Independence Free Graphs and Vertex Connectivity Augmentation *

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Abstract

Given an undirected graph G and a positive integer k, the k-vertex-connectivity augmentation problem is to find a smallest set F of new edges for which $G+F$ is k-vertex-connected. Polynomial algorithms for this problem have been found only for $k \leq 4$ and a major open question in graph connectivity is whether this problem is solvable in polynomial time in general.

In this paper we develop an algorithm which delivers an optimal solution in polynomial time for every fixed k. In the case when the size of an optimal solution is large compared to k, our algorithm is polynomial for all k. We also derive a min-max formula for the size of a smallest augmenting set in this case. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call k-independence free graphs. We also prove new splitting off theorems for vertex connectivity.

1 Introduction

An undirected graph $G=(V,E)$ is *k-vertex-connected*, or more simply *k-connected*, if $|V| \geq k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected graph. Given a graph $G=(V,E)$

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and a positive integer k, the k-vertex-connectivity augmentation problem is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-connected. This problem (and a number of versions with different connectivity requirements and/or edge weights) is an important and well-studied optimization problem in network design. The complexity of the vertex-connectivity augmentation problem is one of the most challenging open questions of this area. It is open even if the graph G to be augmented is $(k-1)$-vertex-connected. Polynomial algorithms have been developed only for $k = 2, 3, 4$ by Eswaran and Tarjan [5], Watanabe and Nakamura [22] and Hsu [11], respectively. Values of k close to $|V| = n$ are also of interest. The case $k = n-1$ is easy, $k = n-2$ is equivalent to finding a maximum matching, and $k = n-3$ is open. Near optimal solutions can be found in polynomial time for every k, see [13], [12].

In this paper we give an algorithm which delivers an optimal solution in polynomial time for any fixed $k \geq 2$. We also obtain a min-max formula which determines the size of an optimal solution when it is large compared to k. In this case the running time of the algorithm is $O(n^6)$, where n is the size of the input graph. When the size of an optimal solution is small compared to k, the running time is bounded by $O(c_k n^3)$, where c_k is a constant if k is fixed. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call k-independence free graphs. We follow some of the ideas of the approach of [15], which obtained a near optimal solution in the special case when the graph to be augmented is $(k-1)$-connected. We also develop new 'splitting off' theorems for k-vertex-connectivity.

We remark that the other three basic augmentation problems (where one wants to make G k-edge-connected or wants to make a digraph k-edge- or k-vertex-connected) have been shown to be polynomially solvable. These results are due to Watanabe and Nakamura [21], Frank [6], and Frank and Jordán [8], respectively. For more results on connectivity augmentation and its algorithmic aspects, see the survey papers by Frank [7] and Nagamochi [20], respectively. In the remainder of this section, we introduce some definitions and our new lower bounds for the size of an augmenting set which makes G k-vertex-connected. We also state our main min-max results.

In what follows we deal with finite undirected graphs. We shall reserve the term 'graph' for graphs without loops or multiple edges and and use 'multigraph' if loops and multiple edges are allowed. Let $G = (V, E)$ be a multigraph, $v \in V$ and $X \subseteq V - v$. We use $d_G(v)$ to denote the *degree* of v in G and $d_G(v, X)$ for the number of edges of G from v to X. Let $N_G(X)$ denote the set of *neighbours* of X, that is, $N_G(X) = \{v \in V - X : uv \in E$ for some $u \in X\}$, and $n_G(X)$ denote $|N_G(X)|$. (We will supress the subscript G in the above functions when it is obvious to which graph we are referring.) We use X^* to denote $V - X - N_G(X)$. We say that X is a *fragment* of G if $X \neq \emptyset \neq X^*$. A *$k$-deficient fragment* is a fragment X for which $n(X) < k$, for some integer k. For two vertices x, y of G we shall use $\kappa(x, y, G)$ to denote the maximum number of openly disjoint paths from x to y in G. We use $\kappa(G)$ to denote the minimum of $\kappa(x, y, G)$ over all pairs of vertices of G. By Menger's theorem $\kappa(G)$ equals the minimum size of a vertex cut in G, unless G is complete.

Let G be a graph with at least $k+1$ vertices. A *k-augmenting set* for G is a set of edges F such that $G + F$ is k-connected. (When the value of k is obvious we shall refer to F simply as an *augmenting set* for G.) Let $a_k(G)$ denote the size of a smallest k-augmenting set for G. It is easy to see that every k-augmenting set for G must contain at least $k - n(X)$ edges from X to X^* for every fragment X. By summing up these 'deficiencies' over pairwise disjoint k-deficient fragments, we may obtain a useful lower bound on $a_k(G)$, similar to the one used in the corresponding edge-connectivity augmentation problem. Let

$$t_k(G) = \max\{\sum_{i=1}^{r} k - n(X_i) : X_1, \ldots, X_r \text{ are pairwise disjoint fragments in } V\}.$$

Then

$$a_k(G) \geq \lceil t_k(G)/2 \rceil. \tag{1}$$

Another lower bound for $a_k(G)$ comes from 'shredders'. For $K \subset V$ let $b_G(K)$, or simply $b(K)$ when it is clear to which graph we are referring to, denote the number of components in $G-K$. We say that K is an *s-shredder* (or simply a *shredder*) in G if $|K| = s$ and $b(K) \geq 3$. Let $b_k(G) = \max\{b(K) : K \text{ is a } (k-1)\text{-shredder in } G\}$. Since $(G+F)-K$ has to be connected for every k-augmenting set F and every $(k-1)$-shredder K, we have $|F| \geq b(K)-1$. This gives the second lower bound:

$$a_k(G) \geq b_k(G) - 1. \tag{2}$$

These lower bounds extend the two natural lower bounds used for example in [5, 11, 15]. Although these bounds suffice to characterize $a_k(G)$ for $k \leq 3$, there are examples showing that $a_k(G)$ can be strictly larger than the maximum of these lower bounds. For example, if we take G to be the complete bipartite graph $K_{3,3}$ with target connectivity $k=4$, we have $t_4(G) = 6$, $b_4(G) = 3$ and $a_4(G) = 4$. We shall show in Section 3 that $a_k(G) = \max\{b_k(G)-1, \lceil t_k(G)/2 \rceil\}$ when G is a 'k-independence free graph'. We use this result in Section 4 to show that if G is $(k-1)$-connected and $a_k(G)$ is large compared to k, then again we have $a_k(G) = \max\{b_k(G)-1, \lceil t_k(G)/2 \rceil\}$. Our proof technique is to find a set of edges F_1 such that $a_k(G+F_1) = a_k(G) - |F_1|$ and $G+F_1$ is k-independence free. The same result is not valid if we remove the hypothesis that G is $(k-1)$-connected. To see this consider the graph G obtained from $K_{m,k-2}$ by adding a new vertex x and joining x to j vertices in the m set of the $K_{m,k-2}$, where $j < k < m$. Then $b_k(G) = m$, $t_k(G) = 2m+k-2j$ and $a_k(G) = m-1+k-j$. We shall see in Section 7, however, that if we modify the definition of $b_k(G)$ slightly, then we may obtain an analogous min-max theorem for augmenting graphs of arbitrary connectivity. For a $(k-1)$-shredder K of G we define $\delta(K) = \max\{0, \max\{k-d(x) : x \in K\}\}$ and $\hat{b}(K) = b(K) + \delta(K)$. We let $\hat{b}_k(G) = \max\{\hat{b}(K) : K \text{ is a } (k-1)\text{-shredder in } G\}$. It is easy to see that

$$a_k(G) \geq \hat{b}_k(G) - 1.$$

We shall prove in Section 7 that if G is a graph of arbitrary connectivity and $a_k(G)$ is large compared to k, then

$$a_k(G) = \max\{\hat{b}_k(G) - 1, \lceil t_k(G)/2 \rceil\}.$$

Our proof technique is to find a set of edges F_1 such that $a_k(G+F_1) = a_k(G) - |F_1|$ and either $G+F_1$ is $(k-1)$-connected or $G+F_1$ is $(k-2)$-connected and has a special structure. In the former case we apply the result of Section 4 to $G+F_1$. In the latter case we find an optimal k-augmenting set for $G+F_1$ using a result on 'detachments' of 2-connected graphs.

Our proofs are algorithmic and give rise to polynomial algorithms for finding an optimal k-augmenting set in each of the cases mentioned above. In the remaining case, when $a_k(G)$ is small compared to k, we simply check all possible k-augmenting sets (spanned by a small set of vertices) to find an optimal solution. This is the only part where our algorithm is polynomial only if k is fixed.

In what follows, we shall suppress the subscript k in the parameters $t_k(G), b_k(G), \hat{b}_k(G)$ when the value of k is obvious.

2 Preliminaries

In this section we first introduce some submodular inequalities for the function n and then describe the 'splitting off' method. We also prove some preliminary results on edge splittings and shredders.

2.1 Submodular inequalities

The following inequalitites are fundamental to our proof technique. Inequality (4) is well-known, see for example [15].

Proposition 2.1 *In a graph $H = (V,E)$ every pair $X,Y \subseteq V$ satisfies*

$$\begin{aligned} n(X)+n(Y) &= n(X\cap Y)+n(X\cup Y)+|\,(N(X)\cap N(Y))-N(X\cap Y)| \\ &+ |\,(N(X)\cap Y))-N(X\cap Y)|+|\,(N(Y)\cap X))-N(X\cap Y)|. \end{aligned} \tag{3}$$

Proof: Readers may find it helpful to follow the proof given below if they imagine $V(G)$ represented by a 3×3 grid, in which the two pairs of opposite sides represent (X,X^*) and (Y,Y^*), respectively, and the 9 subsquares represent the corresponding partition of $V(G)$ into 9 subsets. Then (3) follows from the following equalities:

$$n(X) = |N(X)\cap Y| + |N(X)\cap N(Y)| + |N(X)\cap Y^*|,$$

$$n(Y) = |X\cap N(Y)| + |N(X)\cap N(Y)| + |X^*\cap N(Y)|,$$

$$n(X\cup Y) = |N(X)\cap Y^*| + |N(X)\cap N(Y)| + |X^*\cap N(Y)|,$$

and

$$n(X\cap Y) = |N(X\cap Y)\cap X| + |N(X\cap Y)\cap Y| + |N(X\cap Y)\cap (N(X)\cap N(Y))\,|.$$

•

Proposition 2.2 *In a graph $H = (V,E)$ every pair $X,Y \subseteq V$ satisfies*

$$n(X)+n(Y) \geq n(X\cap Y)+n(X\cup Y), \tag{4}$$

$$n(X)+n(Y) \geq n(X\cap Y^*)+n(Y\cap X^*). \tag{5}$$

Proof: Inequality (4) follows immediately from (3). Inequality (5) can be proved in a similar way to Proposition 2.1 •

The following inequality is new and may be applicable in other vertex-connectivity problems as well.

Proposition 2.3 *In a graph $H=(V,E)$ every triple $X,Y,Z\subseteq V$ satisfies*

$$\begin{aligned}n(X)+n(Y)+n(Z) \quad \geq \quad & n(X\cap Y\cap Z)+n(X\cap Y^*\cap Z^*)+n(X^*\cap Y^*\cap Z)+\\ & n(X^*\cap Y\cap Z^*)-|N(X)\cap N(Y)\cap N(Z)|. \qquad (6)\end{aligned}$$

Proof: Readers may find it helpful to follow the proof given below if they imagine $V(G)$ represented by a $3\times 3\times 3$ grid, in which the three pairs of opposite faces represent (X,X^*), (Y,Y^*), and (Z,Z^*), respectively, and the 27 subcubes represent the corresponding partition of $V(G)$ into 27 subsets. We have

$$\begin{aligned}n(X) \quad = \quad & |N(X)\cap Y\cap Z|+|N(X)\cap N(Y)\cap Z|+|N(X)\cap Y^*\cap Z|+\\ & +|N(X)\cap Y\cap N(Z)|+|N(X)\cap N(Y)\cap N(Z)|+|N(X)\cap Y^*\cap N(Z)|+\\ & +|N(X)\cap Y\cap Z^*|+|N(X)\cap N(Y)\cap Z^*|+|N(X)\cap Y^*\cap Z^*|,\end{aligned}$$

and

$$\begin{aligned}n(X\cap Y\cap Z) \quad \leq \quad & |X\cap Y\cap N(Z)|+|X\cap N(Y)\cap Z|+|X\cap N(Y)\cap N(Z)|+\\ & +|N(X)\cap Y\cap Z|+|N(X)\cap Y\cap N(Z)|+|N(X)\cap N(Y)\cap Z|+\\ & +|N(X)\cap N(Y)\cap N(Z)|.\end{aligned}$$

The lemma follows from the above (in)-equalities and similar (in)-equalities for $n(Y),n(Z)$, $n(X\cap Y^*\cap Z^*)$, $n(X^*\cap Y^*\cap Z)$ and $n(X^*\cap Y\cap Z^*)$. •

2.2 Extensions and Splittings

In the so-called 'splitting off method' one extends the input graph G by a new vertex s and a set of appropriately chosen edges incident to s and then obtains an optimal augmenting set by splitting off pairs of edges incident to s. This approach was initiated by Cai and Sun [2] for the k-edge-connectivity augmentation problem and further developed and generalized by Frank [6]. Here we adapt the method to vertex-connectivity and prove several basic properties of the extended graph as well as the splittable pairs.

Given the input graph $G=(V,E)$, an *extension* $G+s=(V+s,E+F)$ of G is obtained by adding a new vertex s and a set F of new edges from s to V. Note that F may contain multiple edges even though G does not, and hence $G+s$ may be a multigraph. In $G+s$ we define $X^*=V-X-N_G(X)$ and $\bar{d}(X)=n_G(X)+d(s,X)$ for every $X\subseteq V$. We say that $G+s$ is *(k,s)-connected* if $|V|\geq k+1$ and

$$\bar{d}(X)\geq k \text{ for every fragment } X \text{ of } G. \qquad (7)$$

If, in addition, F is an inclusionwise minimal set with respect to (7), then we say that $G+s$ is a *k-critical extension* of G. In this case, the minimality of F implies that every edge su is *k-critical*, that is, deleting su from $G+s$ destroys (7). (Thus an edge su is k-critical if and only if there exists a fragment X of G with $u\in X$ and $\bar{d}(X)=k$.) A fragment X with $d(s,X)\geq 1$ and $\bar{d}(X)=k$ is called *tight*. A fragment X with $d(s,X)\geq 2$ and $\bar{d}(X)\leq k+1$ is called *dangerous*. Observe that if G is l-connected then for every $v\in V$ we have $d(s,v)\leq k-l$ in any k-critical extension of G. The following lemma characterises when we can have $d(s,v)\geq 2$.

Lemma 2.4 *Let $G+s$ be a k-critical extension of G. Suppose $d(s,v) \geq 2$ for some $v \in V$. Let X be a fragment of G with $v \in X$ and $|X| \geq 2$. Then $\bar{d}(X) > k$. Furthermore $d_{G+s}(v) = k$.*

Proof: If $\bar{d}(X) = k$ then $\bar{d}(X-v) \leq k - d(s,v) + 1 < k$ which contradicts (7). Thus $\bar{d}(X) > k$. Since $G+s$ is k-critical we may choose a tight set Y in $G+s$ with $v \in Y$. The first part of the lemma implies that $Y = \{v\}$. Hence $d_{G+s}(v) = \bar{d}(v) = k$. •

Since the function $d(s,X)$ is modular on the subsets of V in $G+s$, Propositions 2.1, 2.2 and 2.3 yield the following (in)equalities.

Proposition 2.5 *In a graph $G+s$ every pair $X,Y \subseteq V$ satisfies*

$$\begin{aligned}\bar{d}(X)+\bar{d}(Y) &= \bar{d}(X\cap Y)+\bar{d}(X\cup Y)+|(N(X)\cap N(Y))-N(X\cap Y)| \\ &+ |(N(X)\cap Y)-N(X\cap Y)|+|(N(Y)\cap X)-N(X\cap Y)|, \end{aligned} \tag{8}$$

$$\bar{d}(X)+\bar{d}(Y) \geq \bar{d}(X\cap Y)+\bar{d}(X\cup Y), \tag{9}$$

$$\bar{d}(X)+\bar{d}(Y) \geq \bar{d}(X\cap Y^*)+\bar{d}(Y\cap X^*)+d(s,X-Y^*)+d(s,Y-X^*). \tag{10}$$

Proposition 2.6 *In a graph $G+s$ every triple $X,Y,Z \subseteq V$ satisfies*

$$\begin{aligned}\bar{d}(X)+\bar{d}(Y)+\bar{d}(Z) &\geq \bar{d}(X\cap Y\cap Z)+\bar{d}(X\cap Y^*\cap Z^*)+\bar{d}(X^*\cap Y^*\cap Z)+\bar{d}(X^*\cap Y\cap Z^*) \\ &-|N_G(X)\cap N_G(Y)\cap N_G(Z)|+2d(s,X\cap Y\cap Z). \end{aligned} \tag{11}$$

Lemma 2.7 *Let $G+s$ be a (k,s)-connected extension of G. Then there exists a k-augmenting set F of G with $V(F) \subseteq N(s)$.*

Proof: Let F be a set of edges such that $A = N(s)$ induces a complete graph in $H = G+F$. Suppose H is not k-connected. Then there exists a k-deficient fragment X in H. Since A induces a clique in H, we have either $A\cap X = \emptyset$ or $A \cap X^* = \emptyset$. Assuming, without loss of generality, that $A \cap X = \emptyset$, we have $\bar{d}_{G+s}(X) = n_H(X) < k$. This contradicts the hypothesis that $G+s$ is (k,s)-connected. •

We can use Lemma 2.7 to obtain upper and lower bounds of $a_k(G)$ in terms of $d_{G+s}(s)$. The following result is an easy consequence of a theorem of Mader [18, Satz 1]. It was used in [15, p 16] in the special case when G is $(k-1)$-connected.

Theorem 2.8 *Let F be a minimal k-augmenting set for a graph G and let B be the set of those vertices of G which have degree at least $k+1$ in $G+F$. Then F induces a forest on B.*

Lemma 2.9 *Let $G+s$ be a (k,s)-connected extension of G and let A be a minimal k-augmenting set for G in which every edge in A connects two vertices of $N(s)$. Then $|A| \leq d(s)-1$.*

Proof: Let $B = \{v \in N(s) : d_{G+A}(v) \geq k+1\}$ and let $C = N(s) - B$. Since $d_{G+A}(x) = k$ and $d_{G+s}(x) \geq k$, we have $d_A(x) \leq d(s,x)$ for each $x \in C$. By Theorem 2.8, B induces a forest in A. Let $e_A(B)$ and $e_A(C)$ denote the number of those edges of A which connect two vertices of B and of C, respectively. The previous observations imply the following inequality.

$$\begin{aligned} |A| &= e_A(C) + d_A(B,C) + e_A(B) \leq \sum_{x \in C} d_A(x) + |B| - 1 \leq \\ &\leq (d(s) - |B|) + |B| - 1 = d(s) - 1. \end{aligned}$$

This proves the lemma. •

To obtain a lower bound on $a_k(G)$ in terms of $d(s)$, we introduce a new parameter. Let $G = (V,E)$ be a graph. We say that a fragment X of G *separates* a pair of vertices $u, v \in V$ if $\{u,v\} \cap X \neq \emptyset \neq \{u,v\} \cap X^*$. A family $\mathcal{F}$ of fragments of G is *half-disjoint* if every pair of vertices of G is separated by at most two fragments in $\mathcal{F}$. Let $t'(G) = \max\{\sum_{X \in \mathcal{F}} k - n(X)\}$ where the maximum is taken over all half-disjoint families $\mathcal{F}$ of k-deficient fragments in G. Note that every family of pairwise disjoint fragments is half-disjoint and hence $t'(G) \geq t(G)$. Since every k-augmenting set for G must contain at least $k - n(X)$ edges from X to X^* for every fragment X of G, we obtain the lower bound:

$$a_k(G) \geq \lceil t'(G)/2 \rceil. \tag{12}$$

Lemma 2.10 *Let $G+s$ be a k-critical extension of a graph G. Then*

$$\lceil d(s)/2 \rceil \leq a_k(G) \leq d(s) - 1.$$

Proof: The last inequality follows immediately from Lemma 2.9. To verify the first inequality we choose a family $\mathcal{X} = \{X_1, \ldots, X_m\}$ of tight fragments of G such that $N(s) \subseteq \cup_{i=1}^m X_i$ and such that m is minimum and $\sum_{i=1}^m |X_i|$ is minimum. Such a family exists since the edges incident to s in $G+s$ are k-critical. We claim that for every $1 \leq i < j \leq m$ either $X_i \cap X_j = \emptyset$ or at least one of $X_i^* \subseteq N(X_j)$ or $X_j^* \subseteq N(X_i)$ holds. Note that in the latter case no pair of vertices can simultaneously be separated by X_i and X_j.

To verify the claim, suppose that $X_i \cap X_j \neq \emptyset$. Then by the minimality of m the set $X_i \cup X_j$ cannot be tight. Thus (9) implies that $X_i^* \cap X_j^* = \emptyset$. Hence either one of $X_i^* \subseteq N(X_j)$ or $X_j^* \subseteq N(X_i)$ holds or $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both non-empty. In the former case we are done. In the latter case we apply (10) to X_i and X_j and conclude that $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both tight and all the edges from s to $X_i \cup X_j$ enter $(X_i \cap X_j^*) \cup (X_j \cap X_i^*)$. Thus we could replace X_i and X_j in $\mathcal{X}$ by two strictly smaller sets $X_i \cap X_j^*$ and $X_j \cap X_i^*$, contradicting the choice of $\mathcal{X}$. This proves the claim.

To finish the proof of the lemma, observe that $\sum_{i=1}^m k - n(X_i) = \sum_{i=1}^m d(s, X_i) \geq d(s)$. In other words, the sum of 'k-deficiencies' of the fragments in $\mathcal{X}$ is at least $d(s)$. We shall show that $\mathcal{X}$ is half disjoint. Suppose on the contrary that some pair $u, v \in V$ is simultaneously separated by three sets in $\mathcal{X}$, say X_1, X_2, X_3. By the above claim, X_1, X_2, X_3 are pairwise disjoint. This contradicts the fact that they each separate u, v and hence $\{u,v\} \cap X_i \neq \emptyset$ for all $1 \leq i \leq 3$. Hence $\mathcal{X}$ is half-disjoint and $d(s) \leq t'(G)$, as required. •

Let $G+s$ be a (k,s)-connected extension of G. *Splitting off* two edges su, sv in $G+s$ means deleting su, sv and adding a new edge uv. Note that if we perform a sequence of splittings at s

starting with graph $G+s$, and denote the resulting graph by $G'+s$, then G' is the graph obtained from G by adding the split edges. A split is *k-admissible* if the graph obtained by the splitting also satisfies (7). We will also say that the pair of edges su,sv is *k-admissible*, or simply *admissible* when the value k is obvious. Notice that if $G+s$ has no edges incident to s then (7) is equivalent to the k-connectivity of G. Hence it would be desirable to know, when $G+s$ is a k-critical extension and $d(s)$ is even, that there is a sequence of admissible splittings such that s is an isolated vertex in the resulting graph $G'+s$. In this case we would have $|E(G')-E(G)|=d_G(s)/2$, and, using the fact that $a_k(G)\geq d(s)/2$ by Lemma 2.10, the graph G' would be an *optimal* k-augmentation of G. This approach works for the k-edge-connectivity augmentation problem [6] but does not always work in the vertex connectivity case. The reason is that such 'complete splittings' do not necessarily exist. On the other hand, we shall prove results which are 'close enough' to yield an optimal algorithm for k-connectivity augmentation using the splitting off method, which is polynomial for k fixed.

Pairs of edges sx,sy which do not give k-admissible splits can be characterized by tight and dangerous 'certificates' as follows. The proof of the following simple lemma is omitted.

Lemma 2.11 *Let $G+s$ be a (k,s)-connected extension of G and $x,y\in N(s)$. Splitting off the pair sx,sy is not k-admissible in $G+s$ if and only if one of the following holds:*
(i) there exists a tight set T with $x\in T$, $y\in N(T)$,
(ii) there exists a tight set U with $y\in U$, $x\in N(U)$,
(iii) there exists a dangerous set W with $x,y\in W$.

2.3 Local separators and shredders

For two vertices $u,v\in V$ a *uv-cut* is a set $K\subseteq V-\{u,v\}$ for which there is no uv-path in $G-K$. A set $S\subset V$ is a *local separator* if there exist $u,v\in V-S$ such that S is an inclusionwise minimal uv-cut. We also say S is a *local uv-separator* and we call the components of $G-S$ containing u and v *essential components* of S (with respect to the pair u,v). Note that S may be a local separator with respect to several pairs of vertices and hence it may have more than two essential components. Clearly, $N(C)=S$ for every essential component C of S. If S is a local uv-separator and T is a local xy-separator then we say T *meshes* S if T intersects the two essential components of S containing u and v, respectively.

Lemma 2.12 *If T meshes S then S intersects every essential component of T (and hence S meshes T).*

Proof: Suppose S is a uv-separator and let C_u, C_v be the two essential components of S containing u and v respectively. Let C be an essential component of T. We need to show S intersects C. Choose $w\in V(C)$. Without loss of generality, $w\notin S$ and $w\notin V(C_v)$. Choose $t\in T\cap C_v$. Then $t\notin S$. Let P be a path in the subgraph of $C\cup T$ from w to t such that $P\cap T=\{t\}$. Then P contains a vertex of S since S separates w from t. Hence $C\cap S\neq\emptyset$. •

Lemma 2.12 extends a result of Cheriyan and Thurimella [4, Lemma 4.3(1)]. The next lemma extends a key observation from the same paper [4, Proposition 3.1] and will be used when we discuss algorithms in Section 8.

Lemma 2.13 *Let K be a local uv-separator of size $k-1$ and suppose that there exist $k-1$ openly disjoint paths $P_1,...,P_{k-1}$ from u to v in G. Let $Q=\cup_{i=1}^{k-1}V(P_i)$.*
(a) For each component C of $G-K$ either $C\cap\{u,v\}\neq\emptyset$ or C is a component of $G-Q$;
(b) If K has at least three essential components then $K=N(C)$ for some component C of $G-Q$.

Proof: (a) Since K is a local uv-separator of size $k-1$, K contains exactly one vertex from each path $P_1,\ldots,P_{k-1}$. Let C_u,C_v,C be distinct components of K with $u\in C_u$ and $v\in C_v$. Then $Q-K\subseteq C_u\cup C_v$. Thus $C\cap Q=\emptyset$. Hence C is a component of $G-Q$.
(b) Suppose K has at least three essential components. Then we choose C to be an essential component of K distinct from C_u,C_v. Then $K=N(C)$ holds by (a). •

Let K be a $(k-1)$-shredder of G and $G+s$ be a (k,s)-connected extension of G. A component C of $G-K$ is called a *leaf component of K in $G+s$* if $d(s,C)=1$ holds. Note that $d(s,C')\geq 1$ for each component C' of $G-K$ by (7). The next lemma is easy to verify by (7).

Lemma 2.14 *Let $G+s$ be a (k,s)-connected extension of a graph G and K be a $(k-1)$-shredder in G.*
(a) Let C_1,C_2 be leaf components of K in $G+s$. Then there exist $k-1$ openly disjoint paths in the subgraph of G induced by $C_1\cup C_2\cup K$ from every vertex of C_1 to every vertex of C_2.
(b) If $d(s)\leq 2b(K)-2$ then K has at least two leaf components, K is a local separator and every leaf component of K is an essential component of K in G.

Proof: Assertion (a) follows from (7). Assertion (b) follows from the fact that $d(s,C)\geq 1$ for every component C of $G-K$, and from (a). •

We shall use the following lemma to find $(k-1)$-shredders with many components in a graph G when some edge incident to s in $G+s$ belongs to many non-admissible pairs.

Lemma 2.15 *Let $G+s$ be a (k,s)-connected extension of a graph G. Suppose there exist r dangerous sets $W_1,W_2,\ldots,W_r$ and a tight set X_0 in $G+s$ such that $r\geq 3$, $W_i\cap W_j=X_0$, and $W_i\cap W_j^*\cap W_h^*\neq\emptyset$ for all distinct $i,j,h\in\{1,2,\ldots,r\}$. Then $K=N_G(X_0)$ is a $(k-1)$-shredder in G with leaf components $C_0,C_1,\ldots C_r$, where $V(C_0)=X_0$ and $V(C_i)=W_i-X_0$ for all $1\leq i\leq r$.*

Proof: Applying (11) and using the facts that: $d(s,W_i\cap W_j\cap W_h)\geq 1$, since $W_i\cap W_j\cap W_h=X_0$, and X_0 is tight; and $n_G(W_i)=\bar{d}(W_i)-d(s,W_i)\leq k-1$ since W_i is dangerous; we obtain

$$\begin{aligned}
3k+3 \;\geq\;& \bar{d}(W_i)+\bar{d}(W_j)+\bar{d}(W_h)\geq \bar{d}(W_i\cap W_j\cap W_h)+\bar{d}(W_i\cap W_j^*\cap W_h^*)+\\
&+\bar{d}(W_j\cap W_i^*\cap W_h^*)+\bar{d}(W_h\cap W_i^*\cap W_j^*)-|N_G(W_i)\cap N_G(W_j)\cap N_G(W_h)|+\\
&+2d(s,W_i\cap W_j\cap W_h)\\
\geq\;& 4k-|N_G(W_i)\cap N_G(W_j)\cap N_G(W_h)|+2\geq 3k+3. \qquad (13)
\end{aligned}$$

Thus equality must hold throughout. Hence $d(s,X_0)=1$, and $|N_G(W_i)\cap N_G(W_j)\cap N_G(W_h)|=n_G(W_i)=k-1$. Thus $N_G(W_i)=N_G(W_j)$ for all $i,j\in\{1,2,\ldots,r\}$. This implies that $N_G(W_i)\cap W_j=\emptyset$ for all $i,j\in\{1,2,\ldots,r\}$ and hence that $N_G(X_0)\subseteq N_G(W_i)$. Since $\bar{d}(X_0)=k$, $d(s,X_0)=1$ and $n_G(W_i)=k-1$; we have $N_G(X_0)=N_G(W_i)=K$, say, for all $i\in\{1,2,\ldots,r\}$.

The fact that $W_i \cap N_G(W_j) = \emptyset$ for all $i, j \in \{1, 2, \ldots, r\}$ also implies that W_i is the disjoint union of $W_i \cap W_j \cap W_h$ and $W_i \cap W_j^* \cap W_h^*$. Thus $W_i \cap W_j^* \cap W_h^* = W_i - X_0$ for all $i, j, h \in \{1, 2, \ldots, r\}$. Equality in (13) implies that $d(W_i) = k + 1$. Since $n_G(W_i) = k - 1$, we have $d(s, W_i) = 2$. The fact that $d(s, X_0) = 1$ now implies that $d(s, W_i - X_0) = 1$. Since $N_G(W_i) = K$ we have $N_G(W_i^*) \subseteq K$ for all $i \in \{1, 2, \ldots, r\}$. Thus $N_G(W_i - X_0) = N_G(W_i \cap W_j^* \cap W_h^*) \subseteq K$. Since $d(s, W_i - X_0) = 1$ and $|K| = k - 1$ we have $N_G(W_i - X_0) = K$. It follows that K is the required $(k-1)$-shredder in G. •

Note that the existence of a $(k-1)$-shredder K as described in Lemma 2.15 certifies that no pair of edges from s to $\cup_{i=0}^{r} C_i$ is k-admissible since each of the sets $V(C_i) \cup V(C_j)$ is dangerous.

3 Independence Free Graphs

In this section we give a complete solution of the k-connectivity augmentation problem for a special family of graphs which we call k-independence free graphs. This result is a key step in our proofs concerning arbitrary graphs. However, we shall only need a special case of the main result of this section: when we augment the connectivity of a $(k-1)$-connected k-independence free graph by one. This is important from an algorithmic point of view, since, as we shall see in Subsection 8.1, we are able to check whether a $(k-1)$-connected graph is k-independence free. Thus the reader may decide to focus on this special case at first reading.

Let $G = (V, E)$ be a graph and k be an integer. Let X_1, X_2 be disjoint non-empty subsets of V. We say (X_1, X_2) is a *k-deficient pair* if $d(X_1, X_2) = 0$ and $|V - (X_1 \cup X_2)| \leq k - 1$. We say two k-deficient pairs (X_1, X_2) and (Y_1, Y_2) are *independent* if for some $i \in \{1, 2\}$ we have either $X_i \subseteq V - (Y_1 \cup Y_2)$ or $Y_i \subseteq V - (X_1 \cup X_2)$. In this case no edge can simultaneously connect X_1 to X_2 and Y_1 to Y_2 and hence the two pairs give 'independent constraints' in the k-augmentation problem for G. We say G is *k-independence free* if G does not have two independent k-deficient pairs. The following observations follow from these definitions.

1. If (X_1, X_2) is a k-deficient pair in G then X_1 is a k-deficient fragment.
2. If X is a k-deficient fragment in G then (X, X^*) is a k-deficient pair.
3. $(k-1)$-connected chordal graphs are k-independence free.
4. Graphs with minimum degree at least $2k - 2$ are k-independence free.
5. All graphs are 1-independence free and all connected graphs are 2-independence free.
6. A graph with no edges and at least $k + 1$ vertices is not k-independence free for any $k \geq 2$.
7. If G is k-independence free and H is obtained by adding edges to G then H is also k-independence free.
8. A k-independence free graph is l-independence free for all $l \leq k$.

In general, a main difficulty in vertex-connectivity problems is that vertex cuts (and hence tight and dangerous sets) can cross each other in many different ways. In the case of an independence free graph G we can overcome these difficulties and provide both a complete characterisation of the case when there is no admissible split containing a specified edge in an extension of G, and a min/max formula which determines the number of edges in an optimal k-augmentation for G.

Lemma 3.1 *Let $G+s$ be a (k,s)-connected extension of a k-independence free graph G and X,Y be fragments of G.*
(a) If X and Y are tight then either: $X\cup Y$ is tight, $X\cap Y\neq\emptyset$ and $\bar{d}(X\cap Y)=k$; or $X\cap Y^*$ and $Y\cap X^*$ are both tight and $d(s,X-Y^*)=0=d(s,Y-X^*)$.
(b) If X is a minimal tight set and Y is tight then either: $X\cup Y$ is tight, $d(s,X\cap Y)=0$ and $n_G(X\cap Y)=k$; or $X\subseteq Y$; or $X\subseteq Y^*$.
(c) If X is a tight set and Y is a maximal dangerous set then either $X\subseteq Y$ or $d(s,X\cap Y)=0$.
(d) If X is a tight set, Y is a dangerous set and $d(s,Y-X^*)+d(s,X-Y^*)\geq 2$ then $X\cap Y\neq\emptyset$ and $\bar{d}(X\cap Y)\leq k+1$.

Proof: (a) Suppose $X\cap Y^*, Y\cap X^*\neq\emptyset$. Then (10) implies that $\bar{d}(X\cap Y^*)=k=\bar{d}(Y\cap X^*)$ and $d(s,X-Y^*)=0=d(s,Y-X^*)$. Thus $X\cap Y^*$ and $Y\cap X^*$ are both tight. Hence we may assume that either $X\cap Y^*$ or $Y\cap X^*$ is empty. Since G is k-independence free, it follows that $X^*\cap Y^*\neq\emptyset\neq X\cap Y$ (for example if $X\cap Y^*=\emptyset=X^*\cap Y^*$ then $Y^*\subseteq V-(X\cup X^*)$, and (X,X^*) and (Y,Y^*) are independent k-deficient pairs). Thus $X\cup Y$ is a fragment in G. Using (9) we deduce that $X\cup Y$ is tight and $\bar{d}(X\cap Y)=k$.

(b) This follows from (a) using the minimality of X.

(c) Suppose $X\not\subseteq Y$ and $d(s,X\cap Y)\geq 1$. If $X\cap Y^*\neq\emptyset\neq Y\cap X^*$ then we can use (10) to obtain the contradiction

$$2k+1\geq\bar{d}(X)+\bar{d}(Y)\geq\bar{d}(X\cap Y^*)+\bar{d}(Y\cap X^*)+2\geq 2k+2.$$

Thus either $X\cap Y^*$ or $Y\cap X^*$ is empty and, since G is k-independence free, $X^*\cap Y^*\neq\emptyset$. Thus $X\cup Y$ is a fragment in G. Using (9) we deduce that $X\cup Y$ is dangerous contradicting the maximality of Y.

(d) Using (10), we deduce that either $X\cap Y^*$ or $Y\cap X^*$ is empty and, since G is k-independence free, $X\cap Y\neq\emptyset\neq X^*\cap Y^*$. We can now use (9) to deduce that $\bar{d}(X\cap Y)\leq k+1$. •

Using Lemma 3.1 we deduce

Corollary 3.2 *If $G+s$ is a k-critical extension of a k-independence free graph G then $d(s)=t(G)$. Furthermore there exists a unique minimal tight set in $G+s$ containing x for each $x\in N(s)$.*

Proof: Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $\sum_{X\in\mathcal{F}}|X|$ is as small as possible. Since every edge incident to s is k-critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X,Y\in\mathcal{F}$ and suppose that $X\cap Y\neq\emptyset$. By Lemma 3.1(a) we may replace X and Y in $\mathcal{F}$ either by $X\cup Y$, or by $X\cap Y^*$ and $Y\cap X^*$. Both alternatives contradict the minimality of $\sum_{X\in\mathcal{F}}|X|$. Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s)=\sum_{X\in\mathcal{F}}(k-n_G(X))\leq t(G)$. The inequality $d(s)\geq t(G)$ follows easily from (7). Thus $d(s)=t(G)$, as required.

The second assertion of the corollary follows immediately from criticality and Lemma 3.1(b). •

Lemma 3.3 *Let $G+s$ be a k-critical extension of a k-independence free graph G and $x_1,x_2 \in N(s)$. Then the pair sx_1,sx_2 is not k-admissible for splitting in $G+s$ if and only if there exists a dangerous set W in $G+s$ with $x_1,x_2 \in W$.*

Proof: Suppose the lemma is false. Using Lemma 2.11 we may assume without loss of generality that there exists a tight set X_1 in $G+s$ such that $x_1 \in X_1$ and $x_2 \in N_G(X_1)$. Let X_2 be the minimal tight set in $G+s$ containing x_2. Since $x_2 \in N(s)\cap(X_2-X_1^*)$, it follows from Lemma 3.1(a) that $X_1 \cup X_2$ is a tight, and hence dangerous, set in $G+s$ containing x_1,x_2. •

Theorem 3.4 *Let $G+s$ be a k-critical extension of a k-independence free graph G and $x_0 \in N(s)$.*
(a) There is no k-admissible split in $G+s$ containing sx_0 if and only if either: $d(s)=b(G)$; or $d(s)$ is odd and there exist maximal dangerous sets W_1,W_2 in $G+s$ such that $N(s)\subseteq W_1\cup W_2$, $x_0 \in W_1\cap W_2$, $d(s,W_1\cap W_2)=1$, $d(s,W_1\cap W_2^*)=(d(s)-1)/2=d(s,W_1^*\cap W_2)$, and $W_1\cap W_2^*$ and $W_2\cap W_1^*$ are tight.
(b) If there is no admissible split containing sx_0 and $3\neq d(s)\neq b(G)$ then there is an admissible split containing sx_1 for all $x_1 \in N(s)-x_0$.

Proof: Note that since $G+s$ is a k-critical extension, $d(s)\geq 2$.
(a) Using Lemma 3.3, we may choose a family of dangerous sets $\mathcal{W}=\{W_1,W_2,\ldots,W_r\}$ in $G+s$ such that $x_0 \in \cap_{i=1}^r W_i$, $N(s)\subseteq \cup_{i=1}^r W_i$ and r is as small as possible. We may assume that each set in $\mathcal{W}$ is a maximal dangerous set in $G+s$. If $r=1$ then $N(s)\subseteq W_1$ and

$$\bar{d}(W_1^*)=n_G(W_1^*)\leq n_G(W_1)\leq k+1-d(s,W_1)\leq k-1,$$

since W_1 is dangerous. This contradicts the fact that $G+s$ is (k,s)-connected. Hence $r\geq 2$.

Claim 3.5 *Let $W_i,W_j \in \mathcal{W}$. Then $W_i\cap W_j^*\neq\emptyset\neq W_j\cap W_i^*$ and $d(s,W_i-W_j^*)=1=d(s,W_j-W_i^*)$.*

Proof: Suppose $W_i\cap W_j^*=\emptyset$. Since G is k-independence free, it follows that $W_i^*\cap W_j^*\neq\emptyset$ and hence $W_i\cup W_j$ is a fragment of G. The minimality of r now implies that $W_i\cup W_j$ is not dangerous, and hence $\bar{d}(W_i\cup W_j)\geq k+2$. Applying (9) we obtain

$$2k+2\geq \bar{d}(W_i)+\bar{d}(W_j)\geq \bar{d}(W_i\cap W_j)+\bar{d}(W_i\cup W_j)\geq 2k+2.$$

Hence equality holds throughout. Thus $\bar{d}(W_i\cap W_j)=k$ and, since $x_0\in W_i\cap W_j$, $W_i\cap W_j$ is tight.

Choose $x_i\in N(s)\cap(W_i-W_j)$ and let X_i be the minimal tight set in $G+s$ containing x_i. Since $x_i\in N(s)\cap X_i\cap W_i$, it follows from Lemma 3.1(c) that $X_i\subseteq W_i$. Since G is k-independence free, $X_i\not\subseteq N(W_j)$. The asumption that $W_i\cap W_j^*=\emptyset$ now implies that $X_i\cap W_i\cap W_j\neq\emptyset$. Applying Lemma 3.1(b), we deduce that $X_i\cup(W_i\cap W_j)$ is tight. Now $X_i\cup(W_i\cap W_j)$ and W_j contradict Lemma 3.1(c) since $x_0\in W_i\cap W_j$ and W_j is a maximal dangerous set. Hence we must have $W_i\cap W_j^*\neq\emptyset\neq W_j\cap W_i^*$. The second part of the claim follows from (10) and the fact that $x_0\in W_i\cap W_j$. •

Suppose $r=2$. Using Claim 3.5, we have $d(s)=1+d(s,W_1\cap W_2^*)+d(s,W_2\cap W_1^*)$. Without loss of generality we may suppose that $d(s,W_1\cap W_2^*)\leq d(s,W_2\cap W_1^*)$. Then

$$\overline{d}(W_2^*)=d(s,W_1\cap W_2^*)+n_G(W_2^*)\leq d(s,W_2\cap W_1^*)+n_G(W_2)=\overline{d}(W_2)-1\leq k.$$

Thus equality must hold throughout. Hence $d(s,W_1\cap W_2^*)=d(s,W_2\cap W_1^*)=(d(s)-1)/2$, $d(s)$ is odd, $W_1\cap W_2^*$ and $W_2\cap W_1^*$ are tight and the second alternative in (a) holds.

Finally we suppose that $r\geq 3$. Choose $W_i,W_j,W_h\in\mathcal{W}$, $x_i\in(N(s)\cap W_i)-(W_j\cup W_h)$. Then Claim 3.5 implies that $x_i\in W_i\cap W_j^*\cap W_h^*$, and hence $W_i\cap W_j^*\cap W_h^*\neq\emptyset$. Since $G+s$ is k-critical, we may choose a maximal tight set X_0 in $G+s$ with $x_0\in X_0$. Lemma 3.1(c) implies that $X_0\subseteq W_t$ for all $1\leq t\leq r$. Since $x_h\in W_i^*\cap W_j^*\cap W_h$ we have $W_i^*\cap W_j^*\neq\emptyset$. We can use (9) to deduce that $W_i\cap W_j$ is tight. Since $X_0\subseteq W_i\cap W_j$, the maximality of X_0 now implies that $W_i\cap W_j=X_0$ for all $1\leq i<j\leq r$. Applying Lemma 2.15 we deduce that $K=N_G(X_0)$ is a $(k-1)$-shredder in G with $b_G(K)=d(s)$. Since the (k,s)-connectivity of $G+s$ implies that $b(G)\leq d(s)$, we have $b(G)=d(s)$.

(b) Using (a) we have $d(s)$ is odd and there exist maximal dangerous sets W_1,W_2 in $G+s$ such that $N(s)\subseteq W_1\cup W_2$, $x_0\in W_1\cap W_2$, $d(s,W_1\cap W_2)=1$, $d(s,W_1\cap W_2^*)=d(s,W_1^*\cap W_2)=(d(s)-1)/2\geq 2$, and $W_1\cap W_2^*$ and $W_1^*\cap W_2$ are tight. Suppose $x_1\in N(s)\cap W_1\cap W_2^*$ and there is no admissible split containing sx_1. Then applying (a) to x_1 we find maximal dangerous sets W_3,W_4 with $x_1\in W_3\cap W_4$ and $d(s,W_3\cap W_4)=1$. Using Lemma 3.1(c) we have $W_1\cap W_2^*\subseteq W_3$ and $W_1\cap W_2^*\subseteq W_4$. Thus $W_1\cap W_2^*\subseteq W_3\cap W_4$ and $d(s,W_3\cap W_4)\geq 2$. This contradicts the fact that $d(s,W_3\cap W_4)=1$. •

We can use this splitting result to determine $a_k(G)$ when G is k-independence free. We first solve the case when $b(G)$ is large compared to $d(s)$.

Lemma 3.6 *Let $G+s$ be a k-critical extension of a k-independence free graph G and K be a $(k-1)$-shredder in G. If $d(s)\leq 2b(K)-2$ then $d(s,K)=0$.*

Proof: Let $b(K)=b$. Suppose $x\in N(s)\cap K$ and let X be the minimal tight set in $G+s$ containing x. Let $\mathcal{L}=\{X_1,X_2,\ldots,X_r\}$ be the leaf components of K. Since $d(s)\leq 2b-2$ we have $r\geq 2$. Choose $X_i\in\mathcal{L}$ and $x_i\in N(s)\cap X_i$. Then X_i is tight. Since $x\in K=N_G(X_i)$ we have $X\not\subseteq X_i^*$. Using Lemma 3.1(b), we deduce that $X\cup X_i$ is tight, $n_G(X\cap X_i)=k$ and $d(s,X\cap X_i)=0$. Hence $x_i\notin X$ and $N(X)\cap X_i\neq\emptyset$. Since this holds for all $X_i\in\mathcal{L}$ and $x\in X\cap K$, we have

$$|N(X)\cap(X_1\cup X_2\ldots X_r)|\geq r. \tag{14}$$

Furthermore, since $X\cap X_2\neq\emptyset$ and $X\cap X_2\subseteq X\cap X_1^*$ we have $X\cap X_1^*\neq\emptyset$. Using (10) and the fact that $d(s,X-X_1^*)\geq 1$ since $x\in X\cap N_G(X_1)$, it follows that $X^*\cap X_1=\emptyset$. Using symmetry we deduce that $X^*\cap X_i=\emptyset$ for all $X_i\in\mathcal{L}$.

Since $X_1\cup X_2$ is dangerous and $x_1,x_2\notin X^*$, we can use Lemma 3.1(d) to deduce that $\overline{d}(X\cap(X_1\cup X_2))\leq k+1$. Using the facts that $n_G(X\cap X_1)=k=n_G(X\cap X_2)$, $N_G(X\cap(X_1\cup X_2))=N_G(X\cap X_1)\cup N_G(X\cap X_2)$, and $N_G(X\cap X_i)\cap X_i\neq\emptyset$ for each $i\in\{1,2\}$, we have $|N_G(X\cap X_i)\cap X_i|=1$ for each $i\in\{1,2\}$ and $K=N_G(X\cap X_1)\cap N_G(X\cap X_2)$. Thus $x\in N_G(X\cap X_1)$, $K\subseteq X\cup N_G(X)$ and $X^*\cap K=\emptyset$. Since $X^*\cap X_i=\emptyset$ for all $X_i\in\mathcal{L}$, $X^*\cap Y\neq\emptyset$ for some non-leaf component Y of $G-K$. Using (14) and the facts that $N_G(X^*\cap Y)\subseteq(N_G(X)\cap Y)\cup(N_G(X)\cap K)$ and $n_G(X)\leq k-1$, we deduce that

$n_G(X^*\cap Y)\le k-1-r$. Since $G+s$ is (k,s)-connected we have $d(s,Y)\ge d(s,X^*\cap Y)\ge r+1$. Thus

$$\begin{aligned} d(s) &= d(s,Y)+d(s,X_1\cup X_2\ldots X_r)+d(s,(Y_1\cup Y_2\ldots Y_{b-r})-Y)+d(s,K)\\ &\ge (r+1)+r+2(b-r-1)+1\ge 2b. \end{aligned}$$

This contradicts the hypothesis that $d(s)\le 2b-2$. •

Lemma 3.7 *Let $G+s$ be a k-critical extension of a k-independence free graph G such that $b(G)+1\le d(s)\le 2b(G)-2$. Then there exists an admissible split at s such that, for the resulting graph $G'+s$, we have $b(G')=b(G)-1$.*

Proof: Let $b(G)=b$ and let K be a $(k-1)$-shredder in G with $b_G(K)=b$ and, subject to this condition, with the maximum number r of leaves in $G+s$. Let $C_1,C_2,\ldots,C_r$ be the leaf components of K and let $N(s)\cap C_i=\{x_i\}$ for $1\le i\le r$. Since $d(s)\le 2b(G)-2$ we have $r\ge 2$. Since $d(s)\ge b(G)+1$ and $r\ge 2$, we may use Theorem 3.4 to deduce without loss of generality that there is an admissible split in $G+s$ containing sx_1. Choose sw such that sx_1,sw is an admissible split in $G+s$. Splitting sx_1,sw we obtain $G'+s$ where $d_{G'+s}(s)=d_{G+s}(s)-2$ and $G'=G+x_1w$.

Suppose $b(G')=b(G)$. Then G has a $(k-1)$-shredder K' with $b_G(K')=b(G)$ such that x_1,w belong to the same component C' of $G-K'$. (Note that $\{x_1,w\}\cap K'=\emptyset$ by Lemma 3.6.) We shall prove that such a K' cannot exist in G.

Suppose $x_1,x_2,\ldots,x_r\in V(C')$. Since w is also contained in C' we have $d(s,C')\ge r+1$. Since $d(s)\le 2b-2$ it follows that K' has at least $r+1$ leaf components, contradicting the maximality of r. Hence we may assume without loss of generality that

$$x_2\notin C'. \tag{15}$$

Thus K' separates x_1 and x_2. Since, by Lemma 2.14, the subgraph of G induced by $C_1\cup C_2\cup K$ contains $k-1$ openly disjoint x_1x_2-paths, we have

$$K'\subseteq C_1\cup C_2\cup K. \tag{16}$$

Claim 3.8 *K and K' are meshing local separators.*

Proof: Arguing by contradicition we assume that K and K' do not mesh. Let C_2' be the component of $G-K'$ containing x_2. Since every x_1w-path in G contains a vertex of K we have $C'\cap K\ne\emptyset$. Also since G has $(k-1)$ x_1x_2-paths by Lemma 2.14, both C' and C_2' are essential K'-components. Since K and K' do not mesh, we have $C_2'\cap K=\emptyset$. Hence C_2' is a connected subgraph of $G-K$. Since $x_2\in V(C_2')$, this imples that $C_2'\subseteq C_2$ and $K'\cap C_2\ne\emptyset$ (since $K\ne K'$). Since K' does not mesh K, we have $C_1\cap K'=\emptyset$. Thus C_1 is a connected subgraph of $G-K'$. Since $x_1\in V(C')$, it follows that $C_1\subseteq C'$. Since $N(C_1)=K$ we have $K-C'\subseteq K'$. Let C_1' be a leaf component of K' distinct from C_2'. Since $x_1,w\in V(C')$, C' is not a leaf component of K' and hence $C_1'\ne C'$. The assumption that K and K' do not mesh and the fact that C' is an essential K' component intersecting K now gives $K\cap C_1'=\emptyset$. Thus C_1' is a connected subgraph of $G-K$.

Since C_1' and C_2' are leaf components of K', Lemma 2.14 implies that there are $(k-1)$ openly disjoint paths in $C_1'\cup C_2'\cup K'$ from each vertex of C_1' to x_2. Since $K\cap C'\ne\emptyset$, we have $|K\cap(C_1'\cup$

$C_2' \cup K')| \leq k-2$. Thus C_1' is contained in the same component of $G-K$ as x_2, and hence $C_1' \subseteq C_2$. But x_2 is the only s-neighbour in C_2. Thus $d(s, C_1') = 0$, a contradiction. •

Claim 3.9 $r = 2$.

Proof: Suppose $r \geq 3$. By Lemma 3.6, $x_1, x_2 \notin K'$. By Lemma 2.14, the subgraph of G induced by $C_1 \cup C_2 \cup K$ contains $k-1$ openly disjoint x_1x_2-paths. Since K and K' mesh by Claim 3.8, $K' \cap C_3 \neq \emptyset$, so $|K' \cap (C_1 \cup C_2 \cup K)| \leq k-2$. Hence at least one of the above $k-1$ openly disjoint x_1x_2-paths avoids K'. This contradicts (15). •

We can now complete the proof of the lemma. Let C_w be the component of $G-K$ containing w. Since sx_1, sw is an admissible split and C_1 is a leaf component of K, it follows that C_w is not a leaf component of K. Using (16), we deduce that C_w is a connected subgraph of $G-K'$ and hence $C_w \subseteq C'$. Since $d(s, C_w) \geq 2$ and $x_1 \in N(s) \cap (C' - C_w)$ we have $d(s, C') \geq 3$. Since $d(s) \leq 2b-2$, it follows that K' has at least three leaf components. This contradicts the maximality of r by Claim 3.9. Thus K' does not exist and we have $b(G') = b(G) - 1$. •

Lemma 3.10 *Let $G+s$ be a k-critical extension of a k-independence free graph G and p be an integer such that $0 \leq p \leq \frac{1}{2}d(s) - 1$. Then there exists a sequence of p admissible splits at s if and only if $p \leq d(s) - b(G)$.*

Proof: We first suppose that there exists a sequence of p admissible splits at s in G. Let the resulting graph be $G_1 + s$. Then $d_{G_1+s}(s) = d_G(s) - 2p$ and $b(G_1) \geq b(G) - p$. Since $G_1 + s$ is (k,s)-connected we must have $d_{G_1+s}(s) \geq b(G_1)$ and hence $p \leq d(s) - b(G)$.

We next suppose that $p \leq d(s) - b(G)$. We shall show by induction on p that $G+s$ has a sequence of p admissible splits at s. If $p = 0$ then there is nothing to prove. Hence we may assume $p \geq 1$. Since $p \leq \frac{1}{2}d(s) - 1$ we have $d(s) \geq 4$. By Theorem 3.4 there is an admissible split at s. Let the resulting graph be $G_2 + s$. If $p - 1 \leq d_{G_2+s}(s) - b(G_2)$ then we are done by induction. Hence we may assume that $p \geq d_{G_2+s}(s) - b(G_2) + 2 \geq d_G(s) - b(G)$. Hence $p = d_G(s) - b(G)$. Since $p \leq \frac{1}{2}d_G(s) - 1$, we have $d_G(s) \leq 2b(G) - 2$. By Lemma 3.7 there exists an admissible split at s such that the resulting graph $G_3 + s$ satisfies $b(G_3) = b(G) - 1$. It now follows by induction that $G_3 + s$ has a sequence of $p-1$ admissible splits at s. •

Lemma 3.11 *Let $G+s$ be a k-critical extension of a k-independence free graph G. If $d(s) \leq 2b(G) - 2$ then $a_k(G) = b(G) - 1$.*

Proof: Suppose $d(s) = b(G)$. Let K be a $(k-1)$-shredder in G with $b(K) = b(G)$. Then all components of $G-K$ are leaf components. Let F be the edge set of a tree T on the vertices of $N(s)$. We shall show that $G+F$ is k-connected. If not, then we can partition V into three sets $\{X, Y, Z\}$ such that $|Z| = k-1$ and no edge of $G+F$ joins X to Y. Each pair of vertices of $N(s)$ are joined by k openly disjoint paths in $G+F$, consisting of $(k-1)$ paths in G (which exist by Lemma 2.14) and one path in T. Thus either X or Y is disjoint from $N(s)$. Assuming $X \cap N(s) = \emptyset$, we have

$\overline{d}(X) = n(X) \leq k-1$, contradicting the fact that $G+s$ satisfies (7). Hence $G+F$ is a k-connected augmentation of G with $b(G)-1$ edges.

Henceforth we may assume that $d(s) > b(G)$. By Lemma 3.7, there exists an admissible split at s such that, for the resulting graph $G'+s$, we have $b(G') = b(G)-1$. Since $G'+s$ is a k-critical extension of G', the lemma follows by induction on $d_{G+s}(s) - b(G)$. •

Theorem 3.12 *If G is k-independence free then $a_k(G) = \max\{\lceil t(G)/2 \rceil, b(G)-1\}$.*

Proof: Let $G+s$ be a k-critical extension of G. By Corollary 3.2, $d(s) = t(G)$. If $d(s) \leq 3$ then $a_k(G) = \lceil t(G)/2 \rceil$ by Lemma 2.10. Hence we may suppose that $d(s) \geq 4$. If $d(s) \leq 2b(G)-2$ then $a_k(G) = b(G)-1$ by Lemma 3.11. Hence we may suppose that $d(s) \geq 2b(G)-1$.

By Lemma 3.10, there exists a sequence of $\lfloor d(s)/2 \rfloor - 1$ admissible splits at s. Let the resulting graph be $G'+s$. Then $G'+s$ is a k-critical extension of G', $d_{G'+s}(s) \leq 3$, and $a_k(G') = \lceil d_{G'+s}(s)/2 \rceil$ by Lemma 2.10. This gives the required augmenting set F for G with $|F| = \lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$. •

4 Augmenting Connectivity by One

Throughout this section we assume that $G = (V,E)$ is a $(k-1)$-connected graph on at least $k+1$ vertices. We shall show that if $a_k(G)$ is large compared to k, then $a_k(G) = \max\{b(G)-1, \lceil t(G)/2 \rceil\}$. Our proof uses Theorem 3.12 and some results from [15]. With the following result we can verify the desired min-max equality when $b(G)-1 \geq \lceil t(G)/2 \rceil$.

Theorem 4.1 *[15] Suppose G is a $(k-1)$-connected graph such that $b(G) \geq k$ and $b(G)-1 \geq \lceil t(G)/2 \rceil$. Then $a_k(G) = b(G)-1$.*

We will apply Theorem 4.1 to graphs which do not satisfy $b(G)-1 \geq \lceil t(G)/2 \rceil$ using the following concept. A set F of new edges is *saturating* for G if $t(G+F) = t(G) - 2|F|$. Thus an edge $e = uv$ is *saturating* if $t(G+e) = t(G)-2$.

Lemma 4.2 *If F is a saturating set of edges for a $(k-1)$-connected graph G with $b(G+F)-1 = \lceil t(G+F)/2 \rceil \geq k-1$ then $a_k(G) = \lceil t(G)/2 \rceil$.*

Proof: By Theorem 4.1 the graph $G+F$ can be made k-connected by adding a set F' of $\lceil t(G+F)/2 \rceil$ edges. Since F is saturating, we have $t(G) = t(G+F) + 2|F|$. Therefore the set $F \cup F'$ is an augmenting set for G of size $\lceil t(G)/2 \rceil$. Since $a_k(G) \geq \lceil t(G)/2 \rceil$, the lemma follows. •

We shall show that if $a_k(G)$ is large, then we can find a saturating set of edges F for G so that $G+F$ is k-independence free. In order to do this we need to measure how close G is to being k-independence free. We use the following concepts. Since G is $(k-1)$-connected, we have $n_G(X) = k-1$ for every k-deficient fragment of G. Following [15], we call the (inclusionwise) minimal k-deficient fragments in G the *k-cores* of G. A k-core B is *active* in G if there exists a $(k-1)$-cut K with $B \subseteq K$. Otherwise B is said to be *passive*. Let $\alpha(G)$ and $\pi(G)$ denote the numbers of active, respectively passive, k-cores of G. Since G is $(k-1)$-connected, the definition of k-independence implies that G is k-independence free if and only if $\alpha(G)=0$. The following characterisation of active k-cores also follows easily from the above definitions.

Lemma 4.3 *Let B be a k-core in G. Then B is active if and only if $\kappa(G-B)=k-|B|-1$.*

A set $S\subseteq V$ is a *k-deficient fragment cover* for G if $S\cap T\neq\emptyset$ for every k-deficient fragment T. Clearly, S is a k-deficient fragment cover for G if and only if S covers every k-core of G. Note that S is a minimal k-deficient fragment cover for G if and only if the extension $G+s$ obtained by joining s to each vertex of S is k-critical. We shall need the following results from [15].

Lemma 4.4 *(a) Every minimal augmenting set for G induces a forest.*
(b) For every k-deficient fragment cover S for G, there exists a minimal augmenting set F for G with $V(F)\subseteq S$.
(c) If F is a minimal augmenting set for G, $e=xy\in F$, and $H=G+F-e$, then H has precisely two k-cores X,Y. Furthermore $X\cap Y=\emptyset$; $x\in X, y\in Y$; for any edge $e'=x'y'$ with $x'\in X, y'\in Y$, the graph $H+e'$ is k-connected; and, for every k-deficient fragment Z in H, we have $X\subseteq Z$ or $Y\subseteq Z$.

Proof: Assertion (a) is given in [15, p 16].
To prove (b), note that since S covers all k-deficient fragments, G becomes k-connected when we add all edges between the vertices of S.
Assertion (c) follows from [15, Lemma 3.2]. •

Based on these facts we can prove the following lemma.

Lemma 4.5 *Let S be a minimal k-deficient fragment cover for G and let F be a minimal augmenting set with $V(F)\subseteq S$. Let $d_F(v)=1$ and let $e=uv$ be the leaf of F incident with v. Let X and Y be the k-cores of $G+F-e$ and suppose that for a set F' of edges we have $\kappa(x,y,G+F')\geq k$ for some vertices $x\in X, y\in Y$. Then $S-\{v\}$ is a k-deficient fragment cover of $G+F'$.*

Proof: Without loss of generality we may assume that $u\in X$ and $v\in Y$. By the minimality of S, there exists a k-core Z of G such that $Z\cap S=\{v\}$. Since Z is also k-deficient in $G+F-e$, it must contain a k-core of $G+F-e$, so $Y\subseteq Z$ by Lemma 4.4(c). Now, since Y is also k-deficient in G and Z is a k-core in G, we must have $Z=Y$ and $Y\cap S=\{v\}$. For a contradiction suppose that there is a k-deficient fragment P in $G+F'$ which is not covered by $S-\{v\}$. Then $P\cap S=\{v\}$ and so P is also k-deficient in $G+F'+F-e$ and in $G+F-e$. Thus, by Lemma 4.4(c), $Y\subseteq P$ and $y\in P$ hold. Furthermore, since $G+F'+F-e+xy$ is k-connected by Lemma 4.4(c), we must have $x\notin P\cup N(P)$ in $G+F'+F-e$. Thus $x\notin P\cup N(P)$ holds in $G+F'$ as well. This contradicts the fact that $\kappa(x,y,G+F')\geq k$. •

We need some further results from [15].

Lemma 4.6 *[15, Lemma 2.1, Claim I(a)] Suppose $t(G)\geq k$. Then the k-cores of G are pairwise disjoint and the number of k-cores of G is equal to $t(G)$. Furthermore, if $t(G)\geq k+1$, then for each k-core X, there is a unique maximal k-deficient fragment $S_X\subseteq V$ with the properties that $X\subseteq S_X$, and $S_X\cap Y=\emptyset$ for every k-core Y of G with $X\neq Y$. In addition, for two different k-cores X,Y we have $S_X\cap S_Y=\emptyset$.*

Lemma 4.7 *[15, Lemma 2.2] Let K and L be distinct $(k-1)$-cuts in G with $b(K) \geq k$. Then L intersects precisely one component D of $G-K$.*

Lemma 4.8 *Suppose $t(G) \geq k+1$. Let K be a $(k-1)$-shredder in G with $b(K) \geq k$. Then*
(a) if $C = S_X$ for some component C of $G-K$ and for some k-core X then X is passive,
(b) if some component D of $G-K$ contains precisely two k-cores X,Y and no edge of G joins S_X to S_Y then both X and Y are passive.

Proof: (a) Suppose that X is active and let L be a $(k-1)$-cut with $X \subseteq L$. Since $b(K) \geq k$, we have $L \subset K \cup C$ by Lemma 4.7. Since G is $(k-1)$-connected and $L \neq K$, $G-L-C$ is connected. Hence $G-L$ has a component C'' with $C'' \subset C$. Therefore C contains a (minimal) k-deficient set X' with $X \cap X' = \emptyset$, contradicting $C = S_X$.
(b) Suppose X is active and let L be a $(k-1)$-cut with $X \subseteq L$. As in the proof of (a), this implies that $G-L$ has a component C with $C \subseteq D-L$. Since D contains precisely two k-cores, $Y \subset C$ and hence, since S_Y is the unique maximal k-deficient fragment containing Y which is disjoint from every k-core, $C \subseteq S_Y$ must hold. On the other hand, since C is a component of $G-L$, we have $X \subseteq N(C)$ and so $X \cap N(S_Y) \neq \emptyset$. This contradicts our assumption that no edge of G joins S_X to S_Y. •

Recall that an edge $e = uv$ is *saturating* if $t(G+e) = t(G)-2$. We say that two k-cores X,Y form a *saturating pair* if there is a saturating edge $e = xy$ with $x \in X, y \in Y$ and otherwise that the pair X,Y is *non-saturating*.

If $t(G) \geq k+2$ and X,Y are a saturating pair, then every edge xy with $x \in X$ and $y \in Y$ is saturating. (To see this suppose that $e = xy$ is not saturating. Then $t(G+e) \geq t(G)-1 \geq k+1$ and hence the k-cores of $G+e$ are pairwise disjoint by Lemma 4.6. This implies that all k-cores of G other than X,Y are k-cores of $G+e$ and that there is a k-core S in $G+e$ which is disjoint from all k-cores of G other than X,Y. Since S is a k-core in $G+e$, S is k-deficient in G. We may assume that $S \cap X \neq \emptyset$. By applying (4) to S and X and using the minimality of X we can deduce that $X \subseteq S$. Since X,Y is a saturating pair, this implies $S \cap Y^* \neq \emptyset$ and $Y \cap S^* \neq \emptyset$. By applying (5) to S and Y we obtain that $Y \cap S^*$ is k-deficient in G. Since S is k-deficient in $G+e$, we must have $y \in S \cup N_G(S)$ and hence $Y \cap S^*$ is a proper subset of Y. This contradicts the minimality of Y.)

We shall need the following characterisation of saturating pairs.

Lemma 4.9 *[15, p.13-14] Let $t(G) \geq k+2$ and suppose that two k-cores X,Y do not form a saturating pair. Then one of the following holds: (a) $X \subseteq N(S_Y)$, (b) $Y \subseteq N(S_X)$, (c) there exists a k-deficient fragment M with $S_X, S_Y \subset M$, which is disjoint from every k-core other than X,Y.*

For a k-core X let $\nu(X)$ be the number of k-cores Y $(Y \neq X)$ for which the pair X,Y is non-saturating. The following lemma implies that an active k-core cannot belong to many non-saturating pairs.

Lemma 4.10 *Suppose $t(G) \geq k+2$ and let X be an active k-core in G. Then $\nu(X) \leq 2k-3$.*

Proof: Let $\mathcal{Y}$ be the set of cores Y $(Y \neq X)$ for which X,Y is a non-saturating pair, and let $\mathcal{Y}' = \{Y_1, Y_2, \ldots, Y_r\}$ be the set of those cores from $\mathcal{Y}$ for which Lemma 4.9(c) holds (with respect to X). For each Y_i, $1 \leq i \leq r$, let M_i be a k-deficient fragment which is disjoint from every k-core other than X, Y_i. Consider two sets M_i, M_j, $1 \le i < j \le r$. Since $t(G) \ge k+2$, $M_i \cap M_j$ is a k-deficient fragment, and hence $S_X = M_i \cap M_j$ must hold. This implies that each vertex of $V - S_X$ belongs to at most one set M_i.

For a contradiction suppose that $\nu(X) \ge 2k-2$. Let $K = N(S_X)$ and let $\mathcal{Y}'' = \{Y_i \in \mathcal{Y}' : M_i \cap K = \emptyset\}$. Since $|K| = k-1$ and $\nu(X) \ge 2k-2$, it follows from Lemmas 4.6 and 4.9, that $|\mathcal{Y}''| \ge k-1$.

Since X is active, Lemma 4.8(a) implies that $b(K) \le k-1$. Thus, since the vertex set of one of the components of $G-K$ is S_X, and $|\mathcal{Y}''| \ge k-1$, there is a component D of $G-K$ which contains at least two sets Y_i, Y_j from $\mathcal{Y}''$. Consider M_i. Since $S_X \subset M_i$ and $K \cap M_i = \emptyset$, we have $K \subset N(M_i)$. Since $Y_j \subset D$, we have $D - M_i \neq \emptyset$, and hence $D \cap N(M_i) \neq \emptyset$. Hence $n(M_i) \ge |K|+1 = k$, contradicting the fact that M_i is a k-deficient fragment. •

For every passive k-core B_i ($1 \le i \le \pi(G)$) let $\mathcal{F}_i = \{X \subset V : X$ is k-deficient in G, $B_i \subseteq X$, the subgraph $G[X]$ is connected, and X contains at most $4k-8$ active k-cores$\}$. Let $M_i = \cup_{X \in \mathcal{F}_i} X$ and let $T(G) = \cup_{i=1}^{\pi(G)} (M_i \cup N(M_i))$.

Lemma 4.11 *Let B_i be a passive k-core for some $1 \le i \le \pi(G)$ and let $\mathcal{X} = \{X_1, ..., X_t\}$ be a minimal family of members of $\mathcal{F}_i$ for which $\cup_{j=1}^t X_j = M_i$. Then $t \le k$ and $n(M_i) \le k(k-1)$. Moreover, if $\alpha(G) \ge 5k-8$, then M_i intersects at most $k(4k-8)$ active k-cores.*

Proof: First we prove that $t \le k$. For a contradiction suppose that $t \ge k+1$. By the minimality of the family $\mathcal{X}$ we have that $\hat{X}_j = X_j - \cup_{r \neq j} X_r$ is non-empty for all $1 \le j \le t$. Note that the sets $\hat{X}_j$ are pairwise disjoint. By applying (4) to a pair $X_r, X_j \in \mathcal{X}$, and using the facts that $X_r \cap X_j \neq \emptyset$ since $B_i \subseteq X_r \cap X_j$, that $t \ge k+1$, and that G is $(k-1)$-connected, we deduce that $X_r \cap X_j$ is k-deficient in G. Since $B_i \subseteq X_r$ for each $X_r \in \mathcal{X}$, a similar argument shows that $P = \cup_{j \neq r}(X_r \cap X_j)$ is also k-deficient. Note that $M_i - P = \cup_{j=1}^t \hat{X}_j$, so $|M_i - P| \ge t \ge k+1$. Since $X_r = \hat{X}_r \cup (P \cap X_r)$ and $G[X_r]$ is connected, there exists a neighbour of P in $\hat{X}_r$. Since the sets $\hat{X}_r$ are pairwise disjoint, these neighbours are distinct. Hence $n(P) \ge t \ge k+1$, contradicting the fact that P is k-deficient. Thus $t \le k$. Since each neighbour of M_i is a neighbour of some set in $\mathcal{X}$, and $\mathcal{X}$ consists of k-deficient fragments, we have $n(M_i) \le k(k-1)$.

To see the second part of the statement suppose that for some $X_r \in \mathcal{X}$ and for some active k-core A we have $X_r \cap A \neq \emptyset$ and $X_r - A \neq \emptyset \neq A - X_r$. Since $\alpha(G) \ge 5k-8$, X_r contains at most $4k-8$ active k-cores, and the (active) k-cores are pairwise disjoint, we have $|V - (X_r \cup A)| \ge k-1$. Now (4) implies that $X_r \cap A$ is k-deficient, a contradiction. Thus every active k-core A for which $A \cap M_i \neq \emptyset$ satisfies $A \subset X_r$ for some $X_r \in \mathcal{X}$. Hence the definition of $\mathcal{F}_i$ implies that M_i intersects at most $k(4k-8)$ active k-cores. •

We shall use the following lemmas to find a saturating set F for G such that $G+F$ has many passive cores. Informally, the idea is to pick a properly chosen active k-core B and, by adding a set F of at most $2k-2$ saturating edges between the active k-cores of G other than B, make $\kappa(G+F-B) \ge k - |B| = r$. By Lemma 4.3, this will make B passive, and will not eliminate any of the passive k-cores of G. We shall increase the connectivity of $G-B$ by choosing a minimal r-deficient fragment cover S for $G-B$ of size at most $k-1$ and then iteratively add one or two edges so that the new graph has an r-deficient fragment cover properly contained in S. Thus after at most $k-1$ such steps (and adding at most $2k-2$ edges) we shall make B passive. The first lemma tells us how to choose the active k-core B.

Lemma 4.12 *Suppose $\pi(G) \leq 4(k-1)$ and $\alpha(G) \geq 20k(k-1)^2$. Then there exists an active k-core B with $B \cap T(G) = \emptyset$.*

Proof: Since $\alpha(G) \geq 20k(k-1)^2 \geq 5k-8$, Lemma 4.11 implies that for any passive k-core B_i, the set M_i intersects at most $k(4k-8)$ active k-cores, and $N(M_i)$ intersects at most $k(k-1)$ active k-cores. Thus $T(G)$ intersects at most $\pi(G)(k(5k-9)) < 4(k-1)k(5k-5) = 20k(k-1)^2$ active k-cores. Since $\alpha(G) \geq 20k(k-1)^2$, the lemma follows. •

Lemma 4.13 *Suppose $\pi(G) \leq 4(k-1)$ and $\alpha(G) \geq 8k^3+6k^2-23k-16$. Let B be an active k-core in G, $H = G - B$, $r = k - |B|$, and S be a minimal r-deficient fragment cover of H. Suppose every r-deficient fragment Z of H contains an active k-core of G. Then there exists a saturating set of edges F for G such that $|F| \leq 2$, F is not incident with B, and either $\pi(G+F) > \pi(G)$; or $\pi(G+F) = \pi(G)$, B is an active k-core in $G+F$, and $H+F$ has an r-deficient fragment cover S' which is properly contained in S.*

Proof: Since B is active, $\kappa(H) = k-1-|B| = r-1$.
By Lemma 4.4 there exists a minimal r-augmenting set F^* for H such that F^* is a forest and $V(F^*) \subseteq S$. Let $d_{F^*}(v) = 1$ and let $e = uv$ be a leaf of F^*. By Lemma 4.4(c), there exist precisely two r-cores Z,W in $H+F^*-e$ with $u \in Z, v \in W$. Then Z,W are r-deficient in H. By an hypothesis of the lemma, there exist active k-cores X,Y of G with $X \subseteq Z$ and $Y \subseteq W$.

Suppose X and Y form a saturating pair in G. We may choose a saturating edge xy for G with $x \in X$ and $y \in Y$. Then $xy \notin E$ and, since $\kappa(G) = k-1$, we have $\kappa(x,y,G+xy) \geq k$ and $\kappa(x,y,H+xy) \geq r$. Hence either $\pi(G+xy) > \pi(G)$; or every active k-core of G other than X,Y remains active in $G+xy$. If the second alternative holds then B remains active in $G+xy$ and, by Lemma 4.5, $S' = S - v$ is an r-deficient fragment cover in $H+xy$.

Hence we may assume that X,Y is not a saturating pair in G. By Lemma 4.9 either

(i) there exists a k-deficient fragment M in G with $S_X \cup S_Y \subseteq M$ which is disjoint from every k-core other than X,Y, or

(ii) $Y \subseteq N_G(S_X)$ or $X \subseteq N_G(S_Y)$.

Choose $x \in X$ and $y \in Y$ arbitrarily and let $P_1,P_2,\ldots,P_{k-1}$ be $k-1$ openly disjoint xy-paths in G. Let $Q = \cup_{i=1}^{k-1} V(P_i)$. It is easy to see that if some edge of G joins S_X to S_Y, then one of the paths, say P_1, satisfies $V(P_1) \subseteq S_X \cup S_Y$. On the other hand, if no edge of G joins S_X to S_Y, then (ii) cannot hold. Hence (i) holds and, either one of the paths, say P_1, satisfies $V(P_1) \subseteq M$, or each of the $k-1$ paths intersects $N_G(M)$. In the latter case, since $n_G(M) = k-1$, we have $|N_G(M) \cap P_i| = 1$, $V(P_i) \subseteq M \cup N_G(M)$ for all $1 \leq i \leq k-1$, and hence $N_G(M) \subset Q$ and $Q \subset M \cup N_G(M)$. We shall handle these two cases separately.

Case 1. No edge of G joins S_X to S_Y, (i) holds, and we have $N_G(M) \subset Q \subset M \cup N_G(M)$.

Let $C_1,C_2,\ldots,C_p$ be the components of $G - N_G(M)$. Using the properties of M (M intersects exactly two k-cores, M is the union of one or more components of $G - N_G(M)$, and $N_G(M) = k-1$)

we can see that either, one component C_i contains S_X and S_Y and is disjoint from every k-core of G other than X,Y and $M = V(C_i)$, or each of S_X and S_Y corresponds to a component of $G - N_G(M)$ and $M = S_X \cup S_Y$.

Since X and Y are active k-cores, Lemma 4.8, with $K = N_G(M)$, implies that $p \le k-1$. Since $\alpha(G) \ge (k-2)(2k+2)+k+3$, G has at least $(k-2)(2k+2)+1$ active k-cores disjoint from B, X, Y, and $N_G(M)$. Thus some component C_j of $G - N_G(M)$ is disjoint from M and contains at least $2k+3$ active k-cores distinct from B. By Lemma 4.10, there exists a saturating edge xa_1 with $a_1 \in A_1$ for some active k-core $A_1 \subset C_j$, $A_1 \neq B$. If $\pi(G+xa_1) \ge \pi(G)+1$ then we are done. Otherwise all the active k-cores in G other than X,A_1 remain active in $G+xa_1$. Applying Lemma 4.10 again, we may pick a saturating edge ya_2 with $a_2 \in A_2$ for some active k-core A_2 of $G+xa_1$, with $A_2 \subset C_j$, $A_2 \neq B$.

We have $\kappa(x,y,G+xa_1+ya_2) \ge k$, since there is a path from x to y, using the edges xa_1, ya_2, and vertices of C_j only, and thus this path is openly disjoint from Q (since $Q \subseteq M \cup N_G(M)$). Hence $\kappa(x,y,H+xa_1+ya_2) \ge r$. Thus by Lemma 4.5, $S' = S - v$ is an r-deficient set cover in $H+xa_1+ya_2$.

Case 2. Either $V(P_1) \subseteq S_X \cup S_Y$ or (i) holds and $V(P_1) \subseteq M$.

Let us call a component D of $G-Q$ *essential* if D intersects an active k-core other than X, Y or B. Let $D_1, D_2, \ldots, D_p$ be the essential components of $G-Q$. We say that a component D_i is *attached to* the path P_j if $N_G(D_i) \cap V(P_j) \neq \emptyset$ holds. Let $R = S_X \cup S_Y$ if $V(P_1) \subseteq S_X \cup S_Y$ holds and let $R = M$ if $V(P_1) \subseteq M$. Then, R is disjoint from every active k-core other than X,Y.

Claim 4.14 *At most $2k-2$ essential components are attached to P_1.*

Proof: Focus on an essential component D which is attached to P_1 and let $w \in W \cap D$ for some active k-core $W \neq X,Y,B$ which has a vertex in D. There exists a path P_D from w to a vertex of P_1 whose inner vertices are in D. Since $w \notin R$ and $V(P_1) \subseteq R$, we have $D \cap N_G(R) \neq \emptyset$. The claim follows since the essential components are pairwise disjoint and $n(R) \le 2k-2$. •

Suppose that one of the paths P_i intersects at least $4k+4$ active k-cores in G other than X, Y or B. For every such active k-core A intersecting P_i choose a representative vertex $a \in A \cap P_i$. Since the k-cores are pairwise disjoint, the representatives are pairwise distinct. Order the active k-cores intersecting P_i following the ordering of their representatives along the path P_i from x to y. By Lemma 4.10, we may choose a saturating edge xa_1 in G, where a_1 is among the $2k+2$ rightmost representatives and a_1 belongs to an active k-core A_1. If $\pi(G+xa_1) \ge \pi(G)+1$ then we are done. Otherwise all the active k-cores of G other than X,A_1 remain active in $G+xa_1$. Again using Lemma 4.10, we may choose a saturating edge ya_2 in $G+xa_1$, where a_2 is among the $2k+2$ leftmost representatives. By the choice of a_1 and a_2 there exist two openly disjoint paths from x to y in $G+xa_1+ya_2$ using vertices of $V(P_i)$ only. Thus $\kappa(x,y,G+xa_1+ya_2) \ge k$. Hence, by Lemma 4.5, $S' = S - v$ is an r-deficient set cover in $H+xa_1+ya_2$.

Thus we may assume that each path P_i intersects at most $4k+3$ active k-cores in G other than X, Y or B. Hence there are at least

$$\alpha(G) - 3 - (k-1)(4k+3) \ge (8k^3+6k^2-23k-19) - (k-1)(4k+3) = (2k+2)(4k^2-3k-8)$$

active k-cores other than B contained in $G-Q$. Note that since k-cores are minimal k-deficient fragments, they induce connected subgraphs in G. Hence each k-core contained in $G-Q$ is contained in a component of $G-Q$. If some component of $G-Q$ contains at least $2k+3$ active k-cores of G other than B then the lemma follows as in Case 1. Hence we may assume that there are at least $4k^2-3k-8$ essential components in $G-Q$, each containing an active k-core distinct from X, Y, and B.

Using Claim 4.14 we deduce that there are at least $4k^2-3k-8-(2k-2)=(4k+3)(k-2)+1$ essential components D_i with all their attachments on $P_2,P_3,\ldots,P_{k-1}$, each containing an active core other than X,Y,B. Since G is $(k-1)$-connected, $n(D_i)\geq k-1$ and hence D_i has at least two attachments on at least one of the paths $P_2,P_3,\ldots,P_{k-1}$. Relabelling the components $D_1,\ldots,D_p$ and the paths $P_2,\ldots,P_{k-1}$ if necessary, we may assume that D_i has at least two attachments on P_{k-1} for $1\leq i\leq 4k+4$.

Let z_i be the leftmost attachment of D_i on P_{k-1}. Without loss of generality we may assume that $z_1,z_2,\ldots,z_{4k+4}$ occur in this order on P_{k-1} as we pass from x to y. By Lemma 4.10, there exists a saturating edge ya_i where $a_i\in A_i$ for some active k-core $A_i\subseteq D_i$, where $A_i\neq B$ and $1\leq i\leq 2k+2$. If $\pi(G+ya_i)\geq\pi(G)+1$ then we are done. Otherwise every active k-core in G other than Y,A_i remains active in $G+ya_i$. Using Lemma 4.10 again, there exists a saturating edge xa_j where $a_j\in A_j$ for some active k-core $A_j\subseteq D_j$, where $A_j\neq B$ and $2k+3\leq j\leq 4k+4$. Note that z_i is either to the left of z_j or $z_i=z_j$. Hence, using the fact that D_j has at least two attachments on P_{k-1} and by the choice of z_i,z_j, there exist two openly disjoint paths in $G+xa_j+ya_i$, using vertices from $V(P_{k-1})\cup D_i\cup D_j$ only. Therefore $\kappa(x,y,G+xa_j+ya_i)\geq k$, and we are done as above. This completes the proof of the lemma. •

Lemma 4.15 *Suppose $\pi(G)\leq 4(k-1)$ and $\alpha(G)\geq 20k(k-1)^2$. Then there exists a saturating set of edges F for G such that $|F|\leq 2k-2$ and $\pi(G+F)\geq\pi(G)+1$.*

Proof: Let B be an active k-core in G with $B\cap T(G)=\emptyset$. Such a set exists by Lemma 4.12. Let $H=G-B$, and $r=k-|B|$. Since B is active, $\kappa(H)=r-1$. Every r-deficient fragment X in H is k-deficient in G and $N_G(B)\cap X\neq\emptyset$. Hence $N_G(B)$ is an r-deficient fragment cover of H. Let $S\subseteq N_G(B)$ be a minimal r-deficient fragment cover of H. Since B is k-deficient in G, we have $|S|\leq n_G(B)=k-1$.

We shall prove by induction on i that, for $0\leq i\leq k-1$, there exists a saturating set of edges F_i for G such that $|F_i|\leq 2i$, F_i is not incident with B, and either $\pi(G+F_i)\geq\pi(G)+1$; or $\pi(G+F_i)=\pi(G)$, B is an active k-core of $G+F_i$, and $H+F_i$ has an r-deficient fragment cover $S_i\subseteq S$ with $|S_i|\leq|S|-i$. The lemma will follow since the second alternative cannot hold with $|S_i|=0$ (since this would imply that $H+F_i$ is r-connected and hence that B is passive in $G+F_i$).

The statement is trivially true for $i=0$ taking $F_i=\emptyset$. Hence suppose that there exists a set F_i satisfying the above statement for some $0\leq i\leq k-2$. If $\pi(G+F_i)\geq\pi(G)+1$ then we can put $F_{i+1}=F_i$. Hence we may suppose that $\pi(G+F_i)=\pi(G)$, B is an active k-core of $G+F_i$, and $H+F_i$ has an r-deficient fragment cover $S_i\subseteq S$ with $|S_i|\leq|S|-i$. We would like to apply Lemma 4.13 to B and $G+F_i$. To do this we must show that $G+F_i$, B and S_i satisfy the hypotheses of this lemma. We have $\pi(G+F_i)=\pi(G)\leq 4(k-1)$. Thus $\alpha(G+F_i)=\alpha(G)-2|F_i|\geq 8k^3+6k^2-23k-16$.

The last property we need to verify is that every r-deficient fragment Z in $G+F_i-B$ contains at least one active k-core of $G+F_i$. Since F_i is a saturating set for G, and since the k-cores of G are pairwise disjoint, each k-core of $G+F_i$ is a k-core of G. Furthermore, since $\pi(G+F_i)=\pi(G)$, if A is an active k-core of G and A is a k-core of $G+F_i$ then A is an active k-core of $G+F_i$. Since Z is r-deficient in $G+F_i-B$, it is k-deficient in $G+F_i$. Thus Z contains at least one core in $G+F_i$. If Z contains an active k-core in $G+F_i$, then we are done, so suppose that every k-core of $G+F_i$ in Z is passive. Let B_j be such a k-core. Then B_j is a passive k-core in G so $G[B_j]$ is connected. Let C be the component of $G[Z]$ containing B_j and let $Z'=V(C)$. Since Z is k-deficient in G, Z' is k-deficient in G, and $B\subseteq N_G(Z')$. Since $B\cap T(G)=\emptyset$ and $B\subseteq N_G(Z')$, it follows that $Z'\notin\mathcal{F}_j$ and hence Z' contains at least $4k-7$ active k-cores in G. Since $|F_i|\leq 2(k-2)=2k-4$ and each edge of F_i is incident to at most two k-cores of G, it follows that there exists an active k-core A in G with $A\subset Z'$ which is still an (active) k-core in $G+F_i$, contradicting the assumption that every k-core of $G+F_i$ in Z is passive. Hence $G+F_i$, B and S_i satisfy the hypotheses of Lemma 4.13. Thus there exists a saturating set of edges F for $G+F_i$ such that $|F|\leq 2$, F is not incident with B, and either $\pi(G+F_i+F)>\pi(G+F_i)=\pi(G)$; or $\pi(G+F_i+F)=\pi(G+F_i)=\pi(G)$ and $G+F_i+F-B$ has an r-deficient fragment cover S_{i+1} which is properly contained in S_i. Hence the inductive statement holds with $F_{i+1}=F_i\cup F$. •

Lemma 4.16 *Suppose* $t(G)\geq 20k(k-1)^2+(4k-3)(4k-4)$. *Then there exists a saturating set of edges* F *for* G *such that* $G+F$ *is* k*-independence free and* $t(G+F)\geq 2k-1$.

Proof: Since every graph is 1-independence free and every connected graph is 2-independence free, we may suppose that $k\geq 3$. If $\pi(G)\leq 4(k-1)$ then we may apply Lemma 4.15 recursively $4k-3-\pi(G)$ times to G to find a saturating set of edges F_1 for G such that $\pi(G+F_1)\geq 4k-3$. If $\pi(G)\geq 4k-3$ we set $F_1=\emptyset$. Applying Lemma 4.10 to $G+F_1$, we can add saturating edges joining pairs of active k-cores until the number of active k-cores is at most $2k-2$. Thus there exists a saturating set of edges F_2 for $G+F_1$ such that $\alpha(G+F_1+F_2)\leq 2k-2$ and $\pi(G+F_1+F_2)\geq 4k-3$. Applying Lemma 4.10 to $G+F_1+F_2$, we can add saturating edges joining pairs consisting of one active and one passive k-core until the number of active k-cores decreases to zero. Thus there exists a saturating set of edges F_3 for $G+F_1+F_2$ such that $\alpha(G+F_1+F_2+F_3)=0$ and $\pi(G+F_1+F_2+F_3)\geq 2k-1$. •

The main theorem of this section is the following.

Theorem 4.17 *If* $a_k(G)\geq 20k^3$ *then*

$$a_k(G)=\max\{\lceil t(G)/2\rceil, b(G)-1\}.$$

Proof: Since every graph is 1-independence free and every connected graph is 2-independence free, the result follows from Theorem 3.12 if $k\leq 2$. Hence we may suppose that $k\geq 3$. Let $G+s$ be a k-critical extension of G. By Lemma 2.10 we have $d(s)\geq a_k(G)+1\geq 20k^3>k+1$. Hence, by [15, Lemmas 3.4, 3.5] we have $t(G)=d(s)\geq 20k^3$. (This equality will also follow from Lemma 5.2 in Subsection 5.) If $b(G)-1\geq\lceil t(G)/2\rceil$ then $a_k(G)=b(G)-1$ by Theorem 4.1 and we are done. Thus we may assume that $\lceil t(G)/2\rceil>b(G)-1$ holds. We shall show that $a_k(G)=\lceil t(G)/2\rceil$. By Lemma 4.16, there exists a saturating set of edges F for G such that $G+F$ is k-independence free and $t(G+F)\geq 2k-1$. Note that adding a saturating edge to a graph H reduces $\lceil t(H)/2\rceil$ by exactly one and $b(H)$ by at most one. Thus, if $\lceil t(G+F)/2\rceil\leq b(G+F)-1$, then there exists

$F' \subseteq F$ such that $\lceil t(G+F')/2 \rceil = b(G+F') - 1$ and the theorem follows by applying Lemma 4.2. Hence we may assume that $\lceil t(G+F)/2 \rceil > b(G+F) - 1$. Since $G+F$ is k-independence free, we can apply Theorem 3.12 to deduce that $a_k(G+F) = \lceil t(G+F)/2 \rceil$. Using (1) and the fact that $t(G) = t(G+F) + 2|F|$ we have $a_k(G) = \lceil t(G)/2 \rceil$, as required. $\bullet$

Theorem 4.17 gives an affirmative answer to a conjecture of the second author, [16, p 300].

5 Unsplittable Extensions

In this section we consider a k-critical extension $G+s$ of an l-connected graph G on at least $k+1$ vertices in which $d(s)$ is large. We show that $d(s) = t(G)$ and characterise when there is no admissible split containing a given edge at s.

Lemma 5.1 *Let $X,Y \subset V$ be two sets with $X \cap Y \neq \emptyset$. Suppose $d(s) \geq (k-l)(k-1)+4$.*
(a) If X and Y are tight then $X \cup Y$ is tight and $\bar{d}(X \cap Y) = k$.
(b) If X is tight and Y is dangerous then $X \cup Y$ is dangerous.
(c) If $d(s) \geq (k-l+1)(k-1)+4$ and X and Y are dangerous then $X^* \cap Y^* \neq \emptyset$.

Proof: We prove (a). Let X,Y be tight sets with $X \cap Y \neq \emptyset$. By (9) we have

$$2k = \bar{d}(X) + \bar{d}(Y) \geq \bar{d}(X \cap Y) + \bar{d}(X \cup Y). \tag{17}$$

Clearly, $X \cap Y$ is a fragment and hence $\bar{d}(X \cap Y) \geq k$ by (7). Using (17) we have $\bar{d}(X \cup Y) \leq k$. Thus if $X^* \cap Y^* \neq \emptyset$ then $X \cup Y$ is also a fragment and hence is tight and $\bar{d}(X \cap Y) = k$.

Suppose $X^* \cap Y^* = \emptyset$. Since $\bar{d}(X \cup Y) \leq k$, we have $n(X \cup Y) \leq k - d(s, X \cup Y)$. Since G is l-connected and $G+s$ is k-critical, $d(s,v) \leq k-l$ for all $v \in V$. Thus

$$\begin{aligned} d(s) &\leq d(s, X \cup Y) + d(s, N(X \cup Y)) \leq d(s, X \cup Y) + (k-l)n(X \cup Y) \leq \\ &\leq d(s, X \cup Y) + (k-l)(k - d(s, X \cup Y)) = (k-l)k - (k-l-1)d(s, X \cup Y). \end{aligned}$$

Since $k-l-1 \geq 0$ and $d(s, X \cup Y) \geq 1$, this gives $d(s) \leq (k-l)(k-1)+1$, contradicting the hypothesis on $d(s)$.

The proofs of (b) and (c) are similar, using the fact that $d(s, X \cup Y) \geq 2$ in (b) and (c). $\bullet$

The following lemma shows that $d(s) = t(G)$ when $d(s)$ is large.

Lemma 5.2 *If $d(s) \geq (k-l)(k-1)+4$ then $d(s) = t(G)$.*

Proof: Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $|\mathcal{F}|$ is as small as possible. Since every edge incident to s is critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X,Y \in \mathcal{F}$ and suppose that $X \cap Y \neq \emptyset$. By Lemma 5.1(a), $X \cup Y$ is also tight. So replacing X and Y in $\mathcal{F}$ by $X \cup Y$ we contradict the minimality of $|\mathcal{F}|$.

Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s) = \sum_{X \in \mathcal{F}} (k - n(X)) \leq t(G)$. The inequality $d(s) \geq t(G)$ follows easily from (7). Thus $d(s) = t(G)$, as required. $\bullet$

Lemma 5.3 *Let sx_0 be a designated edge of a k-critical extension $G+s$ of G and suppose that there are $q \geq (k-l+1)(k-1)+4$ edges sy ($y \neq x_0$) incident to s for which the pair sx_0, sy is not admissible. Then there exists a $(k-1)$-shredder K in G such that K has $q+1$ leaf components $C_0, C_1, \ldots, C_q$ in $G+s$, where $X_0 = V(C_0)$ is the maximal tight set containing x_0 and $K = N_G(X_0)$.*

Proof: Let X_0 be the maximal tight set in $G+s$ containing x_0. Note that the set X_0 is uniquely determined by Lemma 5.1(a). Let $\mathcal{T} = \{X_1, ..., X_m\}$ be the set of all maximal tight sets which intersect $N(X_0)$. Note that $X_i \cap X_j = \emptyset$ for $0 \leq i < j \leq m$ by Lemma 5.1(a). Thus we have $d(s, \cup_{i=0}^m X_i) = d(s, X_0) + d(s, \cup_{i=1}^m X_i)$.

Since each $X_i \in \mathcal{T}$ contains a neighbour of X_0 and X_0 is tight, we have $m \leq n(X_0) = k - d(s, X_0)$. Since each $X_i \in \mathcal{T}$ is tight and G is l-connected, we have $d(s, X_i) \leq k - l$. So

$$d(s, \cup_{i=0}^m X_i) \leq d(s, X_0) + (k-l)(k - d(s, X_0)) = k(k-l) - d(s, X_0)(k-l-1). \tag{18}$$

Let $M = \{y \in N(s) - x_0 : sx_0, sy \text{ is not admissible}\}$. Since there exist $q \geq (k-l+1)(k-1)+4$ edges incident to s which are not admissible with sx_0, we can use (18) to deduce that $R := M - \cup_{i=0}^m X_i \neq \emptyset$. By Lemma 2.11 and by the choice of $\mathcal{T}$ there exists a family of maximal dangerous sets $\mathcal{W} = \{W_1, ..., W_r\}$ such that $x_0 \in W_i$ for all $1 \leq i \leq r$ and $R \subseteq \cup_{j=1}^r W_i$. Let us assume that $\mathcal{W}$ is chosen so that r is as small as possible. By Lemma 5.1(b), $X_0 \subseteq W_i$ for all $1 \leq i \leq r$. Since $d(s, W_i - X_0) \leq k+1-l-d(s, X_0)$, we can use (18) and the fact that $q \geq (k-l+1)(k-1)+4$ to deduce that $r \geq 2$. For $W_i, W_j \in \mathcal{W}$ we have $W_i^* \cap W_j^* \neq \emptyset$ by Lemma 5.1(c). Since $W_i \cup W_j$ is not dangerous by the maximality of W_i, we may apply (9) to obtain

$$k+1+k+1 \geq \bar{d}(W_i) + \bar{d}(W_j) \geq \bar{d}(W_i \cap W_j) + \bar{d}(W_i \cup W_j) \geq k+k+2. \tag{19}$$

Thus equality holds throughout and $W_i \cap W_j$ is tight. Since X_0 is a maximal tight set and $X_0 \subseteq W_i \cap W_j$ we have $X_0 = W_i \cap W_j$. Furthermore, since we have equality in (19), we can use (8) to deduce that $W_j \cap N(W_i) \subseteq N(W_i \cap W_j)$. So $W_j \cap N(W_i) \subseteq N(X_0)$ and, similarly, $W_i \cap N(W_j) \subseteq N(X_0)$. Hence $N(s) \cap W_i \cap N(W_j) \subseteq \cup_{i=0}^m X_i$. (Note that every $z \in N(s) \cap N(X_0)$ is contained in one of the X_i's by the criticality of $G+s$.) So by the choice of $\mathcal{W}$, $R \cap W_i \cap W_j^* \neq \emptyset$ and $R \cap W_j \cap W_i^* \neq \emptyset$ follows.

By (10),

$$2k+2 = \bar{d}(W_i) + \bar{d}(W_j) \geq \bar{d}(W_i \cap W_j^*) + \bar{d}(W_i^* \cap W_j) + d(s, W_i - W_j^*) + d(s, W_j - W_i^*) \geq 2k+2,$$

and so we have equality throughout. Thus all edges from s to W_i, other than the single edge sx_0, end in $W_i \cap W_j^*$ and $d(s, X_0) = 1$. Hence $R \cap W_j \cap W_i^* = (R \cap W_j) - x_0$. Since $d(s, W_i \cup W_j) = \bar{d}(W_i \cup W_j) - n_G(W_i \cup W_j) \leq k+2-l$, we have $d(s, (W_i \cup W_j) - X_0) \leq k+2-l-d(s, X_0)$. We can now use (18) and the fact that $q \geq (k-l+1)(k-1)+4$ to deduce that $r \geq 3$. Thus $\emptyset \neq (R \cap W_j) - x_0 \subseteq W_j \cap W_i^* \cap W_h^*$ holds for all distinct $i, j, h \in \{1, ..., r\}$. Applying Lemma 2.15 we deduce that $K = N_G(X_0)$ is a $(k-1)$-shredder with $r+1$ leaf components $C_0, C_1, \ldots, C_r$ in $G+s$, where $V(C_0) = X_0$ and $V(C_i) = W_i - X_0$ for $1 \leq i \leq r$.

We complete the proof of the lemma by showing that $M = R$ and hence that $r = q$. Suppose that $M \neq R$. Then $\mathcal{T} \neq \emptyset$ and we may choose $X_1 \in \mathcal{T}$. Since $X_1 \cap N(X_0) \neq \emptyset$, we have $X_1 \cap K \neq \emptyset$. Since $X_1 \cap R = \emptyset$, $N(X_1) \cap C_i \neq \emptyset$ for $0 \leq i \leq r$. Using $r = |R| \geq q - d(s, \cup_{i=0}^m X_i)$, and the facts that $d(s, X_0) = 1$, and $q \geq (k-l+1)(k-1)+4$, we may use (18) to deduce that $r \geq k+2$. This contradicts the fact that X_1 is tight since $\bar{d}(X_1) \geq n_G(X_1) \geq r+1$. •

6 Graphs Containing Shredders with many Components

We show in this section that if $\hat{b}(G)$ and $t(G)$ are large compared to k and $\hat{b}(G)-1 \geq \lceil t(G)/2 \rceil$ then $a_k(G) = \hat{b}(G)-1$. We need several new observations on $(k-1)$-shredders. We assume throughout this section that $G+s$ is a k-critical extension of an l-connected graph G, and that K is a $(k-1)$-shredder of G satisfying $d(s) \leq 2\hat{b}(K)-2$.

Lemma 6.1 *Suppose $\hat{b}(K) \geq 4k+3(k-l)-1$. Then*

(a) the number of components C of $G-K$ with $d(s,C) \geq 3$ is at most $b(K)-2k-1$,

(b) $|N(s) \cap K| \leq 1$, and

(c) if $d(s,x) = j \geq 1$ for some $x \in K$ then $k - d_G(x) = j$.

Proof: Let w be the number of components C of $G-K$ with $d(s,C) \geq 3$. Then $d(s) \geq 3w + (b(K)-w)$. Thus

$$2w \leq d(s) - b(K) \leq 2\hat{b}(K) - 2 - b(K) = 2b(K) + 2\delta(K) - 2 - b(K) = 2b(K) + 3\delta(K) - 2 - \hat{b}(K).$$

Since $\delta(K) \leq k-l$ and $\hat{b}(K) \geq 4k+3(k-l)-1$, we have $w \leq b(K)-2k-1$. This proves (a).

Since $G+s$ is a critical extension of G, each vertex in $N(s)$ is contained in a tight set of $G+s$. Thus (b) will follow from the next claim.

Claim 6.2 *At most one vertex of K belongs to a tight set in $G+s$.*

Proof: Suppose that there exist two distinct vertices $x_1, x_2 \in K$ and tight sets Y_1, Y_2 in $G+s$ such that $x_1 \in Y_1$, $x_2 \in Y_2$. Let $Y = Y_1 \cup Y_2$ and let $\mathcal{D} = \{C : C \text{ is a component of } G-K, C \cap (Y \cup N(Y)) \neq \emptyset\}$. We have $|\mathcal{D}| \leq 2k$, since $\bar{d}(Y) \leq \bar{d}(Y_1) + \bar{d}(Y_2) \leq 2k$ and for every $C \in \mathcal{D}$ either $C - Y \neq \emptyset$, in which case $N(Y) \cap C \neq \emptyset$ holds, or $C \subset Y$, in which case $d(s, C \cap Y) \geq 1$ holds by (7).

Since $\hat{b}(K) \geq 4k+3(k-l)-1$ we have $b(K) \geq 4k+2(k-l)-1$. Thus we may choose a component C' of $G-K$ such that $C' \notin \mathcal{D}$. Then $C' \cap N(Y) = \emptyset$ and hence $x_1, x_2 \notin N(C')$. Hence $n(C') \leq k-3$ and $d(s,C') \geq 3$. Since we have at least $b(K)-2k$ choices for C', this contradicts (a). •

To prove (c), we choose a tight set X containing x. By Claim 6.2, $X \cap K = \{x\}$. If $X = \{x\}$ then, since X is tight, we have $d(s,x) = k - d_G(x)$, as required. Thus we may suppose that $X - K \neq \emptyset$. By Lemma 2.4, $d(s,x) = 1$.

We first consider the case when X intersects two distinct components C_1, C_2 of $G-K$. Since $N_G(C_1 \cap X) \subseteq C_1 \cup K$ and $N_G(C_1 \cap X) \subseteq N_G(X) \cup \{x\}$, we have

$$\bar{d}(X) \geq \bar{d}(C_1 \cap X) - 1 + d(s,x) + d(s, C_2 \cap X) + |N_G(C_2 \cap X) \cap C_2|.$$

If $C_2 \subseteq X$ then $d(s, C_2 \cap X) \geq 1$, and if $C_2 \not\subseteq X$ then $|N_G(C_2 \cap X) \cap C_2| \geq 1$. Since $d(s,x) = 1$ and $\bar{d}(C_1 \cap X) \geq k$, we deduce that $\bar{d}(X) \geq k+1$. This contradicts the fact that X is tight.

Thus X intersects a unique component C of $G-K$. Let $M = C \cap X$. Then $N_G(M) \subseteq C \cup K$. Since $(N_G(M) - \{x\}) \cup N_G(x) \subseteq N_G(X)$, we may use (7) to obtain

$$\begin{aligned} k = \overline{d}(X) &\geq \overline{d}(M) - 1 + d(s,x) + |N_G(x) - (M \cup N_G(M))| \\ &\geq k - 1 + d(s,x) + |N(x) - M - N(M)|. \end{aligned}$$

This implies that $N_G(x) \subseteq M \cup N_G(M)$. Therefore $\hat{b}(K) \leq b(K)+1$, and $x \notin N_G(C')$ for every component $C' \neq C$ of $G-K$. Hence $d(s,C') \geq k - n_G(C') \geq 2$. For C we have $d(s,C) \geq 1$ by (7). This gives $d(s) \geq 2(b(K)-1)+1+d(s,x) = 2b(K) \geq 2\hat{b}(K)-2$. Thus equality must hold throughout $\hat{b}(K) = b(K)+1$ and $\delta(K) = 1$. Since $N(s) \cap K = \{x\}$ by (b), we have $k - d_G(x) = \delta(K) = 1 = d(s,x)$. •

We shall use the following construction to augment G with $\hat{b}(G)-1$ edges in the case when $d(s,K) = 0$ and $b(K) = \hat{b}(G) = b$. Let $C_1,...,C_b$ be the components of $G-K$ and let $w_i = d_{G+s}(s,C_i)$, $1 \leq i \leq b$. Note that $w_i \geq 1$ by (7). Since $d(s) \leq 2b-2$, there exists a tree T on b vertices $C_1, C_2, \ldots, C_b$ with degree sequence $d_1,...,d_b$ such that $d_i \geq w_i$, for $1 \leq i \leq b$. (It will be clear from the context whether the label C_i refers to a component of $G-K$ or a vertex of T.) Let F be a set of edges joining vertices of $N_{G+s}(s)$ with $d_F(v) \geq d_{G+s}(s,v)$ for every $v \in V(G)$ and such that the graph obtained from $(V-K,F)$ by contracting $C_1, C_2, \ldots, C_b$ to single vertices is T. Thus $|F| = |E(T)| = b-1$. We shall say that $G+F$ is a *forest augmentation* of G with respect to K and $G+s$, and prove that $G+F$ is k-connected. Note that since $d_{G+s}(s,K) = 0$, there are no k-deficient fragments of G contained in K by (7).

Lemma 6.3 *Suppose $d(s,K) = 0$ and let $G+F$ be a forest augmentation of G with respect to K and $G+s$. If X is a k-deficient fragment in $G+F$ then $|X \cap K| \geq 2$.*

Proof: We proceed by contradiction. Suppose X is a k-deficient fragment in $G+F$ with $|X \cap K| \leq 1$. Let $X^* = V - X - N_{G+F}(X)$. Replacing X by X^* if necessary, we may assume that

$$|X^* \cap K| \geq |X \cap K|. \tag{20}$$

We first suppose that $L \subseteq X$ for some leaf L of T. Since $d(s,L) \leq d_T(L)$, L is a leaf component of K in $G+s$. Hence $K \subseteq X \cup N_G(X)$ by Lemma 2.14. It follows that $X^* \cap K = \emptyset$. Hence $X \cap K = \emptyset$ by (20) and $K \subseteq N_G(X)$. If X properly intersects some component $C_i \neq L$ of $G-K$ then $n_G(X) \geq k$ follows, contradicting the fact that X is k-deficient in $G+F$. Since $X^* \neq \emptyset$, there exists a component C of $G-K$ for which $C \cap X = \emptyset$. Choose a path P from L to C in T. Let C' be the first component for which the edge on P which enters C' corresponds to an edge in F which connects X to $V-X$. For this component we have $|N_{G+F}(X) \cap C'| \geq 1$, so $n_{G+F}(X) \geq |K|+1 = k$, as required. Thus we may assume that

$$L \cap X \neq L \text{ for each leaf } L \text{ of } T. \tag{21}$$

Choose a component D of $G-K$ such that $D \cap X \neq \emptyset$ and let R be the set of edges of F which are incident with $X \cap D$. Let $e_1,...,e_r$ be the edges incident to D in T, which correspond to the edges in R. Choose r longest paths $P_1,...,P_r$ in T starting at D and containing the edges $e_1,...,e_r$. Let A be the set of all paths P_j, $1 \leq j \leq r$, which contain an edge C_sC_t corresponding to an edge u_jv_j in F with $u_j \in C_s \cap X$ and $v_j \in C_t - X$. For every such path we have $v_j \in N_{G+F}(X)$. Let

$A' = \{v_j \,:\, P_j \in A\}$. Let B be the set of paths P_j, $1 \le j \le r$, which do not belong to A and choose $P_j \in B$. Since the first edge of P_j corresponds to an edge in F which is incident to $D \cap X$, every edge of P_j corresponds to an edge of F joining two vertices of X. In particular, the last edge of P_j is incident to a leaf L_j of T which is distinct from D and for which $X \cap L_j \neq \emptyset$. Since $X \cap L_j \neq L_j$ by (21), we may choose a vertex $w_j \in N_G(X) \cap L_j$. Let $B' = \{w_j \,:\, P_j \in B\}$. Clearly, $|A| = |A'|$, $|B| = |B'|$ and $|A| + |B| = r$. The above observations imply that

$$A' \cup B' \cup (N_G(D \cap X) - (X \cap K)) \cup (N_G(X \cap K) - X) \subseteq N_{G+F}(X). \tag{22}$$

Since $G+s$ is (k,s)-connected, $r \ge k - n_G(D \cap X)$. Since $A', B', N_G(D \cap X)$ are pairwise disjoint, we may deduce that, if $X \cap K = \emptyset$, then X is not k-deficient in $G+F$. Hence $X \cap K = \{x\}$ for some $x \in K$.

Let L be a leaf of T distinct from D. Then L is a leaf component of K in $G+s$ so $N_G(x) \cap L \neq \emptyset$. Hence either $(N_G(x) \cap L) - X \neq \emptyset$, or $X \cap L \neq \emptyset$ and, by (21), $N_G(X) \cap L \neq \emptyset$. It follows that, in both cases, we may choose $y \in N_G(X) \cap L$. Thus

$$A' \cup B' \cup (N_G(D \cap X)) \cup \{y\} \subseteq N_{G+F}(X).$$

Clearly $y \notin N_G(D \cap X)$. Since X is k-deficient in $G+F$, we must have $y \in A' \cup B'$. Thus

$$L \cap (A' \cup B') \neq \emptyset \text{ for each leaf } L \text{ of } T \text{ distinct from } D. \tag{23}$$

The definitions of A', B' now imply that the paths P_j, $1 \le j \le r$, cover T, and hence that each edge of F which is incident with D, is incident with $D \cap X$. Since $V(F) = N_{G+s}(s)$, we have $N_{G+s}(s) \cap D \subseteq X$. Since D can be any component of $G-K$ which intersects X we may deduce that

$$\text{If } D \cap X \neq \emptyset \text{ for some component } D \text{ of } G-K \text{ then } N_{G+s}(s) \cap D \subseteq X. \tag{24}$$

Suppose C is a component of $G-K$ with $C \cap X = \emptyset$. Then (23) implies that C is a leaf of T and $A' \cap C \neq \emptyset$. Furthermore, the argument used in the derivation of (23) gives $A' \cap C = \{y\} = N_G(x) \cap L$. Since $y \in A' \subseteq N_{G+s}(s)$, y is the unique neighbour of s in C. Thus

$$\text{If } C \cap X = \emptyset \text{ for some component } C \text{ of } G-K \text{ then } N_{G+s}(s) \cap C \subseteq N_G(x). \tag{25}$$

Properties (24) and (25) imply that $N_{G+s}(s) \subseteq X \cup N_G(X)$. Thus $N_{G+s}(s) \cap X^* = \emptyset$ and $\bar{d}(X^*) \le n_G(X) < k$. This contradicts the (k,s)-connectivity of $G+s$ and completes the proof of Lemma 6.3. •

Lemma 6.4 *Suppose $d(s,K) = 0$ and $\hat{b}(K) = b(K) \ge 4k + 3(k-l) - 1$. Let $G+F$ be a forest augmentation of G with respect to K and $G+s$. Then $G+F$ is k-connected.*

Proof: We proceed by contradiction. Let X be a k-deficient fragment in $G+F$. Then X^* is also k-deficient so by Lemma 6.3, $|X \cap K| \ge 2$ and $|X^* \cap K| \ge 2$. Since $|V - (K \cup X \cup X^*)| \le V - (X \cup X^*)| \le k-1$, there are at least $b_G(K) - (k-1)$ components C of $G-K$ which are contained in $X \cup X^*$. There is no edge from X to X^* in $G+F$, so for each such component either $C \subseteq X$ or $C \subseteq X^*$ holds. Thus we have $N_G(C) \subseteq K - X^*$ or $N_G(C) \subseteq K - X$, and so $n_G(C) \le k-3$. Hence

$d_G(s,C) \geq 3$ by (7). This contradicts Lemma 6.1(a). •

Our final step is to show how to augment G with $\hat{b}(K)-1$ edges when $d(s,K) \neq 0$. In this case, Lemma 6.1(b) implies that there is exactly one vertex $x \in K$ which is adjacent to s. We use the next lemma to split off all edges from s to x and hence reduce to the case when $d(s,K) = 0$.

Lemma 6.5 *Suppose $d(s,x) \geq 1$ for some $x \in K$ and $d(s) \geq (k+1)(k-l+1)$. Then there exists a sequence of $d(s,x)$ admissible splits at s which split off all edges from s to x.*

Proof: We have $d(s,x) \leq k-l$. Suppose we get stuck after splitting off some copies of sx, i.e. we obtain a graph $H+s$ where some edge sx cannot be split off. Since $d_{H+s}(s) \geq d_{G+s}(s) - 2(k-l-1) \geq (k-l+1)(k-1)+4$, we can use Lemma 5.3 to deduce that there is a $(k-1)$-shredder K' in H with $b_H(K') = d_{H+s}(s)$ and with x in one of the components of $H-K'$. Let u,v be two neighbours of s in H distinct from x and let C_u and C_v be the components of $H-K'$ containing u and v respectively. By Lemma 2.14, there exist $k-1$ openly disjoint paths between u and v in H containing only verticies of C_u, C_v and K', and hence avoiding x. Since all edges of $E(H)-E(G)$ are incident with x, these paths exist in G as well.

Since $b_G(K) \geq \hat{b}_G(K)-(k-l) \geq (d_{G+s}(s)+2)/2-(k-l) \geq k+1 \geq d_{G+s}(s,V-x)-d_{H+s}(s,V-x)+2$, and each component of $G-K$ contains a neighbour of s in G, we can choose the two neighbours u,v of s in $H+s$ to belong to different components in $G-K$. But for such a choice of u,v there do not exist $k-1$ disjoint paths from u to v in $G-x$, contradicting the above claim. •

We can now prove our augmentation result for graphs G for which $\hat{b}(G)$ is large.

Theorem 6.6 *Suppose that G is l-connected, $\hat{b}(G) \geq 4k+4(k-l)-1$, $t(G) \geq (k+1)(k-l+1)$ and $\hat{b}(G)-1 \geq \lceil t(G)/2 \rceil$. Then $a_k(G) = \hat{b}(G)-1$.*

Proof: Let $G+s$ be a k-critical extension of G. Then $d(s) = t(G)$ by Lemma 5.2. Let K be a $(k-1)$-shredder in G with $\hat{b}(K) = \hat{b}(G)$. Then $2\hat{b}(K)-2 \geq t(G) = d(s)$. Suppose $d(s,K) = 0$. Then $\hat{b}(G) = b(K)$. Let $G+F$ be a forest augmentation of G with respect to K and $G+s$. Then $|F| = b(G)-1$ and by Lemma 6.4, $G+F$ is the required k-augmentation of G. Hence we may assume that $d(s,K) \geq 1$.

Applying Lemma 6.1(c), we deduce that $\delta_G(K) = d_{G+s}(s,K) = d_{G+s}(s,x)$ for some $x \in K$. By Lemma 6.5, we can construct a graph $H+s$ by performing a sequence of $d_G(s,x)$ admissible splits at s which split off all edges from s to x in $G+s$. Since we only split edges incident to $x \in K$ to form $H+s$, we have $G-K = H-K$ and so $b_G(K) = b_H(K)$. Hence

$$\begin{aligned} d_{H+s}(s) &= d_{G+s}(s) - 2d_{G+s}(s,x) = d_{G+s}(s) - 2\delta_G(K) \leq 2\hat{b}_G(K) - 2 - 2\delta_G(K) = \\ &= 2b_G(K) + 2\delta_G(K) - 2 - 2\delta_G(K) = 2b_G(K) - 2 = 2b_H(K) - 2. \end{aligned}$$

Thus we have $d_{H+s}(s) \leq 2b_H(K)-2$, and $d_{H+s}(s,K) = 0$. Also note that the splittings add a set F_0 of $\delta_G(K)$ new edges to G to form H, and that $b_H(K) = b_G(K) \geq \hat{b}_G(K)-(k-l) \geq 4k+3(k-l)-1$. Let $H+F_1$ be a forest augmentation of H with respect to K and $H+s$. Then $|F_1| = b_H(K)-1 = b_G(K)-1$, and $H+F_1$ is k-connected by Lemma 6.4. Thus $G+F_0+F_1 = H+F_1$ is the required k-augmentation of G with $\delta_G(K)+b_G(K)-1 = \hat{b}_G(K)-1$ edges. •

We will apply Theorem 6.6 to graphs which do not satisfy $\hat{b}(G)-1 \geq \lceil t(G)/2 \rceil$ using saturating edges. Recall that a set F of new edges is saturating for G if $t(G+F)=t(G)-2|F|$.

Lemma 6.7 *If F is a saturating set of edges for an l-connected graph G with $\hat{b}(G+F) \geq 4k+4(k-l)-1$, $t(G+F) \geq (k+1)(k-l+1)$, and $\hat{b}(G+F)-1 = \lceil t(G+F)/2 \rceil$, then $a_k(G) = \lceil t(G)/2 \rceil$.*

Proof: By Theorem 6.6 the graph $G+F$ can be made k-connected by adding a set F' of $\lceil t(G+F)/2 \rceil$ edges. Since F is saturating, we have $t(G)=t(G+F)+2|F|$. Therefore the set $F \cup F'$ is an augmenting set for G of size $\lceil t(G)/2 \rceil$. Since $a_k(G) \geq \lceil t(G)/2 \rceil$, the lemma follows. •

7 Augmenting Connectivity by at least Two

Throughout this section we assume that $G=(V,E)$ is an l-connected graph on at least $k+1$ vertices and that $l \leq k-2$. We shall show that if $a_k(G)$ is large compared to k, then $a_k(G)=\max\{\hat{b}(G)-1, \lceil t(G)/2 \rceil\}$. Our proof uses Theorems 4.17 and 6.6. We shall show that if $a_k(G)$ is large then either we can add a saturating set of edges F so that $G+F$ is $(k-1)$-connected, or else G has a $(k-2)$-shredder with many components. If the latter occurs then we show directly that we can make G k-connected by adding $\lceil t(G)/2 \rceil$ edges. We will occasionally consider two different extensions of the same graph H. To distinguish between them we shall label one of them as $H+s$ and the other as $H \oplus s$.

Let $G+s$ be a k-critical extension of G. Construct a $(k-1)$-critical extension $G \oplus s$ of G from $G+s$ by deleting a set of edges incident to s. Let $f=(k-l+1)(k-1)+4$ be the bound on the number of non-admissible pairs containing a fixed edge given by Lemma 5.3.

Lemma 7.1 *If $d_{G+s}(s) \geq f(k-l+1)/(k-l)$ then $d_{G+s}(s)-d_{G\oplus s}(s) \geq d_{G+s}(s)/(k-l+1)$.*

Proof: If $d_{G\oplus s}(s) \leq f$ then the lemma is trivial. Otherwise by Lemma 5.2(a) there exists a family $\mathcal{F}$ of pairwise disjoint $(k-1)$-deficient fragments in G such that $d_{G\oplus s}(s)=\sum_{\mathcal{F}}(k-1-n(X))$. Since $G+s$ is (k,s)-connected we have $d_{G+s}(s) \geq \sum_{\mathcal{F}}(k-n(X))$. Hence $d_{G+s}(s) \geq d_{G\oplus s}(s)+|\mathcal{F}|$. Since $d_{G\oplus s}(s,X) \leq k-l$ for each $X \in \mathcal{F}$, we have $|\mathcal{F}| \geq d_{G\oplus s}(s)/(k-l)$. Thus $d_{G+s}(s) \geq d_{G\oplus s}(s) + d_{G\oplus s}(s)/(k-l) = (k-l+1)d_{G\oplus s}(s)/(k-l)$. Hence $d_{G+s}(s)-d_{G\oplus s}(s) \geq d_{G+s}(s)/(k-l+1)$. •

We next perform a sequence of $(k-1)$-admissible splits at s in $G \oplus s$ and obtain $G_1 \oplus s$. We do this according to the following rules. If $d_{G\oplus s}(s) \leq 2f$ then we put $G_1 \oplus s = G \oplus s$. If $d_{G\oplus s}(s) \geq 2f+1$ then we perform $(k-1)$-admissible splits until either $d_{G_1\oplus s}(s) \leq 2f$, or $d_{G_1\oplus s}(s) \geq 2f+1$ and there is no $(k-1)$-admissible split at s in $G_1 \oplus s$. We then add all the edges of $(G+s)-(G\oplus s)$ to $G_1 \oplus s$ and obtain G_1+s. We shall refer to the edges of $(G+s)-(G\oplus s)$ as *new edges* of G_1+s.

Lemma 7.2 *If $d_{G+s}(s) \geq f(k+l-1)$ then G_1+s is a k-critical extension of G_1.*

Proof: Suppose G_1+s is not (k,s)-connected. If $d_{G_1\oplus s}(s) \leq f$ then $G_1 \oplus s = G \oplus s$ and $G_1+s=G+s$, contradicting the assumption that $G+s$ is (k,s)-connected. Hence $d_{G_1\oplus s}(s) \geq f+1$. Choose a minimal fragment X of G_1 such that $d_{G_1+s}(X) < k$. Since $d_{G_1\oplus s}(X) \geq k-1$ we have

$\overline{d}_{G_1+s}(X) = k-1 = \overline{d}_{G_1\oplus s}(X)$ and no new edge of G_1+s is incident with X. Since $\overline{d}_{G+s}(X) \geq k$, there exists an edge sx in $G+s$ with $x \in X$. Then $sx \in E(G\oplus s)$, since no new edge is incident with X. Hence sx is $(k-1)$-critical in $G\oplus s$ so there exists a minimal tight set Y with $x \in Y$ and $\overline{d}_{G\oplus s}(Y) = k-1$. Hence $\overline{d}_{G_1\oplus s}(Y) = k-1$. Working in $G_1 \oplus s$ we may use Lemma 5.1(a) to deduce that $\overline{d}_{G_1\oplus s}(X\cap Y) = k-1$. Since there are no new edges incident to X, this gives $\overline{d}_{G_1+s}(X\cap Y) = k-1$. Now the minimality of X implies that $X \subseteq Y$. Since $\overline{d}_{G\oplus s}(Y) = \overline{d}_{G_1\oplus s}(Y)$, we now deduce that $\overline{d}_{G\oplus s}(X) = \overline{d}_{G_1\oplus s}(X)$. Thus $\overline{d}_{G\oplus s}(X) = k-1$ and the minimality of Y gives $X = Y$. Since no new edge is incident with X this gives $\overline{d}_{G+s}(Y) = \overline{d}_{G\oplus s}(Y) = k-1$. Thus Y is k-deficient in $G+s$, contradicting the fact that $G+s$ is (k,s)-connected.

Criticality of G_1+s follows from the criticality of $G+s$, since splitting off pairs of edges from s cannot increase $\overline{d}(X)$ for any $X \subseteq V$. •

Using Lemma 5.3, we can deduce that either $d_{G_1\oplus s}(s)$ is small or else there exists a $(k-2)$-shredder K in G_1 such that G_1-K has $d_{G_1\oplus s}(s)$ components. In the first case, we show that there exists a sequence of k-admissible splits in G_1+s such that, in the resulting graph $G_1'+s$, G_1' is $(k-1)$-connected and then apply Theorems 6.6 and 4.17. We accomplish this by ensuring that $\kappa(x,y,G_1') \geq k-1$ for every $x,y \in N_{G_1\oplus s}(s)$. This is possible since there are many new edges and hence $d_{G_1+s}(s)$ is large compared to $d_{G_1\oplus s}(s)$. We proceed incrementally using the lemmas below. In the second case, we show directly that we can make G k-connected by adding $\lceil t(G)/2\rceil$ edges.

Henceforth we shall assume that $G_1'+s$ is obtained from G_1+s by performing a sequence of k-admissible splits and that $T \subseteq V$ is a cover of all $(k-1)$-deficient fragments of G_1'. (In proving the theorem we will take $T = N_{G_1\oplus s}(s)$.) Let $|T| = \tau$.

Lemma 7.3 *If $\kappa(u,v,G_1') \geq k-1$ for all $u,v \in T$ then G_1' is $(k-1)$-connected.*

Proof: Suppose G_1' has a fragment X with $n(X) \leq k-2$. Then we may choose $u \in X\cap T$ and $v \in X^*\cap T$, contradicting the fact that $\kappa(u,v) \geq k-1$. •

Lemma 7.4 *Let $sz,sw \in E(G_1'+s)$ and suppose that the pair sz,sw is not k-admissible. If $\kappa(z,w,G_1') \leq k-2$ then there are at most f pairs of edges sz,sx which are not k-admissible in $G_1'+s$.*

Proof: Let $R = \{sx : sz,sx \text{ is not } k\text{-admissible in } G_1'+s\}$. Suppose that $r = |R| > f$. Then by Lemma 5.3, there is a $(k-1)$-shredder K in G_1' with $r+1$ leaf components in $G_1'+s$ such that z as well as each vertex x, $sx \in R$, is in one of these components. By Lemma 2.14, $\kappa(z,x) \geq k-1$ for every such x. Taking $x = w$ gives a contradiction. •

Lemma 7.5 *Suppose that $d_{G_1'+s}(s) \geq (f+1)(2(k-2)(f+2)+\tau)+(k-2)(k-l-2)$. Choose $u,v \in T$ and suppose that $\kappa(u,v,G_1') = m \leq k-2$. Then there exists a sequence of at most two k-admissible splits such that, for the resulting graph $G_1''+s$, we have $\kappa(u,v,G_1'') = m+1$.*

Proof: Let X_u and X_v be the smallest sets which contain u and v, respectively, separate u and v, and have precisely m neighbours. It is well-known that these unique smallest separators exist. Since $n_{G_1'}(X_u) = n_{G_1'}(X_v) = m \leq k-2$, there exist vertices $x \in X_u \cap N_{G_1'+s}(s)$ and $y \in X_v \cap N_{G_1'+s}(s)$. It is also known that there exist m paths $P_1,\ldots,P_m$ from u to v, and two paths P_0 and P_{m+1}, one from

u to x and the other from v to y such that all these $m+2$ paths are vertex-disjoint apart from at u and v. (Note that $u=x$ or $v=y$ is possible.) We may assume, without loss of generality, that $N_{G'_1+s}(s)\cap(V(P_0)-x)=\emptyset$ and $N_{G'_1+s}(s)\cap(V(P_{m+1})-y)=\emptyset$. Let $Q=\cup_{i=1}^{m}V(P_i)-\{u,v\}$. If the pair sx,sy is k-admissible, we have $\kappa(u,v,G'_1+xy)\geq m+1$, as required. If not, we need to choose k-admissible pairs in a more complicated way, as in the proof of Lemma 4.13.

Suppose there exists a path P_i $(1\leq i\leq m)$ with $d_{G'_1+s}(s,V(P_i))\geq 2f+(k-l)+1$. By Lemma 7.4 we may choose an admissible pair sx,sa in G'_1+s such that a is a neighbour of s on P_i as close to v as possible. Lemma 7.4 implies that there are at most f edges from s to $P_i(a,v]$. If $\kappa(u,v,G'_1+xa)\geq m+1$ then we are done. Otherwise we may split sy,sb in G'_1+s+xa, where b a neighbour of s on P_i as close to u as possible. Lemma 7.4 implies that there are at most f edges from s to $P_i[u,b)$. Since $d(s,w)\leq k-l$ for each $w\in V(P_i)$, the vertices x,b,a,y appear on P_i in this order. Hence there exist two vertex-disjoint uv-paths on vertex set $V(P_i)\cup V(P_0)\cup V(P_{m+1})$, showing $\kappa(u,v,G'_1+xa+yb)\geq m+1$, as required. Thus we may assume that no such path exists and hence $d_{G'_1+s}(s,V-Q)>d_{G'_1+s}(s)-m(2f+k-l)\geq(f+1)(2(k-2)(f+1)+\tau)$.

Let H be the graph obtained from G'_1-Q by deleting any edges joining u and v. Let $C_0,C_1,\ldots,C_{p+1}$ be the components of H which each contain at least one neighbour of s, where $u,x\in V(C_0)$ and $v,y\in V(C_{p+1})$. Suppose $d(s,C_j)\geq f+2$ for some $1\leq j\leq p$. We may perform a k-admissible split sx,sa for some $a\in C_j$, and then a k-admissible split sy,sb in G'_1+s+sa for some $b\in C_j$. These admissible pairs exist by Lemma 7.4. It is easy to see that $\kappa(u,v,G'_1+xa+yb)\geq m+1$, as required. Thus we may assume that no such component exists. Similarly, if $d(s,C_0)\geq f+1$, then we may split sy,sc for some $c\in C_0$ which is admissible with sy in G'_1+s, and we again have $\kappa(u,v,G'_1+yc)\geq m+1$, as required. A similar construction holds if $d(s,C_{p+1})\geq f+1$. Hence we have at least $d(s,V-Q)/(f+1)\geq 2(k-2)(f+1)+\tau$ components in H, each containing at least one neighbour of s.

Since each component C_i with $n_{G'_1}(C)\leq k-2$ must contain a vertex from T, and $u,v\in T$, there are at least $2(k-2)(f+1)$ components C_i, $1\leq i\leq p$, with at least $k-1$ attachments on Q. Since $m\leq k-2$, we have at least $2f+2$ components $D_1,\ldots,D_r$ which have two attachments on the same path, P_1 say. We now proceed as in the final part of the proof of Lemma 4.13. Let a_j be the attachment of D_j on P_1 closest to u for $1\leq j\leq r$. We first pick a D_i where a_i is among the $f+1$ attachment vertices a_j closest to u on P_1 and we choose a k-admissible pair sy,sb with $b\in D_i$. This pair exists by Lemma 7.4. Then we pick a D_h where a_h is among the $f+1$ attachment vertices a_j closest to v on P_1 and we choose a k-admissible pair sx,sa with $a\in D_h$. This pair exists by Lemma 7.4. Note that a_i either occurs before a_h on P_1 or $a_i=a_h$. Hence, using the fact that the components D_j have at least two attachments on P_1 and by the choice of a_i,a_h, there exist two openly disjoint uv-paths in $G'_1+xa+yb$, using vertices from $V(P_1)\cup V(P_0)\cup V(P_{m+1})\cup D_i\cup D_h$ only. Therefore $\kappa(u,v,G'_1+xa+yb)\geq m+1$, as required. •

Applying this lemma iteratively to all pairs of vertices in T, starting with $G'_1+s=G_1+s$ and using the fact that f is a decreasing function of l, we obtain:

Corollary 7.6 *Suppose that*

$$d_{G_1+s}(s)\geq(f+1)(2(k-2)(f+2)+\tau)+(k-2)(k-l-2)+2\tau^2(k-l-1).$$

Then there exists a sequence of at most $\tau^2(k-l-1)$ k-admissible splits such that, for the resulting graph G'_1+s, we have $\kappa(G'_1)\geq k-1$.

Theorem 7.7 *If G is l-connected and $a_k(G) \geq 10(k-l+2)^3(k+1)^3$ then $a_k(G) = \max\{\hat{b}(G) - 1, \lceil t(G)/2 \rceil\}$.*

Proof: We have $d_{G+s}(s) = t(G) \geq a_k(G) + 1 \geq 10(k-l+2)^3(k+1)^3$ by Lemmas 2.10 and 5.2. If $\hat{b}(G) - 1 \geq \lceil t(G)/2 \rceil$ then $a_k(G) = \hat{b}(G) - 1$ by Theorem 6.6 and we are done. Thus we may assume that $\lceil t(G)/2 \rceil \geq \hat{b}(G)$ holds. We shall show that $a_k(G) = \lceil t(G)/2 \rceil$. We construct $G \oplus s$, $G_1 \oplus s$, and $G_1 + s$ as above. By Lemma 7.2, G_1 is obtained from G by adding a saturating set F of edges. Note that adding a saturating edge to a graph G_0 reduces $\lceil t(G_0)/2 \rceil$ by exactly one and $\hat{b}(G_0)$ by at most one. Thus, if $\lceil t(G+F)/2 \rceil \leq \hat{b}(G+F) - 1$, then there exists $F' \subseteq F$ such that $\lceil t(G+F')/2 \rceil = \hat{b}(G+F') - 1$ and the theorem follows by applying Lemma 6.7. Hence we may assume that $\lceil t(G_1)/2 \rceil \geq \hat{b}(G_1) - 1$. We have

$$t(G_1) = d_{G_1+s}(s) \geq d_{G+s}(s) - d_{G\oplus s}(s) \geq 10(k-l+2)^2(k+1)^3 \tag{26}$$

by Lemma 7.1. Using Lemma 5.3, we either have $d_{G_1\oplus s}(s) \leq 2f$ or else $d_{G_1\oplus s}(s) \geq 2f+1$ and there exists a $(k-2)$-shredder K in G_1 such that $b_{G_1}(K) = d_{G_1\oplus s}(s)$.

Case 1: $d_{G_1\oplus s}(s) \leq 2f$.

Let $T = N_{G_1\oplus s}(s)$. Then $|T| = \tau \leq 2f$. Corollary 7.6 and the fact that $f \leq (k-l+1)(k+1) - 2$ imply that there exists a sequence of at most $4(k-l+1)^3(k+1)^2$ k-admissible splits in $G_1 + s$ such that, for the resulting graph $G_1' + s$, we have $\kappa(G_1') \geq k-1$. Note that $d_{G_1'+s}(s) \geq 2(k-l+2)^2(k+1)^3$, by (26). Thus there exists a saturating set of edges F for G such that $G_1' = G+F$ is $(k-1)$-connected and $t(G+F) \geq 2(k-l+2)^2(k+1)^3$. As above, we may assume that $\lceil t(G+F)/2 \rceil \geq \hat{b}(G+F) - 1 \geq b(G+F) - 1$ (otherwise we are done by Lemma 6.7). Since $G+F$ is $(k-1)$-connected, we can apply Theorem 4.17 to deduce that $a_k(G+F) = \lceil t(G+F)/2 \rceil$. Using (1) and the fact that $t(G) = t(G+F) + 2|F|$ we have $a_k(G) = \lceil t(G)/2 \rceil$, as required.

Case 2: $d_{G_1\oplus s}(s) \geq 2f+1$ and there is no $(k-1)$-admissible split at s in $G_1 \oplus s$.

By Lemma 5.3, there exists a $(k-2)$-shredder K in G_1 such that $b_{G_1}(K) = d_{G_1\oplus s}(s)$ and hence each component of $G_1 - K$ is a leaf component. Using Lemma 2.14, and the fact that $N_{G_1\oplus s}(s)$ covers all $(k-1)$-deficient fragments X in G_1, we deduce:

Claim 7.8 *G_1 is $(k-2)$-connected.*

Since $G_1 + s$ is k-critical, Claim 7.8 and Lemma 2.4 imply:

Claim 7.9 *For all $v \in V$ we have $d_{G_1+s}(s,v) \leq 2$. Furthermore $d_{G_1+s}(s,v) = 2$ if and only if $d_{G_1}(v) = k-2$.*

Let $G_2 + s$ be the graph obtained from $G_1 + s$ by splitting off as many k-admissible pairs of edges sx, sy as possible in $G_1 + s$ such that x and y belong to the same component of $G_1 - K$. Then $G_2 + s$ is a k-critical extension of G_2. Let $C_1, C_2, \ldots, C_r$ be the components of $G_2 - K$. Note that these components have the same vertex sets as the components of $G_1 - K$ and hence

$$r = d_{G_1\oplus s}(s) \geq 2f+1. \tag{27}$$

Let $d_{G_2+s}(s, C_i) = d_i$. Relabelling if necessary, we have $d_1 \geq d_2 \geq \ldots \geq d_r$.

Claim 7.10 $d_{G_2+s}(s,K)=0$.

Proof: Suppose G_2+s has an edge sx with $x\in K$. By criticality there exists a fragment X of G_2 such that $x\in X$ and $d_{G_2+s}(X)=k$. Since, by Caim 7.8, $x\in N_{G_1}(C_i)$ for all $1\le i\le r$, we have $x\in N_{G_2}(C_i)$. Hence either $N_{G_2}(X)\cap C_i\neq\emptyset$, or $C_i\subseteq X$ and $d_{G_2+s}(s,X\cap C_i)\ge 1$, for all $1\le i\le r$. Thus $d_{G_2+s}(X)\ge r>k$, a contradiction. •

Using Lemma 6.7 we may suppose that

$$\hat{b}(G_2)\le\lceil t(G_2)/2\rceil=\lceil d_{G_2+s}(s)/2\rceil. \tag{28}$$

Claim 7.11 $d_1\le(\sum_{i=2}^r d_i)-1$.

Proof: Suppose $d_1\ge(\sum_{i=2}^r d_i)$. Since $d_1+(\sum_{i=2}^r d_i)=d_{G_2+s}(s)\ge r\ge 2f+1$ by Claim 7.10 and (27), we have $d_1\ge f+1$. Since there is no k-admissible pair of edges joining s to C_1 in G_2+s, it follows from Lemma 5.3 that there is a $(k-1)$-shredder $\hat{K}$ in G_2 with each of the d_1 neighbours of s in C_1 in distinct components of $G_2-\hat{K}$ and at least one other component containing the remaining neighbours of s in G_2+s. Thus $b(G_2)\ge d_1+1$, and $\hat{b}(G_2)\ge b(G_2)\ge d_1+1\ge(d_{G_2+s}(s)/2)+1$. This contradicts (28). •

Claim 7.12 *Suppose X is a fragment in G_2 with $|X\cap K|\le|X^*\cap K|$.*
(a) If $n_{G_2}(X)=k-2$, then either $X=C_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_p}$ for some $\{i_1,i_2,\ldots,i_p\}\subset\{1,2,\ldots,r\}$; or $X=Z_i\subset C_i$ for some $1\le i\le r$;
(b) If $n_{G_2}(X)=k-1$, then either $X=Z_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_p}$ for some $\{i_1,i_2,\ldots,i_p\}\subseteq\{1,2,\ldots,r\}$ and $Z_{i_1}\subseteq C_{i_1}$; or $X=Z_{i_1}\cup Z_{i_2}$ for some $1\le i_1<i_2\le r$, $Z_{i_1}\subseteq C_{i_1}$, $Z_{i_2}\subseteq C_{i_2}$, and $n_{G_2}(Z_{i_1})=k-2=n_{G_2}(Z_{i_2})$.

Proof: Suppose $X\cap K\neq\emptyset$. Then $X^*\cap K\neq\emptyset$. Since $N_{G_2}(C_i)=K$ by Claim 7.8, it follows that $C_i\not\subseteq X$ and $C_i\not\subseteq X^*$ for all $1\le i\le r$, and hence that $n_{G_2}(X)=|V-(X\cup X^*)|\ge r>k$. Thus we may suppose that $X\cap K=\emptyset$. Let

$$S=\{i\,:\,X\cap C_i\text{ is a proper subset of }C_i,\ 1\le i\le r\}.$$

Since the claim holds when $S=\emptyset$ we may suppose that $|S|\ge 1$. Let $Z_i=X\cap C_i$ for $i\in S$. By Claim 7.8, $n_{G_2}(Z_i)\ge k-2$. Hence $|N_{G_2}(X)\cap(K\cup C_i)|\ge k-2$ and $|N_{G_2}(X)\cap C_i|\ge 1$ for all $i\in S$. The claim now follows using the hypothesis of (a) and (b) that $n_{G_2}(X)=k-2$ and $n_{G_2}(X)=k-1$, respectively. •

Claim 7.13 *For each i, $1\le i\le r$, there exists a unique minimal subset $Y_i\subseteq V(C_i)$ such that $n_{G_2}(Y_i)=k-2$.*

Proof: The existence of such a set follows from the fact that $n_{G_2}(C_i)=k-2$. To prove uniqueness we suppose to the contrary that X_1 and X_2 are two minimal subsets of C_i satisfying $n_{G_2}(X_1)=k-2=n_{G_2}(X_2)$. Then $n_{G_1}(X_1)=k-2=n_{G_1}(X_2)$, since G_1 is $(k-2)$-connected by Claim 7.8, and the operation used in going from G_1 to G_2 (splitting off pairs of edges from s) cannot decrease $n(X_i)$. Let sw be the unique edge of $G_1 \oplus s$ from s to C_i. Since $G_1 \oplus s$ is $(k-1,s)$-connected, we must have $w \in X_1 \cap X_2$. Since $X_1 \cup X_2 \subseteq C_i$, $X_1 \cup X_2$ is a fragment of G_2, and hence we have $n_{G_2}(X_1 \cup X_2) \geq k-2$, by Claim 7.8. Submodularity of n_{G_2}, now implies that $n_{G_2}(X_1 \cap X_2) \leq k-2$, contradicting the minimality of X_1 and X_2. •

For each i, $1 \leq i \leq r$, choose two distinct edges sy_i, sy_i' in $G_2 + s$ with $y_i, y_i' \in Y_i$. Note that these edges exist by the (k,s)-connectivity of G_2. Furthermore, by Claim 7.9, $y_i = y_i'$, if and only if $Y_i = \{y_i\}$ and $d_{G_2}(y_i) = k-2$.

We are now ready to construct the required augmentation of G. Let $G_2 \oplus s$ be the graph obtained from $G_2 + s$ by adding an extra edge from s to C_2 if $d_{G_2+s}(s)$ is odd. Thus $d_{G_2 \oplus s}(s) = 2\lceil t(G_2)/2 \rceil$ is even. First we try to define a good augmenting set by a method similar to forest augmentation. Since we need to increase the connectivity of G_2 by two, we now look for a loopless 2-connected multigraph H on r vertices whose degree sequence is $d_1, d_2', \ldots, d_r$, where $d_2' = d_{G_2 \oplus s}(s, C_2)$ (so d_2' is either d_2 or d_2+1, depending on whether $d_{G_2+s}(s)$ is even or odd). If such a multigraph exists, it leads to a good augmenting set in a natural way, as we shall see in Subcase 2.1. However, such a graph may not exist, as the following example shows. Let G be obtained from $K_{r,k-2}$ by replacing some vertex v in the r-set by a copy of $K_{k-1,4}$ and then connecting each vertex of the $(k-2)$-set to each vertex of the $(k-1)$-set. It can be seen that the degree sequence defined by the corresponding extension $G_2 \oplus s$ of G is $4, 2, 2, \ldots, 2$. There is no loopless 2-connected multigraph with this degree sequence. When such a multigraph does not exist, we need a somewhat more involved method to define the augmenting set. This will be described in Subcase 2.2.

Subcase 2.1 There exists a loopless 2-connected multigraph H on r vertices with degree sequence $d_1, d_2', \ldots, d_r$.

Let F be a set of edges joining the components of $G_2 - K$ such that $d_F(v) = d_{G_2 \oplus s}(s, v)$ for all $v \in V$ and such that the graph obtained from $(V-K, F)$ by contracting each component C_i to a single vertex c_i, is H. Since H is 2-connected, each vertex $c_i \in V(H)$ has at least two distinct neighbours in H, and thus each component C_i is joined to at least two other components by edges of F. Furthermore, since H is loopless, each edge of F is incident with two distinct components of $G_2 - K$. Let y_i, y_i' be the neighbours of s in C_i as defined after Claim 7.13. Since we are free to interchange the end vertices of the edges of F within each component, we may choose F to have the additional property that, for each $1 \leq i \leq r$, the two edges of F incident to y_i and y_i' join C_i to different components of $G_2 - K$. We can now use Claim 7.12 to deduce that $G_2 + F$ is k-connected. Suppose to the contrary that $G_2 + F$ has a fragment X with $n_{G_2+F}(X) \leq k-1$. Replacing X by X^* if necesssary we may assume that $|X \cap K| \leq |X^* \cap K|$. By Claim 7.8, $n_{G_2}(X) \geq k-2$ and by Claim 7.12, we have one of the following four alternatives.
(a1) $n_{G_2}(X) = k-2$ and $X = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \leq r-1$. Suppose $p \leq r-2$. Then the 2-connectivity of H implies that there are two edges of F from X to distinct components C_{j_1}, C_{j_2} disjoint from X. Hence $n_{G_2+F}(X) \geq k$. Suppose $p = r-1$. There are at least two edges from X to C_{i_r}, where C_{i_r} is the unique component of $G_2 - K$ disjoint from X. If C_{i_r} has only one vertex then $N_{G_2+F}(X) = V - X$ and X is not a fragment. If all edges of F join X to the same vertex $v \in C_{i_r}$, then we have $n_{G_2}(C_{i_r} - v) \leq k-1$ and $d_{G_2+s}(s, C_{i_r} - v) = 0$, contradicting the (k,s)-connectivity of $G_2 + s$. Thus at least two edges of F join X to distinct vertices of C_{i_r} and we again have $n_{G_2+F}(X) \geq k$.

(a2) $n_{G_2}(X) = k-2$ and $X = Z_i \subset C_i$ for some $1 \le i \le r$. By Claim 7.13, $y_i, y_i' \in X$. Since y_i, y_i' are joined by F to distinct components C_{j_1}, C_{j_2} disjoint from C_i, we again have $n_{G_2+F}(X) \ge k$.
(b1) $n_{G_2}(X) = k-1$, and $X = Z_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \le r$ and $Z_{i_1} \subseteq C_{i_1}$. Suppose $2 \le p \le r-1$. Then the 2-connectivity of H implies that there is at least one edge of F from $X - C_{i_1}$ to a component C_{j_1} disjoint from X. Hence $n_{G_2+F}(X) \ge k$. Suppose $p = r$. Since $G_2 + s$ is (k,s)-connected, it has an edge from s to a vertex $v \in X^* \subseteq C_{i_1} - Z_{i_1}$. Since all edges of F are incident to distinct components v is joined by an edge of F to some vertex of $X - C_{i_1}$, and again we have $n_{G_2+F}(X) \ge k$. Suppose $p = 1$. Since $G_2 + s$ is (k,s)-connected, it has an edge from s to at least one vertex $v \in Z_{i_1}$. Since all edges of F are incident to distinct components, v is joined by an edge of F to some component distinct from C_{i_1}, and again we have $n_{G_2+F}(X) \ge k$.
(b2) $n_{G_2}(X) = k-1$ and $X = Z_{i_1} \cup Z_{i_2}$ for some $Z_{i_1} \subseteq C_{i_1}$, $Z_{i_2} \subseteq C_{i_2}$, and $n_{G_2}(Z_{i_1}) = k-2 = n_{G_2}(Z_{i_2})$. By Claim 7.13, $y_{i_1}, y_{i_1}' \in Z_{i_1}$. Since y_{i_1}, y_{i_1}' are joined by F to two distinct components C_{j_1}, C_{j_2} disjoint from C_{i_1}, at least one of these components is also disjoint from C_{i_2} and we again have $n_{G_2+F}(X) \ge k$.

Thus $G_2 + F$ is k-connected. Putting $F_0 = E(G_2) - E(G)$, we deduce that $F_0 \cup F$ is the required augmenting set of edges for G of size $\lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$.

Subcase 2.2 There is no loopless 2-connected multigraph on r vertices with degree sequence $d_1, d_2', \ldots, d_r$.

Hakimi [10] characterised the degree sequences of loopless 2-connected multigraphs, see also [14, Corollary 3.2].

Theorem 7.14 *There exists a 2-connected loopless multigraph with degree sequence $d_1 \ge d_2 \ge \ldots \ge d_r \ge 2$ if and only if $d_1 + d_2 + \ldots + d_r$ is even and $d_1 \le d_2 + d_3 + \ldots + d_r - 2r + 4$.*

This characterisation implies that in Subcase 2.2 we have either: $d_1 \ge d_2'$ and $d_1 \ge d_2' + d_3 + \ldots + d_r - 2r + 5$; or $d_1 = d_2' - 1$ and $d_2' \ge d_1 + d_3 + \ldots + d_r - 2r + 5$. Since $d_{G_2 \oplus s}(s) = d_1 + d_2' + d_3 + \ldots + d_r$ and $d_{G_2 \oplus s}(s)$ is even, both alternatives imply that

$$d_{G_2 \oplus s}(s) \le 2d_1 + 2r - 4. \tag{29}$$

We shall use the following concept to find a good augmenting set in this subcase. Let $H_0 = (V,E)$ be a multigraph, $s \in V$, and $m_1, m_2, \ldots, m_q$ be a partition of $d_{H_0}(s)$. Then an $(m_1, m_2, \ldots, m_q)$-*detachment of H_0 at s* is a multigraph H_1 obtained from H_0 by 'splitting' s into q vertices $s_1, s_2, \ldots, s_q$ with degrees $m_1, m_2, \ldots, m_q$, respectively. We refer to $s_1, s_2, \ldots, s_q$ as the *pieces* of s in H_1. Note that the graph H used in Subcase 2.1 can be viewed as a loopless 2-connected $(d_1, d_2', d_3 \ldots, d_r)$-detachment at s of the graph H_0 consisting of exactly one vertex s incident with $d_{G_2 \oplus s}(s)/2$ loops. Inequality (29) tells us that if this detachment H does not exist, then d_1 is 'large' compared to $d_{G_2 \oplus s}(s)$. We modify our approach in this case by finding a loopless 2-connected $(d_2', d_3, \ldots, d_r)$-detachment of the multigraph obtained from $(G_2 \oplus s) - K - \cup_{i=2}^r C_i$ by adding a suitable number of loops to s. The pieces of s in the detachment will represent the components $C_2, C_3, \ldots, C_r$. We use the following lemma from [14] to construct the required detachment.

Given a multigraph H and $v_1, v_2, \ldots, v_m \in V(H)$, let $b(v_1, v_2, \ldots, v_m)$ be the number of components of $H - \{v_1, v_2, \ldots, v_m\}$.

Lemma 7.15 *[14, Corollary 3.3] Let $H_0 = (V,E)$ be a multigraph, $s \in V$ and $m_1, m_2, \ldots, m_q$ be a partition of $d(s)$ into at least two positive integers, such that $m_1 \ge m_2 \ge \ldots \ge m_q \ge 2$. Let $e(u)$*

denote the number of loops incident to each vertex u in H_0. Then H_0 has a loopless 2-connected $(m_1, m_2, \ldots, m_q)$-detachment at s if and only if
(a) H_0 is 2-edge-connected,
(b) $b(v) + e(v) = 1$ for all $v \in V - s$,
(c) $m_2 + m_3 + \ldots + m_q \geq b(s) + e(s) + q - 2$, and
(d) $d(s, V - v) + e(s) \geq q + b(s, v) - 1$ for all $v \in V - s$.

Let $G_3 + s$ be the multigraph obtained from $(G_2 \oplus s) - K - \cup_{i=2}^{r} C_i$ by adding $p = (d_{G_2 \oplus s}(s) - 2d_1)/2 - 1$ loops at s. Note that p is a non-negative integer by Claim 7.11 and the fact that $d_{G_2 \oplus s}(s)$ is even. Applying Lemma 7.15 to $G_3 + s$ we deduce:

Claim 7.16 *$G_3 + s$ has a loopless 2-connected $(d_2^*, d_3, \ldots, d_{r-1})$-detachment H_1 at s, where $d_2^* = d_2' + d_r - 2$.*

Proof: We have $d_2^* + d_3 + \ldots + d_{r-1} = d_{G_2 \oplus s}(s) - d_1 - 2 = 2p + d_1 = d_{G_3 + s}(s)$ so $(d_2^*, d_3, \ldots, d_{r-1})$ partitions $d_{G_3+s}(s)$. Since $G_2 \oplus s$ is (k, s)-connected and G_3 is connected and loopless, it follows that $G_3 + s$ satisfies Lemma 7.15(a) and Lemma 7.15(b). Using $d_i \geq 2$ for all $3 \leq i \leq r-1$ and (29), we have $d_2' + d_r \leq d_{G_2 \oplus s}(s) - d_1 - 2(r-3) \leq d_{G_2 \oplus s}(s) - d_{G_2 \oplus s}(s)/2 + r - 2 - 2(r-3) = d_{G_2 \oplus s}(s)/2 - r + 4$. Thus $d_3 + \ldots + d_{r-1} = d_{G_2 \oplus s}(s) - d_1 - d_2' - d_r \geq d_{G_2 \oplus s}(s) - d_1 - d_{G_2 \oplus s}(s)/2 + r - 4 = 1 + e(s) + r - 4$, proving that Lemma 7.15(c) holds for $G_3 + s$. To show that Lemma 7.15(d) holds focus on a vertex v of C_1. Considering the graph $G_2 - (K + v)$ and using Claim 7.9, we have $\hat{b}(G_2) \geq b_{G_3}(v) + r - 1 + \beta$, where $\beta = 2$ if $d_{G_2 \oplus s}(s, v) = 2$ and $\beta = 0$, otherwise, since if $d_{G_2 \oplus s}(v) = 2$ then $d_{G_2}(v) = k - 2$. By (28), $\hat{b}(G_2) \leq \lceil t(G_2)/2 \rceil = d_{G_2 \oplus s}(s)/2$. Hence $d_{G_2 \oplus s}(s)/2 \geq b_{G_3}(v) + r - 1 + \beta$. Thus

$$
\begin{aligned}
d_{G_3+s}(s, V(C_1) - v) + e(s) &= d_1 - d_{G_2 \oplus s}(s, v) + e(s) \\
&= d_{G_2 \oplus s}(s)/2 - 1 - d_{G_2 \oplus s}(s, v) \\
&\geq b_{G_3}(v) + r - 1 + \beta - 1 - d_{G_2 \oplus s}(s, v) \\
&\geq (r-2) + b_{G_3+s}(s, v) - 1,
\end{aligned}
$$

as required. •

Label the detached vertices of H_1 as $c_2, c_3, c_4 \ldots, c_{r-1}$ where $d_{H_1}(c_i) = d_i$ for $3 \leq i \leq r-1$ and $d_{H_1}(c_2) = d_2^*$. The edge $e = c_j y_1$ is in $E(H_1)$ for some $2 \leq j \leq r-1$. We next subdivide the edge e with a new vertex c_r to form the multigraph H_2, and then 'flip' some edges from c_2 to c_r in H_2 preserving 2-connectivity and increasing the degree of c_r up to d_r while maintaining the property that y_1 and y_1' are joined to different pieces of s. We use the following result to accomplish this.

Lemma 7.17 *[14, Corollary 2.17] Let $t \geq 3$ be an integer. Let H be a loopless 2-connected multigraph, $x, y \in V(H)$ and $xz_i \in E(H - y)$ for $1 \leq i \leq t$. If $t \geq d(y) - d(y, x) + 1$, then $H - xz_i + yz_i$ is loopless and 2-connected for some i, $1 \leq i \leq t$.*

We construct the new multigraph H_3 from H_2 as follows. If $d_r = 2$ then we put $H_3 = H_2$. If $d_r \geq 3$ then we use Lemma 7.17 to find a set of edges $S = \{c_2 z_i \in E(H_2) \,:\, 1 \leq i \leq d_r - 2\}$ such that $c_2 y_1' \notin S$ and $H_3 = H_2 - S + \{c_r z_i \,:\, 1 \leq i \leq d_r - 2\}$ is 2-connected and loopless. This is possible since $d_{H_2}(c_r)=2$, $d_{H_2}(c_r,c_2)\le 1$, and $d_{H_2}(c_2)=d'_2+d_r-2\ge d_r+d_r-2$. In H_3 we have $y_1c_r\in E(H_3)$, $y'_1c_r\notin E(H_3)$, $d_{H_3}(c_i)=d_i$ for $3\le i\le r$, and $d_{H_3}(c_2)=d'_2$. (Note that we could have used Lemma 7.15 directly to construct a 2-connected loopless detachment with the same degree sequence as H_3 from G_3+s plus one extra loop at s. The reason we go via H_1 and H_2 is to ensure that y_1 and y'_1 are adjacent to distinct pieces of s in H_3.)

Let F be a set of edges joining the components of G_2-K such that $d_F(v)=d_{G_2\oplus s}(s,v)$ for all $v\in V-K$ and such that the graph obtained from $(V-K,F)$ by contracting $C_2,\dots,C_r$ to $c_2,c_3,\dots,c_r$, respectively, is H_3. Since H_3 is 2-connected, each vertex c_i in H_3 has at least two distinct neighbours. Since H_3 is loopless, every edge of F which is incident to a component C_i, $2\le i\le r$, is incident to distinct components of G_2-K. Let y_i,y'_i be the neighbours of s in C_i as defined after Claim 7.13. Since we are free to interchange the end vertices of the edges of F within each component, C_i, for $2\le i\le r$ we may choose F to have the additional property that, for $2\le i\le r$, the two edges of F incident to y_i and y'_i join C_i to different vertices of $G-K-C_i$, which either belong to different components of $G-K-C_i$, or both belong to C_1. Furthermore, since y_1 and y'_1 are joined to different detached vertices in H_3, the two edges of F incident to y_1 and y'_1 join C_1 to different components of G_2-K-C_1.

We can now use Claim 7.12 to deduce that G_2+F is k-connected as in Subcase 2.1. Putting $F_0=E(G_2)-E(G)$ we deduce that $F_0\cup F$ is the required augmenting set of edges for G of size $\lceil d_{G+s}(s)/2\rceil=\lceil t(G)/2\rceil$. •

8 Algorithmic aspects and corollaries

In this section we discuss the algorithmic aspects of our results and also show that our main theorems imply (partial) solutions to a number of conjectures in this area.

8.1 Algorithms

The proofs of our min-max theorems (Theorems 4.17 and 7.7) are algorithmic and lead to a polynomial algorithm which finds an optimal augmenting set with respect to k for any l-connected input graph G and target $k\ge l+1$, provided $a_k(G)\ge 10(k-l+2)^3(k+1)^3$ (or $a_k(G)\ge 20k^3$, if $k=l+1$). As we shall see, the running time in this case can be bounded by $O(n^6)$, even if k is part of the input. Our algorithm for the general case first decides whether $a_k(G)$ is large, compared to k, or not. Since, by Lemma 2.10, $a_k(G)$ is large if and only if $d(s)$ is large in a k-critical extension $G+s$ of G, the first step is to create such an extension. If $a_k(G)$ is small then our algorithm performs an exhaustive search on all possible augmenting sets F with $V(F)\subseteq N(s)$ and outputs the smallest augmenting set which makes G k-connected. The number of possibilities depends only on k, since $|N(s)|$ is also small. We shall present the algorithm as a sequence of sub-algorithms. Most of the steps of these algorithms are easy to implement in polynomial time by network flow techniques.

8.1.1 CRITICAL EXTENSION

Input: A graph G and an integer $k \geq 1$.
Output: A k-critical extension $G+s$ of G.
Step 1. Add a new vertex s to G and $\max\{1, k-d(v)\}$ edges from s to each vertex v of G. (This gives a (k,s)-connected extension $G \oplus s$ of G by Lemma 2.4.)
Step 2. Delete edges incident to s greedily until the the remaining graph $G+s$ is a k-critical extension. (We check whether each edge deletion preserves (k,s)-connectivity using a max-flow computation.)

8.1.2 EXHAUSTIVE SEARCH

Input: A k-critical extension $G+s$ of a graph G.
Output: An optimal k-augmenting set for G.
For each set of edges F with $V(F) \subseteq N(s)$, check whether $G+F$ is k-connected. Choose the smallest such k-augmenting set.

The following lemma implies that the output of EXHAUSTIVE SEARCH is indeed an optimal k-augmentation for G.

Lemma 8.1 *Let $G+s$ be a (k,s)-connected extension of a graph G. Then there exists an optimal k-augmenting set F for G with $V(F) \subseteq N(s)$.*

Proof: Let $S = N(s)$ and let F be an optimal augmenting set with respect to k for which $c(F) = \sum_{uv \in F} |\{u,v\} - S|$ is as small as possible. Suppose $c(F)$ is positive and let $uv \in F$ be an edge with $\{u,v\} - S \neq \emptyset$. Since F is optimal, we have $\kappa(G+F-uv) = k-1$ and, by Lemma 4.4(c), it follows that $G+F-uv$ has precisely two k-cores (i.e. minimal k-deficient fragments) X,Y. Clearly, X and Y are k-deficient fragments in G. Thus, since $G+s$ is (k,s)-connected, we must have $S \cap X \neq \emptyset \neq S \cap Y$. Lemma 4.4(c) also implies that by taking $F' = F - uv + xy$ for a pair x,y of vertices with $x \in S \cap X$ and $y \in S \cap Y$ we have that $G+F'$ is k-connected. Now $|F'| = |F|$ and $c(F') < c(F)$, contradicting the choice of F. This proves that $c(F) = 0$ must hold, and hence the required augmentning set exists. •

It follows that, if $a_k(G)$ is small, then we only need to perform c_k k-connectivity tests, where $c_k = O(2^{\binom{a_k(G)}{2}})$ depends only on k, to find an optimal k-augmentation for G using CRITICAL EXTENSION and EXHAUSTIVE SEARCH. If $a_k(G)$ is large then our augmentation algorithm has several sub-algorithms, according to the different subcases in our proofs. In what follows we give a sketch of these algorithms to verify that they can be run in polynomial time. We do not attempt to work out the details of an efficient implementation.

8.1.3 CORES

Input: A $(k-1)$-connected graph $G = (V,E)$.
Output: The set $\mathcal{C}$ of all k-cores and the set $\mathcal{A}$ of all active k-cores in G.
For each non-adjacent pair $u,v \in V$ such that $\kappa(u,v) = k-1$, find the minimal (with respect to inclusion) sets X_u, X_v such that $u \in X_u$, $v \in X_v$ and $n(X_u) = k-1 = n(X_v)$. Let $\mathcal{C}'$ be the union of

the sets $\{X_u, X_v\}$ over all pairs u, v, and let $\mathcal{C}$ consist of the minimal members of $\mathcal{C}'$. Let $\mathcal{A} = \{X \in \mathcal{C} : \kappa(G - X) = k - 1 - |X|\}$.

Note that CORES can be used to test if a $(k-1)$-connected graph G is k-independence free by checking whether G has any active k-cores. We do not know if there is a polynomial algorithm to determine whether an arbitrary graph is k-independence free.

Cheriyan and Thurimella [4] give a polynomial algorithm for determining $b_k(G)$ for a $(k-1)$-connected graph G and finding all $(k-1)$-shredders K in G with $b_G(K) = b(G)$. We can use this to give a polynomial algorithm for finding an optimal k-augmentation of a $(k-1)$-connected k-independence free graph. Note, however, that it is unlikely that there exists an efficient algorithm to determine $b_k(G)$ for an arbitrary graph G. This follows since the problem of determining whether $b_k(G) \geq k$ for some $1 \leq k \leq |V|$ is NP-complete by [1].

8.1.4 INDEPENDENCE FREE AUGMENTATION

Input: A $(k-1)$-connected k-independence free graph G.
Output: An optimal k-augmenting set F for G with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$.
We first construct a k-critical extension $G + s$ of G using CRITICAL EXTENSION. We have $d(s) = t(G)$ by Corollary 3.2. We construct the required set F by finding a sequence of admissible splits at s (as in the proofs of Lemmas 3.7, 3.10 and 3.11 and Theorem 3.12) to give $G_1 + s$ with $d_{G_1+s}(s) \in \{3, b(G_1)\}$. We then put $F = F_1 \cup F_2$ where $F_1 = E(G_1) - E(G)$ and F_2 is the edge set of a tree with $V(F_2) = N_{G_1+s}(s)$.

We next give algorithms for finding optimal k-augmentations for a graph G when $a_k(G)$ is large. The first two algorithms determine whether G has a *dominating shredder*, that is to say a $(k-1)$-shredder K with $\hat{b}_G(K) = \hat{b}(G)$ and $2\hat{b}(K) - 2 \geq t(G)$, and find an optimal k-augmenting set when G does have such a shredder.

8.1.5 DOMINATING SHREDDER

Input: A k-critical extension $G + s$ of an l-connected graph G for which $d_{G+s}(s) \geq k(k - l + 1) + 2$.
Output: We find a dominating shredder K in G or deduce that no such shredder exists.
We construct a family $\mathcal{K}$ of $(k-1)$-shredders in such a way that $|\mathcal{K}|$ is polynomial in $|V|$ and, if there is a dominating shredder K in G, then $K \in \mathcal{K}$. Once we have $\mathcal{K}$, we complete the algorithm by computing $\hat{b}(K')$ for all $K' \in \mathcal{K}$.
For each triple x, u, v, where $x \in V$ and $u, v \in N_{G+s}(s) - x$, first we try to split off all copies of the edges sx (if there are any). Suppose that all copies can be split off, and let the resulting graph be $G_x + s$. Then we try to find a set $\{P_1, P_2, \ldots, P_{k-1}\}$ of openly disjoint uv-paths in G_x. If we succeed, then we let $Q(x, u, v) = \cup_{i=1}^{k-1} P_i$, $\mathcal{C}(x, u, v) = \{C : C \text{ is a component of } G - Q(x, u, v)\}$,

$$\mathcal{K}_1(x, u, v) = \{N_G(C) : C \in \mathcal{C}(x, u, v) \text{ and } n_G(C) = k - 1\},$$

$$\mathcal{K}_2(x, u, v) = \{N_G(C) \cup \{q\} : C \in \mathcal{C}(x, u, v),\, n_G(C) = k - 2,\, q \in Q - \{u, v\}\}.$$

Let $\mathcal{K}$ be the union of the sets $\mathcal{K}_1(x, u, v) \cup \mathcal{K}_2(x, u, v)$ over all choices of x, u, v. Clearly, $|\mathcal{K}| \leq \binom{n}{3} n^2$.

Lemma 8.2 *If G has a dominating shredder K then $K \in \mathcal{K}$.*

Proof: Suppose there is a $(k-1)$-shredder K with $d(s) \le 2\hat{b}(G) - 2 = 2\hat{b}(K) - 2$. Then Lemma 6.1 implies that $|N(s) \cap K| \le 1$, and if $x \in N(s) \cap K$ then $d_G(x) = k - d(s,x)$, $\hat{b}(K) = b(K) + d(s,x)$, and we can split off all copies of sx (in any order) by admissible splittings. By splitting off these copies $d(s)$ is reduced by $2d(s,x)$ and $\hat{b}(K)$ is reduced by $d(s,x)$. Hence $d(s) \le 2b(K) - 2$ holds in the resulting graph G_x. This implies that K has at least two leaf components C, C' in G_x. By Lemma 2.14 there exist $k-1$ openly disjoint paths from $u \in N(s) \cap C$ to $v \in N(s) \cap C'$. Clearly, $Q \subseteq K \cup C \cup C'$ and $K \subset Q$ hold, where Q is the union of the vertex sets of these paths. Moreover, since $x \in K$, the components of $G - K$ and $G_x - K$ are the same. Lemma 6.1 also implies that $G - K$ has at least $2k + 1 \ge 3$ components D with $d_{G+s}(s, D) \le 2$, and hence $n_G(D) \ge k - 2$. Thus there is a component D' of $G - K$, which is a component of $G - Q$, and satisfies that either $K = N_G(D')$ or $K = N_G(D') + q$ for some $q \in Q - \{u, v\}$.

It follows that for some triple x, u, v we have $K \in \mathcal{K}_1(x,u,v) \cup \mathcal{K}_2(x,u,v)$, as required. •

Note that if DOMINATING SHREDDER finds a dominating shredder K when $l = k-1$, then we have $d(s,K) = 0$ and $b_G(K) = \hat{b}(K)$ by Theorem 4.1.

8.1.6 DOMINATING SHREDDER AUGMENTATION

Input: A k-critical extension $G+s$ of an l-connected graph G for which $d_{G+s}(s) \ge k(k-l+1)+2$, and a dominating shredder K for G.
Output: An optimal augmenting set F for G with $|F| = \hat{b}(G) - 1$.
We construct F by splitting off all edges from s to K and then adding a forest augmentation, as described in Lemma 6.5 and after Lemma 6.1.

8.1.7 LARGE AUGMENT BY ONE

Input: A k-critical extension $G+s$ of a $(k-1)$-connected graph $G = (V,E)$ for which $d_{G+s}(s) \ge 20k^3 + 1$.
Output: An optimal augmenting set F for G with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$.
We use DOMINATING SHREDDER, DOMINATING SHREDDER AUGMENTATION, CORE, and the proof techniques of Lemmas 4.2, 4.15 and 4.16 to find a saturating set of edges F_1 such that either F_1 is an optimal k-augmenting set for G with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$, or $G + F_1$ is k-independence free and has no dominating shredder. In the former case we put $F = F_1$. In the latter case we use INDEPENDENCE FREE AUGMENTATION to find a k-augmenting set F_2 for $G + F_1$ and put $F = F_1 \cup F_2$.

Note that when we increase the number of passive k-cores by making an active k-core passive in LARGE AUGMENT BY ONE, we do not need to compute $T(G)$. We choose an arbitrary active k-core B and, if we fail to make B passive (which means $B \cap T(G) \neq \emptyset$), then we choose a different active k-core.

8.1.8 LARGE AUGMENT

Input: A k-critical extension $G+s$ of a graph $G = (V,E)$ for which $d_{G+s}(s) \ge 10(k-l+2)^3(k+1)^3 + 1$.
Output: An optimal augmenting set F for G with $|F| = \max\{\hat{b}(G) - 1, \lceil t(G)/2 \rceil\}$.

We use DOMINATING SHREDDER, DOMINATING SHREDDER AUGMENTATION, and the proof techniques of Lemmas 6.7 and 7.5 to find a saturating set of edges F_1 such that either F_1 is an optimal k-augmenting set for G with $|F_1| = \max\{\hat{b}(G)-1, \lceil t(G)/2 \rceil\}$, or $G+F_1$ is $(k-1)$-connected, has no dominating shredder, and $d_{G+s}(s) - 2|F_1| \geq 20k^3+1$, or $G+F_1$ has a k-augmenting set F_2 of size $\lceil t(G+F_1)/2 \rceil$ (which can be constructed using detachments as in the proof of Case 2 of Theorem 7.7). In the first case we put $F = F_1$. In the second case we use LARGE AUGMENT BY ONE to find a k-augmenting set F_3 for $G+F_1$ of size $\lceil t(G+F_1)/2 \rceil$ and put $F = F_1 \cup F_3$. In the third case we put $F = F_1 \cup F_2$.

8.1.9 AUGMENT

Input: An l-connected graph G and an integer $k > l$.
Output: An optimal k-augmenting set F for G.
Construct a k-critical extension $G+s$ for G using CRITICAL EXTENSION. If $k = l+1$ and $d_{G+s}(s) \geq 20k^3+1$ then apply LARGE AUGMENT BY ONE. If $l \leq k-2$ and $d_{G+s}(s) \geq 10(k-l+2)^3(k+1)^3+1$ then apply LARGE AUGMENT. Otherwise apply EXHAUSTIVE SEARCH.

As noted above, most of the steps of the above algorithms are easy to implement in polynomial time by network flow techniques. The only exception is finding the required loopless 2-connected detachments as in the proof of Case 2 of Theorem 7.7. We shall not discuss this in this paper but remark that there is a simple algorithm which finds H, if it exists, and we also have a similarly simple and efficient algorithm which finds H_3, when H does not exist.

Before stating our bound on the running time of our algorithm AUGMENT, we note that by inserting a preprocessing step, which works in linear time, we can make the input graph sparse, and hence reduce the running time, as follows. Let $G = (V,E)$ and k be the input of our problem. Let $n = |V|$ and $m = |E|$. It was shown in [3] and [19] that $G = (V,E)$ has a spanning subgraph $G' = (V,E')$ with $|E'| \leq k(n-1)$ satisfying $\kappa(u,v,G') \geq \min\{k, \kappa(u,v,G)\}$ for each pair $u,v \in V$. It can be seen that by replacing G by G' we do not change the k-deficient fragments (or their deficiencies) and that for any augmenting set F the graph $G+F$ is k-connected if and only if $G'+F$ is k-connected. Thus we can work with G' and assume that $m = O(kn)$. Note also that $d(s) = O(kn)$ in any k-critical extension $G+s$ of G. By using these facts and efficient network flow algorithms for the basic operations (such as finding admissible splittings, checking whether an edge is k-critical, etc) we can conclude with the following theorem.

Theorem 8.3 *Given an l-connected graph G and a positive integer k, our algorithm AUGMENT finds an optimal k-augmenting set. If $a_k(G) \geq 10(k-l+2)^3(k+1)^3$ then the running time is $O(kn^5)$. Otherwise the running time is $O(c_k n^3)$.*

We close this subsection by noting that we can also use the theory developed in this paper to derive a near optimal algorithm for the vertex connectivity augmentation problem which is similar to the one given in [13].

8.1.10 NEAR OPTIMAL AUGMENT

Input: An l-connected graph G and an integer $k > l$.
Output: A k-augmenting set F for G with $|F| \leq a_k(G) + \frac{1}{2}k(k-l+1)+1$.

Construct a k-critical extension $G+s$ for G using CRITICAL EXTENSION. We first suppose that $d_{G+s}(s) \geq k(k-l+1)+2$. We use DOMINATING SHREDDER to determine if G has a dominating shredder. If it does then we use DOMINATING SHREDDER AUGMENTATION, to find an optimal k-augmenting set for G. If G does not have a dominating shredder then, by Lemma 5.3, we can split off edges from s such that, in the resulting graph G_1+s, we have either $d_{G_1+s}(s) > k(k-l+1)+2$ and G_1 has a dominating shredder, or $k(k-l+1)+1 \leq d_{G_1+s}(s) \leq k(k-l+1)+2$. In the former case we can use DOMINATING SHREDDER and DOMINATING SHREDDER AUGMENTATION, to find an optimal k-augmenting set for G. In the latter case, Lemma 2.7 implies that we may construct a minimal augmenting set F_1 for G_1 with $V(F_1) \subseteq N_{G_1+s}(s)$. Let $F = F_1 \cup F_2$, where $F_2 = E(G_1) - E(G)$. Lemma 2.9 implies that $|F_2| \leq d_{G_1+s}(s) - 1$. Since $t(G) = d_{G+s}(s)$ and $t(G_1) = d_{G_1+s}(s)$ we have $|F| \leq \frac{1}{2}t(G) + \frac{1}{2}d_{G_1+s}(s) \leq a_k(G) + \frac{1}{2}k(k-l+1)+1$. Finally, if $d_{G+s}(s) < k(k-l+1)+2$, then we construct a minimal augmenting set F for G with $V(F) \subseteq N_{G+s}(s)$. Lemma 2.10 implies that $|F| \leq a_k(G) + \frac{1}{2}d_{G+s}(s) \leq a_k(G) + \frac{1}{2}k(k-l+1)+1$.

The running time of NEAR OPTIMAL AUGMENT is $O(n^6)$.

8.2 Corollaries

Our main results (Theorems 4.17 and 7.7) imply (partial) solutions to several related conjectures. The extremal version of the connectivity augmentation problem is to find, for given parameters n,k,t, the smallest integer m for which every k-connected graph on n vertices can be made $(k+t)$-connected by adding m new edges. Several special cases of this problem were solved in [17] and it was conjectured that (at least if n is large enough compared to k) the extremal value of m for $t \geq 2$, $k \geq 2$ is $\lceil nt/2 \rceil$ (or $\lfloor nt/2 \rfloor$, depending on the parities of n,k,t). Since $\hat{b}(G)-1 \leq n$, the min-max equality of Theorem 7.7 shows that if n is large enough and $t \geq 2$ then $a_k(G)$ is maximised if and only if G is (almost) k-regular. This proves the conjecture (when n is large compared to k), by noting that such (almost) regular graphs exist for $k \geq 2$.

A different version of this problem, when the graphs to be augmented are k-regular, was studied in [9]. It was conjectured there that if G is a k-regular k-connected graph on n vertices, and n is even and large compared to k, then G can be made $(k+1)$-connected by adding $n/2$ edges. If G is k-regular, $b(K) \leq k$ for any cut of size k. Thus if n is large enough, we have $\max\{b(G)-1, \lceil t(G)/2 \rceil\} = n/2$. Now the conjecture follows from Theorem 4.17.

A similar question is whether $a_k(T) = \lceil (\sum_{v \in V(T)} (k-d(v))^+)/2 \rceil$ holds when graph T is a tree, where $x^+ = \max\{0,x\}$ for all integers x. It is known that the minimum number of edges needed to make a tree k-edge-connected (or an arborescence k-edge- or k-vertex-connected) is determined by the sum of the (out)degree-deficiencies of its vertices. As above, using the fact that $\hat{b}(G)-1 \leq n$, Theorem 7.7 implies (when n, and hence also $a_k(T)$, is large compared to k) that if $k \geq 3$ then $a_k(T) = \lceil t(T)/2 \rceil$. That is, $a_k(T)$ is determined by the total deficiency of a family of pairwise disjoint subsets of $V(T)$. Since T is a tree, each member X of this family induces a forest. This implies that there exists a vertex $v \in X$ with $k-d(v) \geq k-n(X)$. Therefore we can find a family consisting of singletons with the same total deficiency. This yields an affirmative answer to our question provided $k \geq 3$ and n is large compared to k. Note that the answer is negative for $k=2$.

Frank and Jordán [8, Corollary 4.8] prove that every $(k-1)$-connected graph $G=(V,E)$ can be made k-connected by adding a set F of new edges such that (V,F) consists of vertex-disjoint paths. They conjectured that such an F can be found among the optimal augmenting sets as well.

We can verify this, provided $a_k(G)$ is large enough. In this case we may use the min-max formula of Theorem 4.17. If $a_k(G) = \lceil t(G)/2 \rceil$ then an optimal augmenting set is a collection of vertex-disjoint paths of length one or two. If $a_k(G) = b(G) - 1$, then a careful analysis of the forest augmentation method shows that we can find an optimal augmenting set F satisfying $d_F(v) \leq 2$ for all $v \in V$. Since F is a forest, it induces vertex-disjoint paths, as claimed.

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