(|\xi_1|, z)\overline{\sigma}_z(\xi_1, z) + g(|\xi_1|, z)\overline{\tau}_{xz}(\xi_1, z), \tag{2.12}$$

where the functions $f(|\xi_1|, z)$, $g(|\xi_1|, z)$, $s(|\xi_1|, z)$ and $r(|\xi_1|, z)$ have to be determined. Such relations exist for every z because the problem is linear. In particular, the first of these relations turns into (2.6) at $z = h$ as $\overline{\tau}_{xz}(\xi_1, h) = 0$.

Since displacements $u_z(x, z)$, $u_x(x, z)$ and stresses $\sigma_z(x, z)$, $\tau_{xz}(x, z)$ may be taken smooth at every point and decreasing as $|x| \to \infty$, the functions

$$f^*(|\xi_1|, z) = f(|\xi_1|, z)/G_0(z), \quad g^*(|\xi_1|, z) = g(|\xi_1|, z)/G_0(z), \tag{2.13}$$

$$s^*(|\xi_1|, z) = s(|\xi_1|, z)/G_0(z), \quad r^*(|\xi_1|, z) = r(|\xi_1|, z)/G_0(z) \tag{2.14}$$

have to be continuous functions of the variable z for every real ξ_1. In addition, $G_0(z)|\xi|u_z = f(|\xi|, z)Q$ when $\sigma_z = Q\exp(ix\xi)$, $\tau_{xz} = 0$, and $G_0(z)|\xi|u_x = g(|\xi|, z)T$ when $\sigma_z = 0$, $\tau_{xz} = T\exp(ix\xi)$. Because the corresponding displacements should be non-zero for any fixed $z > 0$, this requires that $f(|\xi_1|\,, z)$ and $g(|\xi_1|\,, z)$ have no zeros for $\xi_1 \neq 0$ and, therefore, do not change their signs. It also follows from (2.10) that

$$f(|\xi_1|, 0) = s(|\xi_1|, 0) = r(|\xi_1|, 0) = g(|\xi_1|, 0) = 0. \tag{2.15}$$

Substituting (2.11) and (2.12) into (2.8), then expressing $\overline{\sigma}_x$, $d\overline{\tau}_{xz}/dz$ and $d\overline{\sigma}_z/dz$ by means of (2.7), (2.9), (2.11) and (2.12) in terms of $\overline{\sigma}_z$ and $\overline{\tau}_{xz}$, we obtain after elementary manipulations

$$\begin{gathered}\left[G_0\left(\frac{f}{G_0}\right)' - |\xi_1|\,A_1\right]\overline{\sigma}_z(\xi_1, z) - \frac{i\xi_1}{|\xi_1|}\left[G_0\left(\frac{s}{G_0}\right)' - |\xi_1|\,A_2\right]\overline{\tau}_{xz}(\xi_1, z) = 0,\\ \frac{i\xi_1}{|\xi_1|}\left[G_0\left(\frac{r}{G_0}\right)' - |\xi_1|\,\widehat{A}_2\right]\overline{\sigma}_z(\xi_1, z) + \left[G_0\left(\frac{g}{G_0}\right)' - |\xi_1|\,A_3\right]\overline{\tau}_{xz}(\xi_1, z) = 0,\end{gathered} \tag{2.16}$$

where the prime denotes differentiation with respect to z,

$$A_1 = A_1(s,r) = \frac{1-2\nu(z)}{2(1-\nu(z))^2} - \frac{\nu(z)}{1-\nu(z)}\,(r+s) - 2rs,$$

$$A_2 = A_2(s,\ f,g) = f - \frac{\nu(z)}{1-\nu(z)}g - 2gs, \quad \widehat{A}_2 = A_2(r,\ f,g),$$

$$A_3 = A_3(s,r,g) = \frac{1}{1-\nu(z)} + s + r - 2g^2.$$

Equating to zero the coefficients in (2.16), we get the equations

$$G_0\,(f/G_0)' = |\xi_1|\,A_1(s,r), \quad G_0\,(g/G_0)' = |\xi_1|\,A_3(s,r,g), \tag{2.17}$$

$$G_0\,(s/G_0)' = |\xi_1|\,A_2(s,\ f,g), \quad G_0\,(r/G_0)' = |\xi_1|\,A_2(r,\ f,g), \tag{2.18}$$

which manifest that for given z the pliability functions really depend on $p = |\xi_1|$ only.

Equations (2.18) give

$$G_0\left[\frac{s(p,z)-r(p,z)}{G_0}\right]' = -2p\,[s(p,z)-r(p,z)]\,g(p,z). \tag{2.19}$$

Because $s(p,0) - r(p,0) = 0$, it is seen that $r(p,z) = s(p,z)$. Then the pliability functions are solutions of the initial-value problem for the system of nonlinear differential equations

$$G_0\,(f/G_0)' = pA_1(s,s), \quad G_0\,(s/G_0)' = pA_2(s,\ f,g), \quad G_0\,(g/G_0)' = pA_3(s,s,g) \tag{2.20}$$

for $0 \leqslant z \leqslant h$ with $f(p,0) = s(p,0) = g(p,0) = 0$.

Since the above initial-value problem is equivalent to the corresponding problem of elasticity, one might expect that a bounded solution exists. Making the changes (2.13) and (2.14), the existence and uniqueness of the solution may be proved rigorously in the standard way by employing integral equations (see the Picard–Lindelöf theorem (**7**)) even though the right side may possess points of ordinary discontinuity. Successive approximations converge to the solutions at least in the vicinity of the point $z = 0$ for all values p. This solution can be continued at any $z \in [0,h]$. In a neighbourhood of $p = 0$, the solution on the whole interval $[0,h]$ can be written in the form of series which converge absolutely (**7**) and whose terms are certain homogeneous polynomials of the parameter p. Thus the series are analytic functions of p and may be extended analytically for all real p. When $\nu = \frac{1}{2}$ and p is fixed, we obtain in the process of proof the following estimate which will be helpful later:

$$f(p,h) = 2p^3(\gamma_2(h) - p^2\Lambda) + \widetilde{f}(p,h), \tag{2.21}$$

$$\left|\widetilde{f}(p,h)\right| \leqslant M\sum_{n=4}^{\infty}\frac{K^{n-1}\gamma_{n-1}(h)}{(n-1)!}\,p^{n+1}, \tag{2.22}$$

where K and M are constants,

$$\gamma_n(z) = G(z)\int_0^z \frac{(z-t)^n}{G(t)}\,dt, \quad \Lambda = 4G(h)\int_0^h \frac{\gamma_1^2(t)}{G(t)}\,dt, \tag{2.23}$$

$$\Lambda \leqslant 2\delta_1\gamma_2(h) \leqslant 2\delta_0 h\gamma_2(h) \quad \text{and} \quad \delta_n = \sup_{0\leqslant z\leqslant h}\gamma_n(z). \tag{2.24}$$

Expanding the pliability functions into series in ascending powers of p, we derive by substituting into (2.20) and equating coefficients of like powers of p

$$f(p,z) = \sum_{k=0}^{\infty} p^{2k+1} f_{2k+1}(z), \tag{2.25}$$

$$g(p,z) = \frac{pG(z)}{1-\nu(z)} \int_0^z \frac{dt}{G(t)} + \sum_{k=1}^{\infty} p^{2k+1} g_{2k+1}(z), \tag{2.26}$$

$$s(p,z) = \sum_{k=1}^{\infty} p^{2k} s_k(z), \tag{2.27}$$

where $f_k(z)$, $g_k(z)$ and $s_k(z)$ satisfy certain recurrencd equations omitted here. In particular,

$$f_1(z) = \frac{pG(z)}{2\,(1-\nu(z))} \int_0^z \frac{1-2\nu(t)}{(1-\nu(t))\,G(t)}\,dt,$$

$$f_3(z) = -\frac{2G(z)}{1-\nu(z)} \int_0^z \frac{\nu(t)}{G(t)} s_1(t)\,dt,$$

$$s_1(z) = \frac{G(z)}{1-\nu(z)} \int_0^z \left[\int_0^t \frac{1-2\nu(u)}{2\,(1-\nu(u))\,G(u)}\,du - \frac{\nu(t)}{1-\nu(t)} \int_0^t \frac{du}{G(u)} \right] dt.$$

The Maclaurin series (2.25) to (2.27) converge uniformly for small p because the pliability functions are analytic functions of p. If $\nu = \frac{1}{2}$, then $f_1(z) = 0$ and $p^3 f_3(z)$ is the leading term of the expansion (2.25). On changing the order of integration we obtain in this case

$$f(p,z) = f_{\text{in}}(p,z) = 2p^3 G(z) \int_0^z \frac{(z-t)^2}{G(t)}\,dt + O(p^5). \tag{2.28}$$

We observe that the behaviour of the pliability function $f(p,h)$ in the vicinity of the origin is different in the cases of incompressibility and compressibility. This affects the efficiency of algorithms, as well as the asymptotic behaviour of the solution as a layer becomes thin.

The asymptotic expansions as $p \to \infty$ are sought as series in descending powers of p at points where $G(z)$ and $\nu(z)$ are analytic (see (**8**, §36.2)). Equating coefficients of like powers leads to algebraic equations. It is seen from (2.25) that for $z > 0$ the functions $f(p,z)$ and $g(p,z)$ are positive in a neighbourhood of the point $p = 0$ and, therefore (see above), for all $p > 0$ as well. Then the leading terms of the asymptotic expansions for $f(p,z)$ and $g(p,z)$ should be positive. This observation enables us to select the correct solution whose two leading terms are

$$f(p,z) \sim 1 + \frac{(2-\nu)\,G' + \nu' G}{2\,(1-\nu)\,G} \frac{1}{p} + O\left(\frac{1}{p^2}\right), \tag{2.29}$$

$$s(p,z) \sim \frac{1-2\nu(z)}{2\,(1-\nu(z))} + \frac{(1-\nu)\,[G/\,(1-\nu)]'}{2G} \frac{1}{p} + O\left(\frac{1}{p^2}\right), \tag{2.30}$$

$$g(p,z) \sim 1 + \frac{(1-\nu)\,[G/\,(1-\nu)]'}{2G} \frac{1}{p} + O\left(\frac{1}{p^2}\right). \tag{2.31}$$

We note that the complete asymptotic expansions for $p \to \infty$ contain, in addition to descending powers of p, rapidly decreasing terms of order $O\left(p^m \exp\left(-p\varsigma z\right)\right)$ which take into account the initial conditions as well as points where $G(z)$ is discontinuous or varies very rapidly.

3. Reducing dual equations to regular equations in the case of a flat circular indenter

The normal displacement u_z of the upper boundary $z = h$ is given by the two-dimensional Fourier inverse of (2.5) and can be rewritten in cylindrical coordinates by means of the well-known relations between two-dimensional Fourier transforms and Hankel transforms: if

$$q(x_1, x_2) = \sum_{n=-\infty}^{\infty} q_n(r)\, e^{in\theta} \quad \text{and} \quad \overline{q}(\xi_1, \xi_2) = \sum_{n=-\infty}^{\infty} \widetilde{q}_n(p)\, e^{in\phi}, \tag{3.1}$$

with $p = \sqrt{\xi_1^2 + \xi_2^2}$ and $\phi = \arctan\left(\xi_2/\xi_1\right)$, then

$$q_n(r) = \int_0^\infty p\widetilde{q}_n(p)\, J_n(pr)\, dp \quad \text{and} \quad \widetilde{q}_n(p) = \int_0^\infty r q_n(r)\, J_n(pr)\, dr, \tag{3.2}$$

where $J_n(p)$ is the Bessel function of the first kind.

The expression (2.5) is established in section 2 on satisfying the boundary conditions (2.3) and (2.4). One might see that inserting the representations of $u_z(r, \theta, h)$ and $\sigma_z(r, \theta, h)$ in the form of the inverse integral transforms into the boundary conditions (2.1) and (2.2), which are not met yet, gives equivalent dual integral equations. In the case of a flat indenter $w(r, \theta) = 0$ and $\nu = \frac{1}{2}$, these dual integral equations are written due to (3.1) and (3.2) as

$$R \int_0^\infty \widetilde{\sigma}_{z0}(p)\; f_{\text{in}}(p/R, h)\; J_0(p\rho)\, dp = 2G(h) u_0(\rho), \quad 0 \leqslant \rho \leqslant 1, \tag{3.3}$$

$$\int_0^\infty p\widetilde{\sigma}_{z0}(p)\; J_0(p\rho)\, dp = 0, \qquad 1 < \rho < \infty, \tag{3.4}$$

and

$$R \int_0^\infty \widetilde{\sigma}_{z1}(p)\; f_{\text{in}}(p/R, h)\; J_1(p\rho)\, dp = 2G(h) u_1(\rho), \quad 0 \leqslant \rho \leqslant 1, \tag{3.5}$$

$$\int_0^\infty p\widetilde{\sigma}_{z1}(p)\; J_1(p\rho)\, dp = 0, \qquad 1 < \rho < \infty, \tag{3.6}$$

where $\rho = r/R$, $u_0(\rho) = -c$, $u_1(\rho) = -\rho R c_1$ and $\widetilde{\sigma}_{zn}(p)$ are the unknown Hankel transform of the nth harmonics of the integrable contact stress $\sigma_z(\rho R, \theta, h) = \sigma_{z0}(\rho) + \sigma_{z1}(\rho)\cos\theta$,

$$\sigma_{zn}(\rho) = \int_0^\infty p\widetilde{\sigma}_{zn}(p)\; J_n(p\rho)\, dp. \tag{3.7}$$

Further we denote

$$\chi^{\gamma}_{\mu,\nu}(p, t) = Y_\nu(pt) J_\mu(p\gamma) - Y_\mu(p\gamma) J_\nu(pt), \tag{3.8}$$

where $Y_\nu(pt)$ is the Bessel function of the second kind,

$$\gamma = 2\left[\frac{1}{\pi} \lim_{p\to 0} \frac{f_{\text{in}}(p/R, h)}{p^3}\right]^{\frac{1}{3}} = \frac{2}{R}\left[\frac{2G(h)}{\pi} \int_0^h \frac{(h-z)^2}{G(z)}\, dz\right]^{\frac{1}{3}}, \tag{3.9}$$

and employ the discontinuous integral (evaluated by means of the known integral (**9**, equation 2.13.22.5))

$$\int_0^\infty p\chi_{2,2}^\gamma(p,t)J_0(p\rho)\,dp = \begin{cases} \Omega_0/(\gamma t)^2, & 0<\rho<t-\gamma, \\ 0, & \rho>t-\gamma, \end{cases} \tag{3.10}$$

where

$$\Omega_0 = \frac{2[(t^2+\gamma^2-\rho^2)^2-2\gamma^2t^2]}{\pi\sqrt{(t^2-(\rho+\gamma)^2)(t^2-(\rho-\gamma)^2)}}.$$

Applying the inverse formula for the Hankel transform and the operation $t^{-2}(d/dt)t^2$ gives us the operator transforming $J_0(p\rho)$ into $p\chi_{2,1}^\gamma(p,t)$,

$$\mathbf{S}_0[J_0(p\rho)] = p\chi_{2,1}^\gamma(p,t) \quad \text{with } \mathbf{S}_0(\,\cdot\,) = \frac{1}{\gamma^2t^2}\frac{d}{dt}\int_0^{t-\gamma}(\,\cdot\,)\,\Omega_0\,\rho\,d\rho. \tag{3.11}$$

The solution of the dual integral equations (3.3) and (3.4) is sought in the form

$$\frac{R\widetilde{\sigma}_{z0}(p)}{\pi\gamma G(h)} = -p\int_\gamma^\alpha s\omega_0(s)\,\chi_{2,1}^\gamma(p,s)\,ds \tag{3.12}$$

$$= -\alpha\omega_0(\alpha)\chi_{2,2}^\gamma(p,\alpha) + \int_\gamma^\alpha s^2\chi_{2,2}^\gamma(p,s)\,d\left(\frac{\omega_0(s)}{s}\right), \tag{3.13}$$

where $\alpha = 1+\gamma$ and $\omega_0(s)$ is some auxiliary function possessing an integrable derivative. It will be shown in section 4 that this representation leads to the integrable contact stress $\sigma_{z0}(\rho)$.

Putting (3.13) in (3.4) and interchanging the order of integration, we ascertain by means of the integral (3.10) that (3.4) is satisfied identically.

Applying the operator $\mathbf{S}_0$, defined by (3.11), to (3.3) yields

$$R\int_0^\infty p\widetilde{\sigma}_{z0}(p)\, f_{\text{in}}(p/R,h)\chi_{2,1}^\gamma(p,t)\,dp = 2G(h)\mathbf{S}_0[u_0(\rho)], \quad \gamma\leqslant t\leqslant 1+\gamma. \tag{3.14}$$

Substituting (3.12) into (3.14) and taking into account the inversion formula for the Weber–Orr transform (**10**),

$$\varpi(t) = \int_0^\infty \frac{p\widehat{\varpi}(p)\chi_{\nu,\mu}^\gamma(p,t)}{J_\nu^2(p\gamma)+Y_\nu^2(p\gamma)}\,dp, \quad \widehat{\varpi}(p) = \int_\gamma^\infty s\varpi(s)\chi_{\nu,\mu}^\gamma(p,t)\,ds, \quad \nu = \mu+\frac{1}{2}\pm\frac{1}{2},$$

we obtain a Fredholm integral equation of the second kind with a symmetric kernel,

$$(\mathbf{I}+\mathbf{L}_0)\,\omega_0 = \psi_0(t), \quad \gamma\leqslant t\leqslant 1+\gamma, \tag{3.15}$$

with

$$\mathbf{L}_n\omega = \int_\gamma^{1+\gamma} s\omega(s)L_n(t,s)\,ds, \quad \psi_0(t) = -\mathbf{S}_0(u_0(\rho)), \tag{3.16}$$

$$L_n(t,s) = \int_0^\infty p\,[F(p)-1]\,\frac{\chi_{2,n+1}^\gamma(p,t)\chi_{2,n+1}^\gamma(p,s)}{J_2^2(p\gamma)+Y_2^2(p\gamma)}\,dp, \tag{3.17}$$

where $2F(p) = \pi\gamma p f_{\rm in}(p/R,h)\left[J_2^2(p\gamma)+Y_2^2(p\gamma)\right]$; note that $F(p) > 0$, $F(p)-1 = O(p^2)$ as $p \to 0$ and $F(p)-1 = O(p^{-j})$ as $p \to \infty$ with $j \geqslant 1$. The asymptotic formula (2.29) together with asymptotic expansions of Bessel functions enables us to establish that the kernel $L_n(t,s)$ $(n = 0, 1)$ possesses a logarithmic singularity at points $t = s$ when $G'(h) \neq 0$ and $L_n(t,s)$ is continuous when $G'(h) = 0$.

For the flat indenter, the right-hand part of the Fredholm equation (3.15) can be evaluated exactly,

$$\frac{1}{c}\psi_0(t) = \mathbf{S}_0(1) = \lim_{p\to 0}\mathbf{S}_0(J_0(p\rho)) = \lim_{p\to 0} p\chi_{2,1}^{\gamma}(p,t) = \frac{2t}{\pi\gamma^2}. \tag{3.18}$$

The dual integral equations (3.5) and (3.6) are treated in a similar way. We use the discontinuous integral

$$\int_0^{\infty} p\chi_{2,1}^{\gamma}(p,t)J_1(p\rho)\,dp - \frac{2t}{\pi\rho\gamma^2} = \begin{cases}\Omega_1/(\rho t\gamma^2), & 0<\rho<t-\gamma,\\ 0, & \rho > t-\gamma,\end{cases} \tag{3.19}$$

evaluated from (3.10) by integration by parts, where

$$\Omega_1 = \frac{2[\gamma^2(\rho^2+t^2)-(\rho^2-t^2)^2]}{\pi\sqrt{((\rho+\gamma)^2-t^2)((\rho-\gamma)^2-t^2)}}.$$

Inverting (3.19) gives

$$\mathbf{S}_1[J_1(p\rho)] = p\chi_{2,2}^{\gamma}(p,t) \quad \text{with } \mathbf{S}_1(\cdot) = -\frac{t}{\gamma^2}\frac{d}{dt}\frac{1}{t^2}\int_0^{t-\gamma}(\cdot)\,\Omega_1\,d\rho.$$

These relations and the substitution

$$\frac{\widetilde{\sigma}_{z1}(p)}{\pi\gamma G(h)} = -p\int_{\gamma}^{\alpha} s\omega_1(s)\chi_{2,2}^{\gamma}(p,s)\,ds \tag{3.20}$$

$$= \alpha\omega_1(\alpha)\chi_{2,1}^{\gamma}(p,\alpha) - \frac{2\omega_1(\gamma)}{\pi p} - \int_{\gamma}^{\alpha}\frac{1}{s}\chi_{2,1}^{\gamma}(p,s)\,d[s^2\omega_1(s)] \tag{3.21}$$

enable us to derive the Fredholm integral equation of the second kind,

$$(\mathbf{I}+\mathbf{L}_1)\,\omega_1 = -\mathbf{S}_1(u_1(\rho)) = \psi_1(t), \quad \gamma \leqslant t \leqslant 1+\gamma, \tag{3.22}$$

where the operator is defined by (3.16) and (3.17). For the flat indenter, the right-hand side of the equation again can be evaluated exactly,

$$\frac{1}{c_1}\psi_1(t) = \mathbf{S}_1(\rho) = \lim_{p\to 0}\mathbf{S}_1(2p^{-1}J_1(p\rho)) = 2\lim_{p\to 0}\chi_{2,2}^{\gamma}(p,t) = \frac{t^4-\gamma^4}{\pi\gamma^2 t^2}. \tag{3.23}$$

Now it is readily seen from (3.22) that $\omega_1(\gamma) = 0$. It may be shown that $\omega_1(\gamma) = 0$ for any piecewise differentiable function $u_1(\rho)$ which is zero at the point $\rho = 0$.

The operators $\mathbf{I}+\mathbf{L}_n$, $n = 0, 1$, are strictly coercive in the Hilbert space induced by the inner product

$$(a(t), b(t)) = \int_{\gamma}^{1+\gamma} t a(t) b(t)\,dt. \tag{3.24}$$

To prove this, we note that

$$(\omega(t), (\mathbf{I}+\mathbf{L}_n)\,\omega) = \int_0^\infty \frac{pF(p)\widehat{\omega}^2(p)}{J_2^2(p\gamma)+Y_2^2(p\gamma)}\,dp. \tag{3.25}$$

Then the Parseval theorem for the Weber–Orr transforms,

$$\int_0^\infty \frac{p\widehat{\omega}^2(p)}{J_2^2(p\gamma)+Y_2^2(p\gamma)}\,dp = \int_\gamma^\infty t\omega^2(t)\,dt = \|\omega\|^2\,, \tag{3.26}$$

leads to the estimate

$$\|\omega\|^2\, m_- \leqslant (\omega(t), (\mathbf{I}+\mathbf{L}_n)\,\omega) \leqslant \|\omega\|^2\, m_+, \tag{3.27}$$

where $m_- = \inf F(p) > 0$ and $m_+ = \sup F(p) < \infty$. This indicates that the Fredholm integral equations possess unique solutions which can be determined with iterative or projective methods (**11**). The norm of the integral operator $\mathbf{L}_n$ is estimated in the same way,

$$\|\mathbf{L}_n\| = \sup_{\|\omega\|=1} |(\omega(t), \mathbf{L}_n\omega)| \leqslant \sup|F(p)-1|\,, \quad n = 0, 1. \tag{3.28}$$

It follows from the properties of the kernels $L_n(t,s)$ that the solutions are continuous. One may readily ascertain that the integral operators $\mathbf{L}_n$ transform continuous functions into functions possessing an integrable derivative. Then the solutions possess integrable derivatives as was presupposed. These derivatives are continuous when $G'(h) = 0$ or have logarithmic singularities when $G'(h) \neq 0$.

4. Indenter displacement and contact stresses

In order to find relations between the indenter displacement and the imbedding force $\mathcal{P}$, we use the equilibrium conditions which are expressed directly via $\widetilde{\sigma}_{zn}(p)$ and the arm d,

$$2\pi R^2\widetilde{\sigma}_{z0}(0) = -\mathcal{P}, \quad 2\pi R^3 \lim_{p\to 0}\frac{\widetilde{\sigma}_{z1}(p)}{p} = -\mathcal{P}d.$$

On substituting (3.12) and (3.20) these conditions become

$$\mathcal{P} = \frac{4\pi G(h)R}{\gamma}\int_\gamma^{1+\gamma} s^2\omega(s)\,ds, \quad \mathcal{P}d = \frac{\pi G(h)R^3}{\gamma}\int_\gamma^{1+\gamma}\frac{s^4-\gamma^4}{s}\omega_1(s)\,ds. \tag{4.1}$$

The contact stresses are evaluated by applying Hankel transforms to (3.13) and (3.21),

$$\frac{\gamma R}{2G(h)}\sigma_{z0}(\rho) = \frac{2\gamma^2(1+\gamma)^2-(1-\rho^2+2\gamma(1+\gamma))^2}{(1+\gamma)\sqrt{(1-\rho^2)[(1+2\gamma)^2-\rho^2]}}\omega_0(1+\gamma)$$
$$+\int_{\rho+\gamma}^{1+\gamma}\frac{(s^2+\gamma^2-\rho^2)^2-2\gamma^2s^2}{\sqrt{(s^2-(\rho+\gamma)^2)(s^2-(\rho-\gamma)^2)}}\,d\left(\frac{\omega_0(s)}{s}\right),$$

$$\frac{\gamma\rho}{2G(h)}\sigma_{z1}(\rho) = -\frac{2\gamma^2(1+\gamma)+(1-\rho^2)(1-\rho^2+4\gamma+3\gamma^2)}{\sqrt{(1-\rho^2)[(1+2\gamma)^2-\rho^2]}}\,\omega_1(1+\gamma)$$
$$-\int_{\rho+\gamma}^{1+\gamma}\frac{\gamma^2(\rho^2+s^2)-(\rho^2-s^2)^2}{\sqrt{(s^2-(\rho+\gamma)^2)(s^2-(\rho-\gamma)^2)}}\,d(s^2\omega_1(s)).$$

Then we find a very simple expression for the stress intensity factor at the edge of the indenter

$$\lim_{r\to R} \sigma_z(r,\theta,h)\sqrt{R^2-r^2} = -2G(h)\sqrt{\gamma(1+\gamma)}\,[\omega_0(1+\gamma)+\omega_1(1+\gamma)R\cos\theta]\,. \tag{4.2}$$

The structure of the Fredholm integral equations (3.15) and (3.22) is convenient for deriving asymptotic solutions when there are limit relations of the form

$$\lim_{\gamma\to 0} \frac{8 f_{\text{in}}(p/R,h)}{\pi p^3\gamma^3} = 1 \tag{4.3}$$

or

$$\lim_{\gamma\to\infty} f_{\text{in}}(p/R,h) = 1. \tag{4.4}$$

One might observe that these limit relations involve $\lim_{\gamma\to\infty}\|\mathbf{L}_n\| = \lim_{\gamma\to 0}\|\mathbf{L}_n\| = 0$, $n = 0, 1$. Then we establish for the flat indenter as $\gamma\to 0$ or $\gamma\to\infty$: $\omega_n(t)\sim\psi_n(t)$,

$$\frac{\mathcal{P}}{2G(h)Rc} \sim P = \frac{(1+\gamma)^4-\gamma^4}{\gamma^3} \sim \begin{cases} 4 & \text{if } \gamma\to\infty, \\ \gamma^{-3} & \text{if } \gamma\to 0, \end{cases} \tag{4.5}$$

$$\frac{\mathcal{P}d}{c_1 G(h)R^3} \sim M = \frac{(1+2\gamma)^3(1+2\gamma+4\gamma^2)}{6\gamma^3(1+\gamma)^2} \sim \begin{cases} 16/3 & \text{if } \gamma\to\infty, \\ \gamma^{-3}/6 & \text{if } \gamma\to 0, \end{cases} \tag{4.6}$$

$$\lim_{r\to R} \sigma_z(r,\theta,h)\sqrt{R^2-r^2} \sim -\frac{\mathcal{P}}{2\pi R^2} \begin{cases} R+3d\cos\theta & \text{if } \gamma\to\infty, \\ 4\gamma^{3/2}(R+6d\cos\theta) & \text{if } \gamma\to 0. \end{cases} \tag{4.7}$$

By virtue of the formulae (2.21) to (2.24) and the inequality

$$\gamma_n(h) = 3G(h)\int_0^h \frac{(h-t)^2}{G(t)}\left(G(t)\int_0^t \frac{(h-s)^{n-3}}{G(s)}\,ds\right)dt \leqslant 3\delta_0 h^{n-3}\gamma_2(h),$$

for $n \geqslant 3$, we have

$$\frac{8 f_{\text{in}}(p/R,h)}{\pi\gamma^3 p^3} = 1 + \varphi\left(\frac{p}{R},h\right),$$

$$\left|\varphi\left(\frac{p}{R},h\right)\right| \leqslant \frac{2\delta_0 h}{R^2}p^2 + \frac{3M\delta_0}{2}\sum_{n=4}^{\infty}\frac{K^{n-1}h^{n-4}p^{n-3}}{R^{n-3}(n-1)!}$$

$$\leqslant \frac{\delta_0}{h}\left[2p^2\left(\frac{h}{R}\right)^2 + \frac{MK^3p}{4}\frac{h}{R}\exp\left(Kp\frac{h}{R}\right)\right].$$

The above estimate elucidates the dependence of the limit (4.3) on $G(z)$, h and R. Since

$$\gamma \leqslant 2\left(\frac{2\delta_0}{\pi h}\right)^{1/3}\frac{h}{R}, \tag{4.8}$$

this allows us to infer that the asymptotic solutions obtained above for $\gamma \to 0$ may be employed under one of the sufficient conditions

$$\frac{\delta_0}{h} = O(1), \quad \frac{h}{R} \ll 1, \tag{4.9}$$

$$\left(\frac{\delta_0}{R}\right)^{1/3} = \left[\frac{1}{R}\sup\left(G(z)\int_0^z \frac{dt}{G(t)}\right)\right]^{1/3} \ll 1, \quad \frac{h}{R} = O(1). \tag{4.10}$$

Coarse upper estimates for δ_0/h are given by

$$\frac{\delta_0}{h} \leqslant \frac{1}{h}\sup\frac{zG(z)}{\inf_{0\leqslant t\leqslant z} G(t)} \leqslant \frac{\sup G(z)}{\inf G(z)}. \tag{4.11}$$

It is seen that $\delta_0/h \leqslant 1$ for a monotone decreasing $G(z)$. Then (4.9) is valid as $h/R \to 0$.

The requirements (4.10) are fulfilled when h is fixed and R is sufficiently large. Then the expression (4.7) implies that the condition of perfect contact between the flat indenter and a very thin incompressible layer is $d \leqslant R/6$. This is different from that obtained for a thin compressible layer, namely $d \leqslant R/4$ (**5**).

Note that (4.10) remains valid when $(\delta_0/h)^{1/3} \ll 1$, $h/R = O(1)$. This shows that for a varying shear modulus the concept of a thin layer might hold even as the geometric characteristics are of the same order. For instance, we observe such a situation for $G(z) = G_0\, e^{-az}$, $ah \gg 1$.

If h is fixed and $R \to 0$, then the condition (4.4) is fulfilled and we can use the asymptotic solutions for $\gamma \gg 1$. However, it is unclear what simple restrictions should be imposed on the varying shear modulus $G(z)$, the indenter radius R and the layer thickness h to indicate the range of applicability for the asymptotic formulae. In general, all terms of the asymptotic expansion for $f_{\text{in}}(p/R, h) - 1$, including exponentially decreasing terms, should be examined to be small as $p/R \gg 1$ and $1/p = O(1)$. For instance, for the layer with the shear modulus

$$G(z) = \begin{cases} G_0, & 0 \leqslant z \leqslant h_0, \\ G_1, & h_0 \leqslant z \leqslant h, \end{cases} \tag{4.12}$$

where G_0 and G_1 are constants, all power terms of the asymptotic expansion discussed in section 2 are equal to zero. In this case one might ascertain by a change $z = z_1 R$ that, when $h - h_0 = R$ and $h/R \to \infty$, $f_{\text{in}}(p/R, h)$ turns into the pliability function for a layer of unit thickness bonded to an elastic half-space with different elastic properties. This shows that in order to get $f_{\text{in}}(p/R, h) \to 1$ we must impose the additional requirement $(h - h_0)/R \to \infty$ which is caused by the exponentially decreasing terms.

5. Homogeneous layer indented by a flat indenter

The pliability function $f_{\text{in}}(p, h)$ for a homogeneous layer can be written explicitly,

$$f_{\text{in}}(p/R, h) = \frac{\sinh 2p\lambda - 2p\lambda}{\cosh 2p\lambda + 2p^2\lambda^2 + 1}, \quad \lambda = \frac{h}{R}.$$

Then $\gamma = [16/(3\pi)]^{\frac{1}{3}}\lambda$, $m_+ = 1{\cdot}05754$ and $m_- = 0{\cdot}74279$.

For both limiting cases of a thin layer ($\gamma \ll 1$) and a thick layer ($\gamma \gg 1$), we can use the asymptotic formulae derived in the preceding section when $\lambda \ll 1$ and $\lambda \gg 1$, respectively.

Since the operators in the Fredholm equations (3.15) and (3.22) are self-adjoint, their spectrums, by virtue of (3.27), are localized within the interval $[m_-, m_+]$ of the positive semi-axis. Then we can write

$$(\mathbf{I} - \mathbf{B}_n)\,\omega_n = \eta\psi_n(t), \quad n = 0, 1, \tag{5.1}$$

$$\mathbf{B}_n = (1-\eta)\mathbf{I} - \eta\mathbf{L}_n, \quad \eta = 2/(m_+ + m_-), \tag{5.2}$$

and solve equations (5.1) with the iterative algorithm (**11**):

$$\omega_{n,k+1}(t) = \mathbf{B}_n\omega_{n,k}(t) + \eta\psi_n(t), \tag{5.3}$$

$$\|\omega_n(t) - \omega_{n,k}(t)\| \leqslant \|\omega_{n,0}(t)\|\theta, \quad \theta = q\,\|\omega_n(t) - \omega_{n,k-1}(t)\|/\|\omega_{n,0}(t)\|, \tag{5.4}$$

$$q = (m_+ - m_-)/(m_+ + m_-) = 0{\cdot}17481. \tag{5.5}$$

Taking $\omega_{n,0}(t) = \psi_n(t)$ and making the next iteration, we obtain the approximate solutions

$$\widehat{\omega}_n(t) = \psi_n(t) - \frac{\eta}{\pi\gamma^2}\int_0^\infty [F(p) - 1]\frac{\chi^\gamma_{2,n}(p,t)\chi^\gamma_n(p)}{J_2^2(p\gamma) + Y_2^2(p\gamma)}\,dp, \quad n = 0, 1, \tag{5.6}$$

$$\chi^\gamma_0(p) = 2(1+\gamma)^2\chi^\gamma_{2,2}(p, 1+\gamma), \tag{5.7}$$

$$\chi^\gamma_1(p) = (1+\gamma)^4\chi^\gamma_{2,3}(p, 1+\gamma) + \gamma^4\chi^\gamma_{2,1}(p, 1+\gamma)/(1+\gamma); \tag{5.8}$$

the error factor θ has the estimate $|\theta| \leqslant q\,\|\mathbf{L}_n\|/m_- \leqslant 0{\cdot}06053$, $\pi\,\|\psi_0(t)\| = \sqrt{P/\gamma}$ and $\pi\,\|\psi_1(t)\| = \sqrt{M/\gamma}$, where P and M are defined in (4.5) and (4.6), respectively.

Approximate formulae relating the indenter displacement with the imbedding force $\mathcal{P}$ now follow from (4.1). Denoting

$$\widetilde{\mathcal{P}} = \frac{\mathcal{P}}{2G(h)Rc} \quad \text{and} \quad \mathcal{M} = \frac{\mathcal{P}d}{G(h)R^3c_1},$$

we write

$$\widetilde{\mathcal{P}} \approx \widetilde{\mathcal{P}}_1 = P - \frac{4\,(1+\gamma)^4\,\eta}{\gamma^3}\int_0^\infty \frac{[F(p)-1][\chi^\gamma_{2,2}(p, 1+\gamma)]^2}{p[J_2^2(p\gamma) + Y_2^2(p\gamma)]}\,dp, \tag{5.9}$$

$$\mathcal{M} \approx \mathcal{M}_1 = M - \frac{\eta}{\gamma^3}\int_0^\infty \frac{[F(p)-1]\chi_1^2(p,\gamma)}{p[J_2^2(p\gamma) + Y_2^2(p\gamma)]}\,dp. \tag{5.10}$$

Upper estimates for the relative errors of the approximate formulae (5.9) and (5.10) can be readily found by using the Schwartz inequality together with (5.4),

$$\left|\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_1\right| \leqslant \frac{2\pi}{\gamma}\int_\gamma^{1+\gamma} s^2|\omega_1(s) - \widehat{\omega}_1(t)|ds \leqslant \pi\sqrt{\gamma P}\,\|\omega_1(s) - \widehat{\omega}_1(t)\|, \tag{5.11}$$

or $\left|\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_1\right| \leqslant \theta P = \varepsilon_P\widetilde{\mathcal{P}}_1$, where $\varepsilon_P = \theta P/\widetilde{\mathcal{P}}_1$. In the same manner $|\mathcal{M} - \mathcal{M}_1| \leqslant \varepsilon_M\mathcal{M}_1$, where $\varepsilon_M = \theta M/\mathcal{M}_1$. Calculations show that for all values of λ the upper estimates of the relative errors, ε_P and ε_M, do not exceed 6·14 per cent and 6·38 per cent, respectively.

Evaluating the integrals asymptotically for $\lambda \to 0$ (**12**), (5.9) and (5.10) become

$$\widetilde{\mathcal{P}}_1 = \frac{1}{\beta^3}\left[1 + \sum_{k=1}^{5} D_k \beta^k\right] + O(\beta^3) \quad \text{with } \beta = \frac{\gamma}{\gamma+1} = \frac{2^{\frac{4}{3}}\lambda}{2^{\frac{4}{3}}\lambda + (3\pi)^{\frac{1}{3}}}, \tag{5.12}$$

$D_1 = D_2 = 1{\cdot}9235$, $D_3 = -2{\cdot}5201$, $D_4 = -2{\cdot}4109$ and $D_5 = 7{\cdot}1567$, and

$$\mathcal{M}_1 = \frac{1}{\beta^3}\left[\frac{1}{6} + \sum_{k=1}^{6} B_k \beta^k\right] + O(\beta^4), \tag{5.13}$$

with $B_1 = 0{\cdot}7309$, $B_2 = 1{\cdot}6926$, $B_3 = 0{\cdot}4598$, $B_4 = -2{\cdot}8566$, $B_5 = 13{\cdot}2328$ and $B_6 = -10{\cdot}6227$.

Taking the asymptotic expansions of the integrals and results of numerical calculations, we construct the next approximate formulae for $\lambda \leqslant 4$:

$$\widetilde{\mathcal{P}} \approx \widetilde{\mathcal{P}}_{\text{ap}} = P + 0{\cdot}9235(1+3\gamma)\gamma^{-2} + 1{\cdot}22 \ln\beta - 1{\cdot}17\beta^2 + 2{\cdot}1\beta^8, \tag{5.14}$$

$$\mathcal{M} \approx \mathcal{M}_{\text{ap}} = M + \frac{0{\cdot}9235(1+5\gamma)}{4\gamma^2} - 0{\cdot}285(1{\cdot}03 + 0{\cdot}82\beta^2)^2 \ln\beta - 2{\cdot}7631\beta^2 + 3{\cdot}66\beta^4. \tag{5.15}$$

For $\lambda \geqslant 4$, $\widetilde{\mathcal{P}}$ can be evaluated by Alexandrov's asymptotic formulae (**4**) and $\mathcal{M}_1$ by the approximate formula

$$\mathcal{M} \approx \mathcal{M}_{\text{ap}} = \frac{16}{3} + \frac{19{\cdot}5}{\lambda^4}. \tag{5.16}$$

Another highly effective approximate solution can be derived from (5.1) if we write

$$\omega_n = (\mathbf{I} - \mathbf{B}_n)^{-1}\,\eta\psi_n(t) = -\eta q^{-1} Q_l(q^{-1}\mathbf{B}_n)\psi_n(t) + \eta q^{-1}\mathbf{U}\psi_n(t),$$

where $\mathbf{U} = (q^{-1}\mathbf{I} - q^{-1}\mathbf{B}_n)^{-1} + Q_l(q^{-1}\mathbf{B}_n)$, q is defined in (5.5), $Q_l(x)$ is some polynomial and the operator $\mathbf{U}$ is a function of the self-adjoint operator $\mathbf{B}_n/q$ whose spectrum is localized within the interval $[-1, 1]$. In accordance with the spectral theory of self-adjoint operators (**13**),

$$\|\mathbf{U}\| \leqslant \sup_{|x|\leqslant 1}\left|\frac{1}{x - 1/q} - Q_l(x)\right|. \tag{5.17}$$

This estimate is minimized by choosing $Q_l(x)$ as the polynomial of least deviation from $1/(x-1/q)$. Explicit formulae for this polynomial and its maximal deviation are known (**14**),

$$Q_l(x) = \frac{1}{x - 1/q} - \frac{q^{l+2}\cos[l \arccos x + \phi(x)]}{(1-q^2)(1+\sqrt{1-q^2})^l},$$

$$\sup_{|x|\leqslant 1}\left|\frac{1}{x-1/q} - Q_l(x)\right| = \frac{q^{l+2}}{(1-q^2)(1+\sqrt{1-q^2})^l} = \theta_l,$$

say, where $\phi(x) = \arccos[(x-q)/(qx-1)]$. Then the approximate solution

$$\widetilde{\omega}_n = -\eta q^{-1} Q_l(q^{-1}\mathbf{B}_n)\psi_n(t) \tag{5.18}$$

has the error whose norm is less than $\widetilde{\theta}_l\,\|\psi_n(t)\|$, with $\widetilde{\theta}_l = \mu\theta_l/q$.

For $l = 1$,

$$-\eta q^{-1} Q_1(q^{-1}\mathbf{B}_n) = \eta(1-q^2)^{-1/2}\mathbf{I} + \eta(1-q^2)^{-1}\mathbf{B}_n, \tag{5.19}$$

$$\widetilde{\omega}_n = \eta_0[\eta_1 \psi_n(t) + \widehat{\omega}_n(t)], \tag{5.20}$$

where $\widehat{\omega}_n(t)$ is defined by (5.6), $\eta_0 = \eta/(1-q^2) \approx 1{\cdot}1459248$ and $\eta_1 = \sqrt{1-q^2} - \eta \approx -0{\cdot}126305$. The error factor for this solution is $\widetilde{\theta}_1 \approx 0{\cdot}017645$.

The formulae relating the indenter displacement with $\mathcal{P}$ and the arm d become

$$\widetilde{\mathcal{P}} \approx \widetilde{\mathcal{P}}_2 = \eta_0[\eta_1 P + \widetilde{\mathcal{P}}_1] \quad \text{and} \quad \mathcal{M} \approx \mathcal{M}_2 = \eta_0[\eta_1 M + \mathcal{M}_1]. \tag{5.21}$$

The upper estimates of the relative errors $|1 - \widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}_2| \times 100$ per cent and $|1 - \mathcal{M}/\mathcal{M}_2| \times 100$ per cent do not exceed 1·77 per cent.

Results of approximate calculations for $\widetilde{\mathcal{P}}$ and $\mathcal{M}$ are summarized in Table 1, where $\widetilde{\mathcal{P}}_{\mathrm{A}}$ are the corresponding values calculated by Alexandrov's asymptotic solutions; $\widetilde{\mathcal{P}}_n$ and $\mathcal{M}_n$, $n = 1, 2$, are results of calculations using (5.9), (5.10) and (5.21); $\widetilde{\mathcal{P}}_{\mathrm{as}}$ and $\mathcal{M}_{\mathrm{as}}$ are the values calculated according to the asymptotic formulae (5.12) and (5.13); $\widetilde{\mathcal{P}}_{\mathrm{ap}}$ and $\mathcal{M}_{\mathrm{ap}}$ are found from the approximations (5.14), (5.15) and (5.16).

We see that the discrepancies of the approximations (5.14) from (5.9) do not exceed $\frac{1}{3}$ per cent as $\lambda \leqslant 4$, and the approximations (5.15) and (5.16) from (5.10) are less than 0·3 per cent for all λ.

The formulae for $\widetilde{\mathcal{P}}_1$ and $\mathcal{M}_1$, as seen from (4.5) and (4.6), are asymptotically correct as $\lambda \to \infty$ and $\lambda \to 0$. The formulae for $\widetilde{\mathcal{P}}_2$ and $\mathcal{M}_2$ are preferable in the interval $0{\cdot}02 \leqslant \lambda < 2$. As $\lambda \to \infty$ and $\lambda \to 0$, the limiting errors of $\widetilde{\mathcal{P}}_2$ and $\mathcal{M}_2$ are 0·12 per cent.

For $\lambda \geqslant 2$, $\widetilde{\mathcal{P}}_1$ is in excellent agreement with Alexandrov's formula which gives the exact asymptotic expansion in this range. The maximal discrepancy is 0·32 per cent for $\lambda = 2$ and becomes negligibly small as $\lambda \geqslant 3$. The formula for $\widetilde{\mathcal{P}}_2$ also gives very accurate results in this range.

Alexandrov's asymptotic solution for small λ is approximate. This solution for $0{\cdot}06 \leqslant \lambda < 2$ is essentially different from our error controlled solution. The discrepancy becomes small as $\lambda \to 0$.

Table 1 $\widetilde{\mathcal{P}}$ and $\mathcal{M}$: results of calculations

λ	$\widetilde{\mathcal{P}}_{\mathrm{A}}$	$\widetilde{\mathcal{P}}_2$	$\widetilde{\mathcal{P}}_1$	$\widetilde{\mathcal{P}}_{\mathrm{as}}$	$\widetilde{\mathcal{P}}_{\mathrm{ap}}$	$\mathcal{M}_2$	$\mathcal{M}_1$	$\mathcal{M}_{\mathrm{as}}$	$\mathcal{M}_{\mathrm{ap}}$
0·1	949·60	1024·3	1010·7	1010·7	1009·8	222·07	218·16	218·18	218·08
0·2	165·16	199·75	199·00	199·01	198·83	53·382	52·136	52·127	52·277
0·3	63·109	87·564	87·008	86·953	87·057	27·189	26·530	26·480	26·597
0·4	32·777	52·003	51·619	51·502	51·728	18·226	17·804	17·748	17·819
0·5	20·021	36·095	35·824	35·681	35·892	13·969	13·674	13·648	13·654
0·75	8·446	20·317	20·185	20·139	20·179	9·421	9·286	9·388	9·259
1	4·693	14·654	14·358	14·534	14·328	7·603	7·543	7·728	7·546
2	7·778	7·829	7·753	8·796	7·734	5·724	5·737	5·651	5·723
3	6·148	6·180	6·152	–	6·134	5·446	5·459	4·950	5·445
4	5·479	5·493	5·481	–	5·490	5·406	5·410	–	5·410
5	5·123	5·128	5·122	–	5·174	5·354	5·358	–	5·364
10	4·504	4·502	4·501	–	4·708	5·338	5·335	–	5·335

For small λ, only the first (leading) terms in the asymptotic expansion (5.12) and Alexandrov's expansion are the same and the difference is caused by the subsequent terms.

Note that the leading asymptotic term of $\widetilde{\mathcal{P}}$ for a very thin layer was first established by intuitive arguments in the paper by Barber (**1**). We observe that Barber's theory gives the error $(587\lambda \pm 2)$ per cent and is valid only for a very thin layer. An error of less than 10 per cent cannot be achieved for $\lambda \geqslant 0{\cdot}021$ and can be ensured for $\lambda \leqslant 0{\cdot}014$.

6. Circular indenter of three-dimensional profile

Consider a circular indenter of three-dimensional profile

$$w(r, \theta) = w_0(r) + \sum_{n=1}^{\infty} \left[w_n(r) \cos n\theta + w_n^*(r) \sin n\theta \right]. \tag{6.1}$$

If a contact stress is expanded into the trigonometric Fourier series

$$\sigma_z(r, \theta, h) = \sigma_{z0}(r) + \sum_{n=1}^{\infty} \left[\sigma_{zn}(r) \cos n\theta + \sigma_{zn}^*(r) \sin n\theta \right],$$

then, for the harmonics $n = 0$ and $n = 1$, we again have the dual equations (3.3), (3.4) and (3.5), (3.6), where now $u_0(\rho) = w_0(R\rho) - c$, and $u_1(\rho) = w_1(R\rho) - c_1 R\rho$ or $u_1(\rho) = w_1^*(R\rho) - c_2 R\rho$. The coefficients of the unknown indenter displacement $-c - c_1 r \cos\theta - c_2 r \sin\theta$ should be determined from the equilibrium equations,

$$2\pi R^2 \overline{\sigma}_{z0}(0) = -\mathcal{P} \quad \text{and} \quad 2\pi R^3 \lim_{p \to 0} \frac{\widetilde{\sigma}_{z1}(p) + i\widetilde{\sigma}_{z1}^*(p)}{p} = -\mathcal{P}d.$$

Using the substitutions (3.12) and (3.20), we obtain

$$\mathcal{P} = \frac{4\pi G(h) R}{\gamma} \int_{\gamma}^{1+\gamma} s^2 \omega(s)\, ds, \tag{6.2}$$

$$\mathcal{P}d = \frac{\pi G(h) R^3}{\gamma} \int_{\gamma}^{1+\gamma} \frac{s^4 - \gamma^4}{s} \left[\omega_1(s) + i\omega_1^*(s) \right] ds, \tag{6.3}$$

where $\omega(s)$ is the solution of the Fredholm equation (3.15) with the right-hand part $\mathbf{S}_0[w_0(R\rho) - c]$; $\omega_1(s)$ and $\omega_1^*(s)$ obey the Fredholm equation (3.22) with the right-hand parts $\mathbf{S}_1[w_1(R\rho) - c_1 R\rho]$ and $\mathbf{S}_1[w_1^*(R\rho) - c_2 R\rho]$, respectively.

Leading terms of the asymptotic expansions as $\gamma \to 0$ can be readily found in the same manner as for the flat indenter:

$$\frac{\gamma^3 \mathcal{P}}{8G(h)R} = \frac{c}{4} - \frac{1}{2\pi R^4} \int_0^R \int_0^{2\pi} r(R^2 - r^2) w(r, \theta)\, d\theta\, dr, \tag{6.4}$$

$$\frac{\gamma^3 \mathcal{P}d}{G(h)R^3} = \frac{c_1 + ic_2}{6} - \frac{2}{\pi R^6} \int_0^R \int_0^{2\pi} (R^4 - 3R^2r^2 + 2r^4) w(r, \theta)\, e^{i\theta}\, d\theta\, dr. \tag{6.5}$$

The corresponding asymptotic formula for the axisymmetric component of the contact stress in the general case is given by

$$\frac{\pi\gamma^3 R^3}{8G(h)}\sigma_{z0}(r) = \left[\frac{2\gamma^2 R^4 - (R^2 - r^2 + 2\gamma R^2)^2}{\sqrt{(R^2 - r^2)[(1+2\gamma)^2 R^2 - r^2]}} + R^2 - r^2\right] q(R) + q_0(r), \tag{6.6}$$

$$q(R) = \frac{c}{2} - \frac{1}{2\pi R^2}\int_0^R \int_0^{2\pi} s w(s,\theta)\, d\theta\, ds,$$

$$q_0(r) = \frac{r^2 - R^2}{2}c - \frac{1}{\pi}\int_0^R \int_0^{2\pi} s w(s,\theta)\, \ln\left|\frac{\max(r,s)}{R}\right| d\theta\, ds.$$

This formula is valid at every point where $w_0(r)$ possesses a continuous derivative, excepting very narrow neighbourhoods of the points $r = r_m$ where $w_0'(r)$ is discontinuous. If the discontinuity is ordinary, then the contact pressure has a singularity of the form $\gamma^2 g(r, r_m, \gamma) \ln|r - r_m|$, where the continuous function $g(r, r_m, \gamma)$ is $o(1)$ as $\gamma \ll 1$.

We note in conclusion that Jaffar's intuitive solution for a rigid sphere (**2**) may by derived from (6.4) and (6.6).

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