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ABSTRACT

In a setting with information asymmetry and a tradable value-weighted market index, ambiguity averse investors hold undiversified portfolios, and assets have nonzero alphas. But when a passive fund offers the risk-adjusted market portfolio (*RAMP*), whose weights depend on information precisions as well as market values, all investors hold the same portfolios as in the economy without model uncertainty and thus engage in index investing. So *RAMP* improves participation and risk sharing. Asset alphas are zero with *RAMP* as pricing portfolio. *RAMP* can be implemented by a fund of funds even if no manager individually has sufficient knowledge to do so.

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1 Introduction

Index investing has long been recommended by distinguished practitioners, such as Jack Bogle and Warren Buffett, and leading scholars, such as William Sharpe and Harry Markowitz, as an attractive means for retail investors to optimize risk and return. In recent years, investors have increasingly followed this recommendation.[1] Market indexes are also important in asset pricing: they are used as pricing portfolios for determining assets' risk premia.

The design of the index is central to how well it serves these roles of facilitating stock market investment and of pricing assets in the cross section. In the Capital Asset Pricing Model (Sharpe 1964; Lintner 1965), the *Value-Weighted Market Portfolio* (*VWMP*) serves both these roles. In equilibrium, all investors hold the *VWMP*, and with the *VWMP* as the pricing portfolio, correctly priced assets have zero alphas.

In practice, however, investors have heterogeneous beliefs and are imperfectly rational, violating the CAPM assumptions. Investors observe different signal values, and their signals can differ in precision. So even if informed investors were rational, if there is some noise in the system, investors would not hold the same portfolio. Indeed, Admati (1985) shows that in such a setting, the *VWMP* is not mean-variance efficient for any information set.

Furthermore, laboratory and field evidence indicates that investors usually face model uncertainties (i.e., they do not know exactly financial market parameters). This can cause investors to perceive the stock market as very risky, potentially discouraging market participation and increasing risk premia.[2] In the absence of model uncertainty, investors glean information from asset prices, which partially aggregates other investors' information.[3] This reduces the conditional variance of asset prices, and encourages market participation.

However, if investors face model uncertainty (also known as ambiguity), they may

[1]For example, the *Wall Street Journal* article "Index Funds Are the New Kings of Wall Street" indicates that "Funds that track broad U.S. equity indexes hit $4.27 trillion in assets as of Aug 31, 2019, according to research firm Morningstar Inc., giving them more money than stock-picking rivals for the first-ever monthly reporting period. Funds that try to beat the market had $4.25 trillion as of that date."

[2]Dimmock, Kouwenberg, and Wakker (2016), Dimmock et al. (2016), and Bianchi and Tallon (2018) measure ambiguity attitudes using artificial events based on Ellsberg urn experiments. Anantanasuwong et al. (2019) directly elicit ambiguity attitudes using an incentivized survey. These studies provide evidence suggesting that ambiguity aversion reduces market participation.

[3]Investor learning from asset prices has been extensively studied in rational expectations equilibrium models, such as Grossman and Stiglitz (1980), Hellwig (1980), and Admati (1985).

find it much harder to extract information from asset prices, since prices depend upon unknown financial market parameters, not just private information. Investors' aversion to uncertainty may cause them to take zero positions in assets they are ambiguous about. As we show, such non-participation may obstruct the aggregation of information in asset prices, and undermine the roles of the *VWMP* as a common holding of investors and as a pricing portfolio.

These points suggest that in a financial market with information asymmetry and ambiguity averse investors, the *VWMP* does not facilitate investor stock market participation, and that assets are not priced according to their loadings on this index.[4] This raises the question of whether a different index can be designed that investors will use and benefit from? If so, how is it constructed? Do investors with heterogeneous information signals and ambiguity aversion all invest in this index? Can assets be priced relative to such an alternative index?

We address these questions in a noisy rational expectations equilibrium setting with ambiguity averse investors who do not know some parameters of the financial market and optimize under worst-case assumptions (Gilboa and Schmeidler 1989). We introduce a new index portfolio, whose weights are functions of the financial market parameters. We then introduce the possibility that there is a passive index fund that knows the true financial market parameters and commits to offering this index portfolio.[5]

The key to the index design is to have asset weights depend appropriately upon the precisions of investor private signals and the precisions of random supply shocks, as well as the market values of assets. Specifically, the index is constructed as the optimal portfolio that an investor who does not have private signals about asset payoffs would choose in a setting without model uncertainty. Relative to the *VWMP*, such a new index has lower investment in more volatile assets (conditional on asset prices). In other words, it is a defensive (low volatility) investing strategy. Therefore, holding this new index is less risky than holding the *VWMP*. We call this index the *Risk-Adjusted Market*

[4]We formally show the dysfunctions of the *VWMP* in Appendix B. In equilibrium, ambiguity averse investors do not hold the *VWMP*, since they perceive that the *VWMP* is extremely risky in the worst-case scenario. So the *VWMP* fails to help ambiguity averse investors with index investing. In addition, since the *VWMP* is not mean-variance efficient for public information, and ambiguity averse investors do not participate in the markets of assets they are ambiguous about, assets have non-zero alphas relative to the *VWMP*.

[5]In the baseline model, we abstract away agency problems and fund fees to focus on the question whether a well-designed index portfolio can mitigate the adverse effects of ambiguity aversion. In Section 4, we consider how such an index investing strategy can be implemented.

Portfolio (*RAMP*).

Investors have common knowledge of how *RAMP* is constructed as a function of financial market parameters. However, investors who are ambiguous about the financial market do *not* know the exact composition of *RAMP* (since they do not know some financial market parameters). *RAMP* does not rely on private information about asset payoffs, so *RAMP* is still (in this sense) a passive index. In particular, *RAMP* can be viewed as a "smart beta" investing strategy, a general approach that recently has gained popularity in investment practice.[6]

However, it is not obvious whether a passive fund offering *RAMP* can attract ambiguity averse investors. *RAMP* may potentially be perceived to be extremely risky given some investors' subjective beliefs about the financial market. Also, different investors have different subjective beliefs about the structure of the financial market and different private signals about asset payoffs. So it is unclear whether one passive fund can be a useful component of the optimal portfolios of all investors.

The main result of this paper is that there is an equilibrium in which investors' asset holdings are exactly the same as those in the economy without model uncertainty. All investors, whether facing model uncertainty or not, hold exactly one share of *RAMP* by delegating the passive component of their portfolios to the index fund. Investors additionally hold investor-specific positions based upon their private information signals. So partial delegation by investors of portfolio choice to the index fund solves the problem of ambiguity aversion and nonparticipation.

A surprising property of the equilibrium that we identify is that investors hold the same number of shares of the fund, provided that they have the same risk tolerance. In equilibrium, investors with the same risk tolerance may disagree heavily for two reasons. First, investors who have different private signals about asset payoffs hold different beliefs. Furthermore, a pair of investors who have non-overlapping subjective belief supports about financial market parameters disagree about the composition of *RAMP*. So investors form different estimates of return and risk of *RAMP*. Nevertheless, these heterogeneity do not lead to different index investments. Furthermore, we show that *RAMP* is the correct portfolio for pricing all the assets in the capital market. In equilibrium, assets' alphas relative to *RAMP* are zero, implying a new version of CAPM security market line under information asymmetry and ambiguity aversion.

[6]Smart beta strategies seek to passively follow indexes that use alternative weighting schemes, such as making weights a function of volatility, rather than weighting solely by traditional market capitalization.

The key intuition for why investors hold *RAMP* as the passive component of their portfolios derives from a new separation theorem that applies in the setting with no model uncertainty. Without model uncertainty, there is a rational expectations equilibrium in which any investor's equilibrium risky asset holding can be decomposed into two components. The first is just *RAMP*, which is a common deterministic component of all investors' equilibrium holdings. The second is the investor's *information-based portfolio*, which includes a non-zero position in an asset if and only if the investor receives a private signal about the asset.

Our new separation theorem differs from the separation theorem derived in the literature in that an investor's optimal portfolio is separated by her conditionally independent signals of asset payoffs.[7] We therefore call it the *Information Separation Theorem*. Specifically, *RAMP* is an investor's equilibrium holding, when she trades based only on public information. In contrast, an investor's information-based portfolio is her optimal portfolio based on her private signals and conditional on asset prices (as exogenous parameters). So any investor's information-based portfolio is independent of public information.

The Information Separation Theorem provides new insight into why the investors' equilibrium asset holdings in the setting where investors are subject to model uncertainty are the same as in the economy without model uncertainty. Consider the strategy profile in which each investor holds exactly one share of the fund, and additionally holds her information-based portfolio (which could be a nullity). Given that all other investors behave as prescribed, no investor has an incentive to deviate.

The key insight is that the fund provides investors with a channel to share risks. Consider, for example, an investor and a vector of parameters of a financial market that is possible according to her subjective belief. Given the value of this vector, the investor would be in a possible world without model uncertainty. Since all other investors are holding one share of the fund and their own information-based portfolios, they are effectively holding the same portfolios as they would in the rational expectations equilibrium in this world, even if they perceive different values of the vector. Hence, the market clearing condition implies that the pricing function is the same as the one in the

[7] This separation holds generally under mean-variance preferences. It derives from the model assumption that investors are price takers, so that an investor's private signal and the price signal are conditionally independent. By Bayes' rule, it follows that investors can optimally construct a portfolio based on each of her signals and then sum all the constructed portfolios together to get the equilibrium asset holdings.

rational expectations equilibrium. Therefore, if the investor knew the parameter values that characterize this possible world, her optimal portfolio choice would consist of *RAMP* and her own information-based portfolio.

Because the fund offers *RAMP* in any possible world, without the knowledge of the exact world, the investor's optimal portfolio choice is to hold one share of the fund together with her own information-based portfolio. Investors may disagree with each other about financial market parameters (owing to having different prior supports of their distributions) and thus the fund's composition and risks. Nevertheless, by holding one share of the fund along with their own information-based portfolios, each investor knows that her position is, in the proposed equilibrium, optimal under each possible value of the financial market parameter vector. When other investors trade as prescribed, the optimal non-informational component of the investment strategy, assuming that the investor knows the financial market parameters, is independent of the particular values of these parameters. This implies that this portfolio component maximizes the investor's utility in the worst-case scenario, or more technically, solves the investor's max-min utility. Put differently, a strong max-min property holds in the equilibrium. Therefore, ambiguity averse investors all hold the same diversified portfolio, *RAMP*, by holding the passive fund.

The above argument makes clear that an investor's willingness to hold the fund relies crucially on equilibrium reasoning. Each investor understands that all other investors hold the fund and their own information-based portfolios. Indeed, if an ambiguity averse investor thought that other investors were not choosing *RAMP* as their non-informational positions, she would not in general find it optimal to hold the fund.

In other words, the fact that investors hold *RAMP* is not just a consequence of benefits to diversification — benefits which are present even in a partial equilibrium setting. The willingness to hold *RAMP* depends on more than this — an understanding that other investors will, in equilibrium, also hold *RAMP*. Hence, it is only in equilibrium, and by virtue of the Information Separation Theorem, that we can conclude that ambiguity averse investors optimally hold the fund offering *RAMP*.

Notably, *RAMP* does not solve the problem of model uncertainty by eliminating it; it solves the problem by providing an attractive vehicle for risk sharing.[8] Although

[8] Mele and Sangiorgi (2015) show that in a model without a passive fund offering *RAMP*, ambiguity averse investors acquire costly information about model parameters, which helps reduce model uncertainty. In comparison with their finding, our results indicate that delegation can serve as a substitute for information acquisition as a means of addressing ambiguity aversion.

investors agree to hold the fund in equilibrium, they disagree on its composition, since investors have heterogeneous uncertainties about the financial market parameters that *RAMP* is a function of. So even when *RAMP* is offered to investors, model uncertainty in equilibrium remains.

Furthermore, compared with a passive fund offering the *VWMP*, a *RAMP* fund encourages more investors to participate in the financial market, increasing the demand for risky assets. Hence, for a given pair of asset payoffs and random supply shocks, the financial market with a *RAMP* fund has higher asset prices and the same price volatility. Therefore, the equilibrium asset risk premia in the financial market with a *RAMP* fund are lower.

Another consequence of the Information Separation Theorem is that all assets have zero alphas relative to *RAMP*. The fact that an investor with no private information optimally holds *RAMP* (through the fund) implies that *RAMP* is mean-variance efficient for all possible values of the ambiguous parameters. So the CAPM security market line holds with *RAMP* as the pricing portfolio regardless of the actual values of the financial market parameters. Because the pricing portfolio does not depend on the realization of the random supply shock, and the weight of each asset in the pricing portfolio are conditional on asset prices, which are publicly observable, the portfolio is potentially observed by an econometrician. This makes the model empirically testable.

Given the potential benefits of *RAMP* to investors, it is important to consider whether the portfolio is implementable in practice as an index fund. One question is whether implementation requires the existence of a single fund that knows all parameters of the financial market, enabling it to offer *RAMP*. We show that this is not a necessary condition for *RAMP* to be made available to investors. Suppose instead that no single agent knows all parameters of the financial market. Nevertheless, as long as each parameter is known by some agents, knowledgeable agents can start competitive specialized funds (e.g., geography-specific funds or industry-specific funds). This enables investors at large to hold *RAMP* by holding all the specialized funds. To put this another way, it is possible to offer *RAMP* by creating a fund of funds. This can enable the formation of *RAMP* funds and improve market participation and risk sharing.

Alternatively, if private agents do not have the knowledge about financial markets needed to offer a *RAMP* fund, government could potentially collect the needed information and make it public. For either government or private parties, the cost of collecting relevant information about the financial market have been declining as information processing technology and data sets improve.

Another practical challenge for the implementation of *RAMP* is helping naïve investors understand its benefits. This suggests that improving financial literacy can also help investors obtain the benefits of market participation and risk sharing via *RAMP*.

Our findings contribute to the growing literature on the economic consequences of index investment. Chabakauri and Rytchkov (2016) find that the introduction of index trading increases volatilities and correlation of stock returns and that such an effect arises from the improved risk sharing. Bond and García (2018) show that as indexing becomes cheaper, indexing increases while individual asset trading decreases, and aggregate price efficiency decreases while relative price efficiency increases. Baruch and Zhang (2018) prove that it is optimal for nonindex investors to index and establish a version of CAMP in a very different economic setting.

All these papers assume that some investors are constrained to make index investing. In contrast, our paper assumes that all investors can freely employ index investing strategy and analyzes how a newly designed market index affects portfolio choice and asset pricing in financial markets with information asymmetry and ambiguity averse investors.

Past literature has analyzed extensively the extent to which ambiguity aversion hinders market participation and leads to inefficiency. Early work mainly considers investor uncertainty about the first moment of asset payoffs and study market participation decisions in partial equilibrium frameworks.[9] Recent papers, such as Condie and Ganguli (2011), Condie and Ganguli (2017), and Illeditsch, Condie, and Ganguli (2019), study the effects of investor ambiguity about the characteristics of private or public information in rational expectations equilibrium settings, and show that asset prices may be informationally inefficient. Our paper differs from this literature in considering investor model uncertainty about general parameters that determine asset payoff distributions in a rational expectations equilibrium setting, and in studying index investing as a way to address ambiguity aversion.

A very different version of the CAPM has been derived in somewhat similar model

[9]See, e.g., Bossaerts et al. (2010), Cao, Wang, and Zhang (2005), Dow and Werlang (1992), Easley and O'Hara (2009), Easley and O'Hara (2010), Epstein and Schneider (2010), and Cao, Han, Hirshleifer, and Zhang (2011)). An exception is Epstein and Schneider (2008) who assume that investors are uncertain about their signals' qualities and so dislike assets for which information quality is poor. These induce ambiguity premia. Another exception is Watanabe (2016), who assumes that investors are ambiguous about the mean of the asset's random supply shock. The focus of Watanabe's paper is on market fragility.

setups where all investors are perfectly rational (see, for example, Easley and O'Hara (2004), Biais, Bossaerts, and Spatt (2010), and the online appendix of Van Nieuwerburgh and Veldkamp (2010)). These papers do not separate investor equilibrium asset holdings and do not obtain zero-alphas conditional on public information (using the *VWMP* as the pricing portfolio) for correctly priced assets. Instead, in these models, the market portfolio for CAPM pricing is the ex-post total supply of the risky assets, the sum of the endowed risky assets and the random supply of risky assets. This market portfolio is mean-variance efficient conditional on the average investor's information set, and so the CAPM security market line holds from the perspective of the average investor.

The version of the CAPM security market line we derive differs in that the pricing portfolio is *RAMP*, which is determined ex ante (prior to the realization of the random supply shocks), and in that risk premia are conditional only upon the public information (market prices). This makes the market portfolio more directly observable to an econometrician. Furthermore, since *RAMP* is mean-variance efficient for any possible parameters of the financial market, the CAPM security market line holds even when investors are ambiguity averse.

Our setting endogenizes investor trust in fund managers (Gennaioli, Shleifer, and Vishny 2015). An insight provided by our approach is that inducing investors to make index investing requires more than investor trust in the honesty and superior knowledge of fund managers about the financial market. It is crucial that investors foresee an equilibrium in which other investors also trust the fund managers and trade accordingly. Off equilibrium, an investor would not be willing to hold the fund, even if she trusted the fund manager.

In a working paper, Li and Wang (2018), a representative investor faces model uncertainty about the financial market and so is uncertain about the composition of the efficient portfolio offered by a fund based on the fund's knowledge of the financial market. They call such an uncertainty the "delegation uncertainty." In their partial equilibrium setting where there is no risk sharing, the delegation uncertainty causes the investor to partially delegate, leading to higher CAPM alphas. By contrast, in our model with *RAMP*, delegation uncertainty is endogenously eliminated by risk sharing among investors: holding one share of the fund and the information-based portfolio is optimal to an investor in every possible world in her subjective belief support, when other investors are employing the same investment strategy. So an investor is not concerned about the exact asset holdings through the fund, which vary across possible worlds in her belief support.

2 The Financial Market

A continuum of investors with measure one, who are indexed by i and uniformly distributed over $[0,1]$ trade assets at date 0 and consume at date 1.

Assets. At date 0, each investor i can invest in a riskfree asset and $N \geq 2$ risky assets.[10] At date 1, the riskfree asset pays r units, and risky asset n pays f_n units of the single consumption good. In addition to trading directly, investors can also hold individual risky assets through a passive fund that commits to offering a portfolio X, which is an N-dimension column vector with the n^{th} element being the shares of the n^{th} risky asset in X.

Letting D_i be the vector of shares of the risky assets held by investor i, and d_i (a scalar) the shares of the fund held by investor i, investor i's effective risky asset holding is $d_i X + D_i$.

Return Information. Let $F = (f_1, f_2, \ldots, f_N)'$ be the vector of risky assets' returns. We assume that all investors share a common uniform improper prior of F, and so no investor has prior information about any risky asset's return.[11] Each investor i receives a vector of private signals S_i about asset returns, $S_i = F + \epsilon_i$, where F and ϵ_i are independent; and ϵ_i and ϵ_j are also independent. Each ϵ_i is normally distributed, with mean zero and precision matrix $\mathbf{\Omega}_i$. Since $\mathbf{\Omega}_i$ is the inverse of the variance-covariance matrix of ϵ_i, it is symmetric and positive definite. Note that we do not require $\mathbf{\Omega}_i$ to be diagonal; hence, investor i's private signal about asset n's payoff may be correlated to that of asset k.

Letting $\mathbf{\Sigma}$ be the matrix of the average precision of private signals, we have

$$\mathbf{\Sigma} = \int_0^1 \mathbf{\Omega}_i \mathrm{d}i. \tag{1}$$

In addition, as is standard, the independence of the errors implies that in the economy as a whole investor private signal errors average to zero, so that the equilibrium pric-

[10] To simplify notation, we assume that the number of risky assets N is common knowledge. However, our results hold in an extension where investors are also ambiguous about the total number of risky assets.

[11] The uniform improper prior assumption is for simplicity. In Section 5.1, we study the case in which investors have (potentially uncertain) normal priors of F. We show that with a modified portfolio offered by the passive fund, our results still hold.

ing function does not depend on the error realizations (though it does depend on their distributions). Therefore, we have

$$\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i = \boldsymbol{\Sigma} F. \tag{2}$$

Random Supply. Let Z denote the random vector of supplies of all risky assets. We assume that Z is independent of F and of ϵ_i (for all $i \in [0,1]$). We further assume that Z is normally distributed with mean 0 and the precision matrix $\boldsymbol{U}$. Here, $\boldsymbol{U}$ is also symmetric and positive definite.

Model Uncertainty and Ambiguity Aversion. Taking the riskfree asset to be the numeraire, let P be the price vector of the risky assets. Also, let $W_i = (w_{i1}, w_{i2}, \ldots, w_{iN})'$ be the endowed shareholdings of investor i. We assume that the aggregate endowments of shares of each stock are strictly positive; that is, $W = \int_0^1 W_i \mathrm{d}i \gg 0$. Then investor i's final wealth at date 1 is

$$\Pi_i = r \left[W_i' - \left(d_i X' + D_i' \right) \right] P + \left(d_i X' + D_i' \right) F. \tag{3}$$

The first term is the return of investor i's investment in the riskfree asset, and the second term is the total return from her investments in risky assets.

Since investor i is risk averse, if she knows all model parameters, at date 0 she maximizes a CARA expected utility function,

$$\mathbb{E}_i u(\Pi_i) = \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{4}$$

The expectation in equation (4) is taken based on investor i's information about asset returns. Since the common prior about asset returns is uninformative, any investor i's information consists of the equilibrium price vector and the realization of a private signal S_i only.

However, investor i may be subject to model uncertainty. We assume that all parameters are common knowledge among investors, except the average private signal precision $\boldsymbol{\Sigma}$, the random supply shock precision $\boldsymbol{U}$, and the aggregate endowments of shares W. Therefore, we characterize a financial market, or a model, by $m = (\boldsymbol{\Sigma}, \boldsymbol{U}, W)$, and the set of all possible financial markets is denoted by $\mathcal{M}$. Let the nonempty set

$\mathcal{M}_i \subseteq \mathcal{M}$ be the support of investor i's subjective belief over possible models. Then, when $\mathcal{M}_i$ is not a singleton, we say that investor i is subject to model uncertainty.[12]

We further assume that when investor i is subject to model uncertainty, she is ambiguity averse. Hence, she chooses an investment strategy (d_i, D_i) to maximize the infimum of her CARA utility. Formally, each investor i's decision problem is[13]

$$\max_{d_i, D_i} \inf_{m_i \in \mathcal{M}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{5}$$

We assume that the passive fund knows the true model but does not have any private information about asset payoffs. Hence, the fund commits to offering the portfolio X based only on its knowledge of the model and the asset prices.[14]

Equilibrium. In our model, an investor i's investment strategy is a mapping from her subjective belief support $\mathcal{M}_i$, her private signals S_i, and asset prices P to the shares of assets she wants to hold directly, D_i, and the shares of the fund she wants to hold, d_i. We are interested in an equilibrium defined as follows.

Definition 1. *A pricing vector P^* and a profile of all investors' risky asset holdings $\{d_i^*, D_i^*\}_{i\in[0,1]}$ constitute an equilibrium, if:*

1. *Given P^*, (d_i^*, D_i^*) solves investor i's maximization problem in equation (5), for all $i \in [0,1]$; and*

2. *P^* clears the market,*

$$\int_0^1 (d_i^* X + D_i^*)\, \mathrm{d}i = W + Z, \quad \textit{for any realizations of } F \textit{ and } Z. \tag{6}$$

3 The Risk-Adjusted Market Portfolio

The conventional stock market index is the value-weighted market portfolio (*VWMP*), which, in our model, is $X = W$. However, as we show in Appendix B, the *VWMP*

[12] We do not specify an ambiguity averse investor i's subjective belief over $\mathcal{M}_i$ and their subjective beliefs about other investors' subjective beliefs, since these will not affect our equilibrium characterization, as will become clear shortly.

[13] An investor's utility in this paper differs slightly from that defined in Gilboa and Schmeidler (1989). Since $\mathcal{M}_i$ may be a non-compact set, the investor maximizes the infimum rather than the minimum of her CARA utility among all possible models in $\mathcal{M}_i$.

[14] The fund behaves non-strategically in our model. Allowing for agency problem would add analytical complexity. However, it is an immediate implication of Proposition 1 that if the passive fund wants to maximize the measure of investors who buy its shares, the optimal portfolio it commits to offering is the one specified in equation (7).

fails to encourage better diversification and risk sharing by ambiguity averse investors, and fails as a benchmark for equilibrium asset pricing. The dysfunctions of the *VWMP* derive from important limitations: assets are weighted in the *VWMP* by their market capitalizations only, so these weights depend upon neither the average precision of private signals nor the precision of random supply, which determine the distribution of price signal.

This raises the question of whether there is an alternative index which, if offered by a fund, ambiguous investors would be willing to hold. If so, this could improve diversification and risk-sharing between investors who are ambiguous about any given asset and those who are not. We will show that a new index design achieves these goals.

The index is formed by adjusting the market capitalization weights in the *VWMP* by the average precision of investor private signals and the precision of random supply. We dub the new index the *Risk-Adjusted Market Portfolio* (*RAMP*), because as we will show, it places lower weight on more volatile stocks than the *VWMP* does. We analyze portfolio choices and asset pricing, when there is a passive fund offering *RAMP*.

Formally, *RAMP* is defined as

$$X = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W. \tag{7}$$

Investors who are subject to model uncertainty do not know $\boldsymbol{\Sigma}$, $\boldsymbol{U}$, or W, and hence do not know the exact composition of X. However, the functional relationship between X and $(\boldsymbol{\Sigma}, \boldsymbol{U}, W)$ that is specified in equation (7) is common knowledge to investors. *RAMP* does not depend on any of the private signals about asset payoffs, and so *RAMP* is a passive asset management product. Because of these features (i.e., passive but weighting assets not only by market capitalization), *RAMP* can be viewed as a type of *smart beta* strategy, an investing approach which has been growing in popularity in investment practice.[15]

More specifically, *RAMP* is a defensive investment strategy in the sense that it underweights high volatility stocks. *RAMP* differs from *VWMP* in that it contains a component $(\boldsymbol{\Sigma U})^{-1}$. It then follows from equation (7) that *RAMP* includes fewer shares of more volatile assets. Therefore, holding *RAMP* will be a defensive investment strategy that can largely reduce the risk of loss.

[15]By the end of December 2017, smart beta funds have surpassed $1 trillion in assets under management.

3.1 Equilibrium Index Investing

Even though *RAMP* is offered based on the true parameters of the financial market, it is not obvious that an ambiguity averse investor will be willing to hold the fund. The investor does not know its exact composition, so holding it entails bearing extra risk. Furthermore, different investors may have different subjective beliefs about the structure of the financial market, which might make it seem unlikely that one market index can attract all investors. For example, when investor i and investor j completely disagree with each other about the financial market, i.e., $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$, it seems plausible that the optimal market index for investor i would differ from that for investor j.

However, we show in this subsection that with the passive fund offering *RAMP*, all investors employ the index investment strategy in equilibrium by holding one share of the fund, and their effective asset holdings are exactly the same as in the economy without model uncertainty. This is stated in Proposition 1.

Proposition 1. *In the model with a passive fund that commits to offering the portfolio X specified in equation (7), there is an equilibrium in which*

1. *All investors will buy one share of the passive fund, and so $d_i^* = 1$ for all $i \in [0, 1]$;*
2. *Any investor i will hold an extra portfolio $D_i^* = \rho \mathbf{\Omega}_i (S_i - rP)$; and*
3. *For any given F and Z, the equilibrium price is*

$$P = \boldsymbol{B}^{-1}\left[F - A - \boldsymbol{C}Z\right], \tag{8}$$

where

$$A = \frac{1}{\rho}\left[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}\right]^{-1} W \tag{9}$$

$$\boldsymbol{B} = r\boldsymbol{I} \tag{10}$$

$$\boldsymbol{C} = \frac{1}{\rho}\mathbf{\Sigma}^{-1}. \tag{11}$$

The intuition of Proposition 1 builds upon the Information Separation Theorem, which applies in the setting without model uncertainty (that is, m is common knowledge among investors). Making use of a standard formula for investor equilibrium asset holdings in traditional rational expectations equilibrium models (see Admati (1985)), we prove the following Information Separation Theorem in Appendix C.

Theorem 1 (The Information Separation Theorem)**.** *When the characteristics of all assets are common knowledge, equilibrium portfolios have three components: a risk-adjusted market portfolio (RAMP); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.*

The Information Separation Theorem states that in a setting without model uncertainty, any investor i's equilibrium risky asset holding can be decomposed into two components. The first component is the optimal portfolio based on the price information gleaned from the stock price only, and hence is common among all investors. Such a component is just *RAMP* (the portfolio X evaluated at the commonly known m). The second component is $\rho\mathbf{\Omega}_i\left(S_i - rP\right)$, the optimal portfolio based on investor i's private signal only, and hence is called the information-based portfolio. Importantly, the price signal and investor i's private signal are conditionally independent, and so these two components can be formed independently.

With the Information Separation Theorem, we are now in a position to derive the intuition for investors' equilibrium investment strategies in the setting with model uncertainty. We consider a strategy profile in which all investors hold one share of the passive fund and their own information-based portfolio. We will then argue that no investor has an incentive to deviate unilaterally from such a strategy.

To build intuition, we first consider investors with min-max preferences, and then show that our conclusions also apply to max-min preferences as well. With min-max preferences, we argue that given other investors' strategies of holding *RAMP* and their own information-based portfolios, an investor's optimal trading strategy is constant across all possible values of financial market parameters in her subjective belief support. Hence, the investor's optimal trading strategy with max-min utility is the same as that with min-max utility.

Specifically, when investor i has a min-max utility, for each possible world $m_i \in \mathcal{M}_i$, she can solve her optimal risky asset holdings, assuming that the equilibrium pricing function is the one in equation (8) with m being m_i. Importantly, because all other investors are holding one share of the fund along with their own direct information-based portfolios, they are effectively holding the risky assets as in the world with m_i being common knowledge. Therefore, in the possible world m_i, the market clearing condition implies that the pricing function is the one specified in equation (8) with m being m_i. That is, investor i's belief about the pricing function is correct. So, she would like to hold the risky assets as in the world m_i. Such risky asset holdings can be implemented by holding one share of the passive fund and her information-based portfolio, so investor i would like to use the investment strategy in Proposition 1. Furthermore, investor i is still uncertain about m, so holding the risk-adjusted market portfolio through holding one share of the fund is strictly preferred.

In the above, for any given possible world, holding one share of the fund and her own information-based portfolio maximizes investor i's expected CARA utility (given that all other investors trade according to the prescribed strategy profile). Since such a trading strategy is optimal across all possible values of financial market parameters, it maximizes investor i's max-min utility. That is, a strong max-min property holds in the equilibrium, and hence, in our model with investors having max-min utilities, the investment strategy of holding one share of the fund and the information-based portfolio is also optimal to investors.

So the Information Separation Theorem helps explain why ambiguity averse investors are willing to hold the fund that offers *RAMP* in an equilibrium. Indeed, the same argument can also be applied when investors have the smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji 2005). Since holding one share of the passive fund and her own information-based portfolio helps an investor to achieve the optimal balance between risks and returns in each possible model in her subjective belief support, the optimal investment strategy will be independent of the utility representations of ambiguity preference.

3.2 Properties of the Equilibrium

The equilibrium characterized in Proposition 1 has several interesting properties describing behavior when a *RAMP* index fund is available. First, investors all hold exactly one share of the fund, even though they have heterogeneous priors about the financial market and thus different beliefs about the fund's composition.[16] This is true even in the extreme case in which two investors, i and j, completely disagree about the financial market parameters; that is, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$. By Proposition 1, both investor i and investor j hold one share of the passive fund, along with their own information-based portfolios. So differences in investors' holdings arise solely from differences in their private signals about asset payoffs, not from differences in their model uncertainties.

Second, Proposition 1 shows that the willingness of investors to buy an index is based

[16]For simplicity in presentation, we focus on the setting where everyone optimally holds one share of the passive fund. We show that this can be generalized to heterogeneous holdings in Section 5.2.

on an understanding of equilibrium risk-sharing, rather than just a partial equilibrium understanding that there can be risk-reduction benefits to the investor in isolation to diversifying her portfolio. Specifically, consider an investor who faces model uncertainty about a subset of traded assets, and views the return distributions as exogenous. Even if she can indirectly trade those assets through a passive fund, it may not be optimal for her to do so, because she cannot calculate the fund's expected return and risk. Therefore, arguments based on the incentive of individuals to diversify do not, under radical ignorance, justify holding of the fund. In contrast, in our equilibrium setting, an investor optimally holds the fund, given her belief that other investors will also do so (together with their direct portfolios). So she is willing to hold the fund too, which achieves the benefit of optimally sharing risks with other investors.[17]

Proposition 1 more broadly suggests that the reason why actual investors often fail to diversify goes beyond investor ambiguity aversion. In particular, for an investor to hold the fund, all other investors need to behave according to the prescribed equilibrium strategy profile. If imperfectly rational investors reason about possible portfolios based solely on partial equilibrium risk and return arguments, portfolios containing assets that investors are ambiguous about might seem extremely risky (or in the limiting case, infinitely risky). Proposition 1 shows that, owing to equilibrium considerations, even ambiguity averse investors, if otherwise rational, will hold such assets. But actual investors may not understand the equilibrium reasoning which underlies this result.

Third, and finally, comparing the equilibrium characterized in Proposition 1 with that in Proposition 5 in the Appendix B, we can see the different effects of *RAMP* and *VWMP* on the asset prices and investors' equilibrium asset holdings. Whether the passive fund offers *RAMP* or *VWMP*, conditional on the asset payoffs, the price volatility is same. This follows from the fact that $\boldsymbol{B}$ and $\boldsymbol{C}$ in the equilibrium pricing function with *RAMP* are equal to $\boldsymbol{B}_V$ and $\boldsymbol{C}_V$ in the equilibrium pricing function with *VWMP*, respectively. For any given asset payoffs (F) and asset random supplies (Z), however, the asset prices are higher in the economy with *RAMP* being the index, because $A < A_V$. So in the model, introducing a passive fund that offers *RAMP* helps mitigate the equity premium puzzle. This also implies that an informed investor's asset holdings based on her private signals are higher in the economy with *VWMP*. On the other hand, while an

[17]The fact that equilibrium rather than just diversification considerations are crucial for the index investment result can be seen more concretely by considering the off-equilibrium possibility that other investors trade in a fashion that causes asset prices to be almost uninformative. In such a scenario, an ambiguity averse investor would not hold the passive fund, because *RAMP* would be perceived as extremely risky.

informed investor's asset holdings based on the price signals are the same in both the economies with *RAMP* and with *VWMP*, an ambiguity averse investor's asset holdings are higher with *RAMP*.

The market clearing condition then implies that when the passive fund offers *VWMP*, the informed investors trade more aggressively based on their private signals to absorb the random supplies, and hence they should receive higher premia, which can be called ambiguity premia. This in turn suggests that when the passive fund offers *RAMP*, the assets' ambiguity premia should disappear, which we verify in Section 3.3.

3.3 CAPM Pricing with *RAMP*

Proposition 1 indicates that the model with ambiguity aversion and a fund that offers *RAMP* has an equilibrium in which investors' effective risky assets holdings are exactly the same as in the rational expectations equilibrium without model uncertainty. This suggests that asset risk premia should not have any ambiguity premia. In addition, as shown in the Information Separation Theorem, *RAMP* is the efficient portfolio conditional only on the price signals. This suggests that a version of CAPM security market line will hold under information asymmetry and ambiguity aversion, with *RAMP* as the benchmark pricing portfolio.

To formally analyze asset alphas relative to *RAMP* as an asset pricing benchmark, we first return to the special case of no model uncertainty. From equation (35), the equilibrium pricing function in this case is

$$P = \frac{1}{r}\left[F - A - \frac{1}{\rho}\mathbf{\Sigma}^{-1}Z\right], \tag{12}$$

where $A = \frac{1}{\rho}[\rho^2(\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}) + \mathbf{\Sigma}]^{-1}W$. We then have

$$\text{diag}(P)^{-1}\mathbb{E}(F) - r\mathbb{1} = \text{diag}(P)^{-1}A. \tag{13}$$

Here, $\mathbb{E}(F)$ is the expected payoff conditional on the equilibrium price. The LHS of equation (13) is just the vector of the risky assets' equilibrium risk premia.

Given a realized equilibrium price, *RAMP* has the value $P'X$. Then the vector of the weights of risky assets in *RAMP* is

$$\omega = \frac{1}{P'X}\text{diag}(P)X.$$

Hence, conditional on the price P, the difference between the expected return of *RAMP* and the riskfree rate is

$$\begin{aligned}\mathbb{E}(R_X) - r &= \omega' \mathrm{diag}(P)^{-1}\mathbb{E}(F) - r \\ &= \frac{1}{P'X} X' \mathrm{diag}(P) \mathrm{diag}(P)^{-1}(A + rP) - r \\ &= \frac{1}{P'X} X' A, \end{aligned} \tag{14}$$

where the expectations are all conditional on the equilibrium price.

The variance of *RAMP* is

$$\mathbb{V}(R_X) = \mathbb{E}\left[\left(\omega' \mathrm{diag}(P)^{-1}\boldsymbol{C}Z\right)\left(\omega' \mathrm{diag}(P)^{-1}\boldsymbol{C}Z\right)'\right] = \left(\frac{1}{P'X}\right)^2 X'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X, \tag{15}$$

and the covariance between all risky assets and *RAMP* is

$$Cov(R, R_X) = \frac{1}{P'X} \mathrm{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X. \tag{16}$$

Let α_X be the CAPM alpha with *RAMP* being the pricing portfolio. From equations (13)-(16), and since $X = \rho(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}A$, we have the following proposition.

Proposition 2 (Risk Premia with Supply Shocks)**.** *In the model with all parameters being common knowledge, asset risk premia satisfy the CAPM security market line where the relevant market portfolio for pricing is RAMP.*

This result may seem surprising, since investors have heterogeneous asset holdings, and since the portfolios held by informed investors are not mean-variance efficient with respect to the public information set. Nevertheless, in equilibrium, there are no extra risk premia incremental to those predicted by the CAPM using *RAMP*.

The CAPM pricing relation using *RAMP* is equivalent to the assertion that *RAMP* is mean-variance efficient conditional only on all public information. This efficiency can be seen from the utility maximization problem of an investor who does not have any private signals about asset payoffs. Such an investor balances the expected returns and the risks of her holdings, and her information consists of the equilibrium price only, which is the only public information in our baseline model. In equilibrium, such an investor holds *RAMP*, implying that *RAMP* is mean-variance efficient conditional only on public information.

Privately informed investors also hold *RAMP* as the passive component of their portfolios; this is the piece that does not depend upon their private signals (except to the extent that their signals are incorporated into the publicly observable market price). In addition, they have other asset holdings to take advantage of the greater safety of assets they have more information about, and for speculative reasons based upon their private information. *RAMP* is not mean-variance efficient with respect to their private information sets, but it is efficient with respect to the information set that contains only publicly available information.

In the special case of asymmetric information but no model uncertainty, *RAMP* is a natural candidate for the CAPM pricing portfolio, because it is the common component in all investors' risky asset holdings. We show that *RAMP* is mean-variance efficient unconditional on any investor's private information. Therefore, the CAPM security market line relation holds without conditioning on private information, with respect to *RAMP*.

What is perhaps more surprising is that when there is a passive fund that offers *RAMP*, even with model uncertainty, *RAMP* is the appropriate CAPM pricing portfolio. To see how the presence of a fund offering *RAMP* affects asset risk premia when there is model uncertainty, consider equilibrium asset holdings as in Proposition 1. This shows that even when investors are uncertain about the precisions of asset supply shocks, they all hold one share of the passive fund, eliminating ambiguity premia. So asset risk premia satisfy the CAPM.

Corollary 1 presents this even more surprising result.

Corollary 1. *In the model where investors are subject to model uncertainty, if a passive fund offers RAMP as specified in equation* (7)*, asset risk premia satisfy the CAPM with RAMP as the pricing portfolio.*

4 Centralized versus Distributed Implementation of the *RAMP* Fund

We have shown that our new index, *RAMP*, can encourage all investors, including ambiguity averse ones, to employ index investment strategy, and thus in equilibrium investor risky asset holdings are exactly same as those in the economy without model uncertainty.

In order to offer *RAMP*, the passive fund needs to have full knowledge about the relevant parameter values that characterize the financial market. Improving information processing technologies and "big data" may have improved the feasibility of this over time. It is also possible that the regulatory powers of government may give it advantages for collecting information relevant for estimating the relevant parameter values. If so, in principle the government itself could offer *RAMP*, or could publicly disclose relevant information that helps others estimate relevant parameters. So potentially either government or private funds can contribute to the offering of *RAMP*, promoting investor participation and risk sharing. There can also be agency problems associated with either public or private fund providers, a topic that we do not focus on in this paper.

By whatever means, if a single agent has access to all relevant parameter values, a centralized approach to offering *RAMP* is straightforward. The agent calculates *RAMP* based on equation (7) and publicly announces the formula for calculating portfolio weights. If there are multiple agents with the requisite information, competition between them can drive fees to a very low level.

Furthermore, even if no single agent knows all the relevant parameters, a decentralized approach can also implement *RAMP*. Suppose that each parameter is known by a nontrivial set of investors. Specifically, suppose that the set of all traded assets can be partitioned into Q subsets. In the partition j, there are $q_j \geq 1$ assets. We assume that there is a positive measure of investors who know all parameters about assets in partition j but do not have any private signals about the payoffs of such assets. We call these investors "Group-j-uninformed investors." (In the extreme case, $Q = N$, and so, in each partition, there is only one asset. Then, we are in the setting described in Section 2.)

We consider the following equilibrium. Each of the Group j uninformed investors commits to offering a portfolio H_j. Here, H_j can be viewed as a local fund, which includes only assets in partition j. For each asset n included in asset partition j, H_j includes exactly the same position as in the portfolio X. Since there are uncountably many funds that are committed to offering H_j, the j^{th} local fund industry is perfectly competitive. Hence, the fund fee should be the same as the marginal cost of offering H_j, which is zero.

Importantly, no local fund will deviate from the equilibrium. As is actually required in the index fund industry, all local funds are required to disclose their asset holdings at the end of the period. Then, if a local fund of Group j that deviates from H_j, its portfolio holding will differ from other Group j local funds' portfolio holdings. Hence, such a deviation is observable ex post and verifiable. Ex post, once a fund's deviation is detected, we assume that the fund will be heavily punished or incur a large reputation cost. It follows that no local fund is willing to deviate from its commitment to invest in H_j. This indeed follows the idea of Nash implementation in the mechanism design literature.

Given all the local funds, *RAMP* can be implemented by creating a fund of funds. Specifically, any investor will first buy one share of the Group j local fund, for each j. By doing so, the investor will form an asset holding $(H_1', H_2', \ldots, H_M')' = X'$, which is exactly *RAMP* specified in equation (7). Then, investors will hold their own information-based portfolios. Obviously, investors are effectively holding one share of the passive fund and their own information-based portfolios, which are their optimal investment strategies in the equilibrium described in Proposition 1.

5 Extensions

We now consider the robustness of our conclusions to allowing for a Gaussian prior or for heterogeneous risk aversions.

5.1 Normal Prior

In the model of Section 2, we assume for simplicity that investors hold a common uniform improper prior. We next verify that similar conclusions hold when we consider ambiguity averse investors' asset holdings with normal priors about asset payoffs. Formally, we assume that the asset payoffs are drawn from the distribution $F \sim \mathcal{N}\left(\overline{F}, \boldsymbol{V}^{-1}\right)$, where $\overline{F}$ is the ex-ante mean and $\boldsymbol{V}$ is the ex-ante precision matrix. Investors, however, may be ambiguous about such a distribution. So, in this case, a model should be characterized by $m = (\boldsymbol{\Sigma}, \boldsymbol{U}, W, \overline{F}, \boldsymbol{V})$.

We assume that in this case, the passive fund commits to offering a portfolio

$$Y = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1} V\left(\overline{F} - rP\right). \tag{17}$$

The portfolio Y is the sum of *RAMP* and a portfolio for contrarian trading. It can be shown to be the optimal portfolio in the setting without model uncertainty based on the prior and the price signal. However, since the prior and the price signal are not conditionally independent (because the prior is a determinant of the price signal), the two components that constitute the portfolio Y are not information-separated.

Proposition 3 below shows that with the fund offering the portfolio Y, all investors will hold exactly one share of the fund, and any investor's overall risky asset holding is the same as that in the setting without model uncertainty.

Proposition 3. *In the model with normal priors, when there is a passive fund that commits to offering the portfolio Y specified in equation (17), there is an equilibrium in which*

1. *All investors will buy one share of the passive fund, and so $d_i^* = 1$ for all $i \in [0,1]$; and*

2. *Any investor i will hold an extra portfolio $D_i^* = \rho \mathbf{\Omega}_i (S_i - rP)$.*

We further analyze asset risk premia with the portfolio Y being the pricing portfolio. While it is rather intractable to show that any asset's alpha is zero, Corollary 2 below argues that the portfolio Y is the optimal portfolio choice of investors who do not have any private information. Since investors with no private information choose their portfolios based only on all public information (in this case, the prior and the price signal), Y is mean-variance efficient conditional on all public signals.

Corollary 2. *In the model with normal priors, when there is a passive fund that commits to offering the portfolio Y, any investor i with $\mathbf{\Omega}_i = \mathbf{0}$ will choose the portfolio Y in equilibrium.*

5.2 Heterogeneous Risk Aversions

In the model of Section 2, investors share a same risk aversion coefficient ρ. This leads to investors' homogeneous holdings of the passive fund. Indeed, in the equilibrium characterized in Proposition 1, all investors hold one share of the passive fund. However, this raises the question of whether investors with different risk tolerances are willing to hold a single common passive fund, and if so, whether differences in risk tolerances, and investor unawareness of other investors' risk tolerances might result in heterogeneous holdings of the passive fund. To evaluate the robustness of our conclusions, we now extend the model to allow for heterogeneous risk tolerances.

We extend the model in Section 2 by assuming that any investor i ($i \in [0,1]$) has the risk aversion coefficient ρ_i. Here, ρ_i is a continuous function of i. Let

$$\overline{\rho} = \int_0^1 \rho_i \mathrm{d}i \quad \text{and} \quad \overline{\mathbf{\Sigma}} = \int_0^1 \rho_i \mathbf{\Omega}_i \mathrm{d}i.$$

Here, $\overline{\rho}$ is the average risk tolerance, and $\overline{\mathbf{\Sigma}}$ is the average precision of investors' private information that is weighted by their risk tolerances. We assume that any investor i

knows ρ_i, but she does not know the distribution of ρ_j or the average risk tolerance $\overline{\rho}$. Therefore, a financial market is characterized by $m = (\overline{\mathbf{\Sigma}}, \boldsymbol{U}, W, \overline{\rho})$. The passive fund cannot evaluate each individual investor's risk tolerance, but it has accurate information about the distribution of investors' risk tolerances; hence, it knows $\overline{\rho}$ and $\overline{\mathbf{\Sigma}}$. Then, the passive fund offers the portfolio

$$\overline{X} = \left[\overline{\rho}\boldsymbol{I} + \left(\overline{\mathbf{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1} W. \tag{18}$$

to all investors. Proposition 4 characterizes investors' equilibrium portfolios in such an extension.

Proposition 4. *In the model with investors' heterogeneous risk tolerances, when there is a passive fund that commits to offering the portfolio $\overline{X}$ specified in equation (18), there is an equilibrium in which*

1. *All investors will buy ρ_i share of each passive fund, and so $d_i^* = \rho_i$ for all $i \in [0, 1]$; and*
2. *Any investor i will hold an extra portfolio $D_i^* = \rho_i \mathbf{\Omega}_i (S_i - rP)$.*

It directly follows from Proposition 4 that with a passive fund offering the portfolio $\overline{X}$, investors hold the same portfolios as they do when they are not facing model uncertainty. Therefore, the conclusion of our baseline model that a wisely designed index can encourage investors to participate in the financial market and engage in index investing is robust.

On the other hand, Proposition 4 shows that the number of shares of the passive fund an investor holds depends on her risk tolerance. This is similar to that in the classic CAPM — all investors invest in the same index but the shares they hold the index fund depend on their risk tolerance.

6 Concluding Remarks

We study here two major roles played by market indexes, facilitating diversified investing, and providing an appropriate asset pricing benchmark, in a financial market with information asymmetry, model uncertainty, and ambiguity aversion. We show that with the Value-Weighted Market Portfolio (*VWMP*) as the available index investing vehicle, ambiguity averse investors do not invest in the index, which hinders diversification and risk sharing. This also implies that in comparison to a market without model uncertainty, informed investors need to hold extra positions to absorb a greater proportion of outstanding shares, including random supplies, and thus they will require ambiguity premia in expected returns. So information asymmetry and ambiguity aversion lead to non-zero alphas of assets relative to *VWMP* as the pricing portfolio.

We derive a new market index that adjusts market value weights to take into account the average precision of investor private signals and the precision of random supply of different assets, i.e., the amount of risk reduction investors obtain by conditioning on price as a signal. We call this index design the *Risk-Adjusted Market Portfolio* (*RAMP*). *RAMP* is a defensive strategy in the sense that, relative to the value-weighted market, it underweights assets that are more volatile.

The ability of investors to invest in a *RAMP* index fund has major implications for equilibrium trading and asset pricing. In equilibrium, regardless of investors' heterogeneity in their subjective beliefs about the financial market, all investors hold the index as the passive (non-information-based) component of their portfolios. That is, *RAMP* induces investors to diversify by employing an index investment strategy. This improves the sharing of risk between investors who face model uncertainty about an asset and those who do not. In equilibrium, all investors' asset holdings are exactly the same as those in the economy without model uncertainty. Finally, the CAPM pricing relationship holds with respect to *RAMP* as the benchmark pricing portfolio, even though investors have asymmetric information and face model uncertainty. These results imply that delegation is a potential way to alleviate the inefficiency caused by model uncertainty.

These properties of *RAMP* derive from an information separation theorem in the financial market without model uncertainty. The information separation theorem says that to attain her optimal asset holdings, an investor first constructs an optimal portfolio based on each of her signals (i.e., price signal and private signal) and then sums all these optimal portfolios together. Then, in the setting with model uncertainty, when other investors are holding a passive fund offering *RAMP* and their information-based portfolios, in any possible financial market in her subjective belief support, an investor's optimal investment strategy is also to hold the fund and her own information-based portfolio. Therefore, providing *RAMP* to all investors facilitates their asset market participation and risk sharing.

The design of *RAMP* has important implications. First, because it underweights high volatility stocks, *RAMP* is a defensive investing strategy. The investment strategy of following *RAMP* can also be viewed as a smart beta strategy, which is gaining increasing popularities. In order to implement *RAMP*, information technology and large data sets have reduced the cost to private fund companies or the government of gathering information about features of the capital market and to offer funds that approximate *RAMP*. Furthermore, educators or policymakers can take steps to inform investors about concepts of market equilibrium and optimal risk sharing, and may consider making *RAMP* available to investors in retirement investing plans.

A Omitted Proofs

Proof of Proposition 1:

We first verify that the market clearing condition holds. Each investor i's effective risky assets holding is

$$d_i^* X + D_i^* = \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Omega}_i\left(S_i - rP\right).$$

Then, using the pricing function (equation (8)), the aggregate demand can be calculated as

$$\begin{aligned}
& \int_0^1 \left(d_i^* X + D_i^*\right) \mathrm{d}i \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma}\left(F - rP\right) \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma}\left(\frac{1}{\rho}\left(\boldsymbol{\Sigma} + \rho^2\boldsymbol{\Sigma U \Sigma}\right)^{-1} W + \frac{1}{\rho}\boldsymbol{\Sigma}^{-1} Z\right) \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma U}\right]^{-1} W + Z \\
= & \rho^2\boldsymbol{\Sigma U}\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma U}\right]^{-1} W + \left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma U}\right]^{-1} W + Z \\
= & W + Z.
\end{aligned}$$

Therefore, the market clears.

We then show that when all other investors choose the prescribed investment strategies, any investor i will not make a unilateral deviation from the prescribed investment strategy either. That is, if $d_j^* = 1$ and $D_j^* = \rho\boldsymbol{\Omega}_j\left(S_j - rP\right)$ for all $j \in [0,1] \setminus \{i\}$, then $d_i^* = 1$ and $D_i^* = \rho\boldsymbol{\Omega}_i\left(S_i - rP\right)$.

In our model, any investor understands that she is small and so her trading will not affect the aggregate demand. Hence, for a fixed financial market $m_i \in \mathcal{M}_i$, investor i knows that all other investors' investment strategies lead to the aggregate asset demand

$$\begin{aligned}
& \int_0^1 \left(d_j^* X + D_j^*\right) \mathrm{d}j \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \int_0^1 \rho\boldsymbol{\Omega}_j\left(S_j - rP\right) \mathrm{d}j \\
= & \left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma U}\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma} F - \rho r P.
\end{aligned}$$

Here, $(\boldsymbol{\Sigma}, \boldsymbol{U}, W) = m_i$.

Then, the market clearing in the financial market m_i implies

$$\begin{aligned}
&\left[\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma}\boldsymbol{U})^{-1}\right]^{-1} W + \rho\boldsymbol{\Sigma}F - \rho r P = W + Z \\
\Rightarrow\ & \rho\boldsymbol{\Sigma}F = \left[\boldsymbol{I} - \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma}\boldsymbol{U})^{-1}\right)^{-1}\right] W + \rho r P + Z \\
\Rightarrow\ & F = \frac{1}{\rho}\left[\boldsymbol{\Sigma}^{-1} - \left(\boldsymbol{\Sigma} + \frac{1}{\rho^2}\boldsymbol{U}^{-1}\right)^{-1}\right] W + r\boldsymbol{\Sigma}^{-1}P + \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}Z.
\end{aligned}$$

Hence, conditional on the price, investor i's belief over F is

$$F|P \sim \mathcal{N}\left(\frac{1}{\rho}\left[\boldsymbol{\Sigma}^{-1} - \left(\boldsymbol{\Sigma} + \frac{1}{\rho^2}\boldsymbol{U}^{-1}\right)^{-1}\right] W + r\boldsymbol{\Sigma}^{-1}P, \frac{1}{\rho^2}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^{-1}\boldsymbol{\Sigma}^{-1}\right).$$

Then, investor i's optimal portfolio in the financial market m_i is

$$\begin{aligned}
&\rho\mathbb{V}(F|P, S_i)^{-1}\left(\mathbb{E}(F|P, S_i) - rP\right) \\
=&\left[\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma}\boldsymbol{U})^{-1}\right]^{-1} W + \rho\boldsymbol{\Omega}_i\left(S_i - rP\right) \\
=&X + D_i^*.
\end{aligned}$$

That is, given $d_j^* = 1$ and $D_j^* = \rho\boldsymbol{\Omega}_j\left(S_j - rP\right)$ for all $j \in [0,1] \setminus \{i\}$, investor i's optimal portfolio in any $m_i \in \mathcal{M}_i$ is $(d_i^*, D_i^*) = (1, \rho\boldsymbol{\Omega}_i(S_i - rP))$.

Note that (d_i^*, D_i^*) is independent of m_i, so we have

$$\begin{aligned}
&\inf_{m_i \in \mathcal{M}_i} \max_{(d_i, D_i)} U(m, (d_i, D_i)) \\
=&\inf_{m_i \in \mathcal{M}_i} U(m, (d_i^*, D_i^*)) \\
\leq&\max_{(d_i, D_i)} \inf_{m_i \in \mathcal{M}_i} U(m, (d_i, D_i)).
\end{aligned}$$

Since generally

$$\inf_{m_i \in \mathcal{M}_i} \max_{(d_i, D_i)} U(m, (d_i, D_i)) \geq \max_{(d_i, D_i)} \inf_{m_i \in \mathcal{M}_i} U(m, (d_i, D_i)),$$

we have

$$\inf_{m_i \in \mathcal{M}_i} \max_{(d_i, D_i)} U(m, (d_i, D_i)) = \max_{(d_i, D_i)} \inf_{m_i \in \mathcal{M}_i} U(m, (d_i, D_i)).$$

Hence, (d_i^*, D_i^*) is a solution to $\max_{(d_i,D_i)} \inf_{m_i \in \mathcal{M}_i} U(m, (d_i, D_i))$, implying that investor i does not have incentives to deviate from the prescribed investment strategy.

Q.E.D.

Proof of Proposition 2:

By equations (14), (15), and (16), we have

$$\begin{aligned} & \frac{\frac{1}{P'X}\operatorname{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}{\left(\frac{1}{P'X}\right)^2 X'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}\frac{X'A}{P'X} \\ = & \frac{\operatorname{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}{X'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}X'A. \end{aligned}$$

This is the RHS of the Security Market Line relation. We want to show that this equals the difference between the risky assets' rates of return and the riskfree asset's rate of return, which is shown to be $\operatorname{diag}(P)^{-1}A$ from equation (13).

Then, we have

$$\begin{aligned} & \frac{\operatorname{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}{X'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X}X'A = \operatorname{diag}(P)^{-1}A \\ \Leftrightarrow\ & \operatorname{diag}(P)^{-1}\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}XX'A = \operatorname{diag}(P)^{-1}AX'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X \\ \Leftrightarrow\ & \boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}XX'A = AX'\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}X. \end{aligned}$$

The last equation holds because $X = \rho(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}A$ and $(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C})^{-1}$ is a symmetric matrix.

Q.E.D.

Proof of Proposition 3:

The proof is similar to that of Proposition 1. Consider the strategy profile that $(d_i^*, D_i^*) = (1, \rho\boldsymbol{\Omega}_i(S_i - rP))$ for all $i \in [0,1]$. We argue that no investor wants to make a unilateral deviation.

Given all other investors' investment strategy, in a model in investor i's subjective belief $m_i = (\boldsymbol{\Sigma}, \boldsymbol{U}, W, \overline{F}, \boldsymbol{V}) \in \mathcal{M}_i$, the market clearing condition implies that

$$F = \frac{1}{\rho}\Sigma^{-1}\left[\boldsymbol{I} - \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma}\boldsymbol{U})^{-1}\right)^{-1}\right]W - \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma}\boldsymbol{U}\right]^{-1}V(\overline{F} - rP) + rP + \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}Z.$$

Hence, $F|P$ is normally distributed with mean

$$\frac{1}{\rho}\mathbf{\Sigma}^{-1}\left[\boldsymbol{I}-\left(\boldsymbol{I}+\frac{1}{\rho^2}(\mathbf{\Sigma}\boldsymbol{U})^{-1}\right)^{-1}\right]W-\mathbf{\Sigma}^{-1}\left[\boldsymbol{I}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\right]^{-1}\boldsymbol{V}(\overline{F}-rP)+rP$$

and the precision

$$\rho^2\mathbf{\Sigma}\boldsymbol{U}\mathbf{\Sigma}.$$

Therefore, conditional on the prior, and the asset prices, and her own private signals (in model m_i), investor i's optimal portfolio is

$$\begin{aligned}&\left[\boldsymbol{I}+\frac{1}{\rho^2}\left(\mathbf{\Sigma}\boldsymbol{U}\right)^{-1}\right]^{-1}W+\rho\left[\boldsymbol{I}+\rho^2\mathbf{\Sigma}\boldsymbol{U}\right]^{-1}V\left(\overline{F}-rP\right)+\rho\mathbf{\Omega}_i\left(S_i-rP\right)\\=&Y+D_i^*,\end{aligned}$$

implying that $(d_i^*, D_i^*) = (1, \rho\mathbf{\Omega}_i\left(S_i - rP\right))$ is optimal in the model m_i.

Furthermore, (d_i^*, D_i^*) is independent of m_i. Hence, by the exactly same argument at the end of the proof of Proposition 1, we show that (d_i^*, D_i^*) is the solution to the investor i's maxmin problem, provided that all other investors trade as prescribed.

Q.E.D.

Proof of Proposition 4:

The proof is also similar to that of Proposition 1. For a given model $m_i = (\overline{\mathbf{\Sigma}}, \boldsymbol{U}, W, \overline{\rho})$, when all other investors trade as prescribed, investor i believes that the market clearing condition is

$$\int_0^1\left(\rho_j\left[\overline{\rho}\boldsymbol{I}+\left(\overline{\mathbf{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1}W+\rho_j\mathbf{\Omega}_j\left(S_j-rP\right)\right)\mathrm{d}j=W+Z,$$

which implies that the pricing function is

$$F=\overline{\mathbf{\Sigma}}^{-1}\left[I-\overline{\rho}\left[\overline{\rho}\boldsymbol{I}+(\overline{\mathbf{\Sigma}}\boldsymbol{U})^{-1}\right]^{-1}\right]W+rp+\overline{\mathbf{\Sigma}}^{-1}Z.$$

Hence, the asset payoffs conditional on asset prices have the distribution

$$F|P\sim\mathcal{N}\left(\overline{\mathbf{\Sigma}}^{-1}\left[I-\overline{\rho}\left[\overline{\rho}\boldsymbol{I}+(\overline{\mathbf{\Sigma}}\boldsymbol{U})^{-1}\right]^{-1}\right]W+rp,\overline{\mathbf{\Sigma}}^{-1}\boldsymbol{U}^{-1}\overline{\mathbf{\Sigma}}^{-1}\right).$$

Then, investor i's optimal portfolio choice in the financial market m_i is

$$\begin{aligned} & \rho_i \mathbb{V}(F|P,S_i)^{-1} \left[\mathbb{E}(F|P,S_i) - rp\right] \\ = \; & \rho_i \left[\overline{\rho}\boldsymbol{I} + \left(\overline{\boldsymbol{\Sigma}}\boldsymbol{U}\right)^{-1}\right]^{-1} W + \rho_i \boldsymbol{\Omega}_i \left(S_i - rP\right) \\ = \; & \rho_i \overline{X} + \rho_i \boldsymbol{\Omega}_i \left(S_i - rP\right). \end{aligned}$$

That is, investor i's optimal investment strategy in the perceived financial market m_i is $(d_i^*, D_i^*) = (\rho_i, \rho_i \boldsymbol{\Omega}_i (S_i - rP))$. Because such an investment strategy is independent of m_i, by the same argument at the end of the proof of Proposition 1, we can show that investor i's optimal investment strategy is (d_i^*, D_i^*).

Therefore, given investor $j \in [0,1] \setminus \{i\}$ employs the investment strategy $(d_j^*, D_j^*) = (\rho_j, \rho_j \boldsymbol{\Omega}_j \left(S_j - rP\right))$, investor i will not deviate, proving that the strategy under consideration is an equilibrium.

Q.E.D.

B A Benchmark: The Value-Weighted Market Portfolio

In this appendix, we discuss the investors' equilibrium asset holdings and the asset pricing implications when the fund is offering the investors with the value-weighted market portfolio (*VWMP*); formally, we assume that $X = W$. We show that ambiguity averse investors are not willing to hold a fund offering *VWMP* as the passive (noninformational) index component of their portfolios, and that *VWMP* is not the appropriate benchmark portfolio for pricing assets. This establishes a benchmark for our analysis when the passive fund is offering *RAMP*.

To keep such a benchmark model tractable, we assume that all assets are independent. That is, $\mathbf{\Omega}_i$ is diagonal for all $i \in [0,1]$. Then, investor i is an informed investor of asset n if and only if the n^{th} diagonal entry of $\mathbf{\Omega}_i$ is strictly positive. Let λ_n be the measure of informed investors of asset n; we assume that $\lambda_n \in (0,1)$, for all $n \in \mathcal{Q}$. Let $\text{diag}(\lambda)$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being λ_n.

For simplicity, we assume that the private signals of all informed investors of asset n have the same precision $\kappa_n > 0$. Let $\mathbf{\Omega}$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being κ_n. Letting $\mathbf{\Sigma}$ be the matrix of the average precision of private signals, we have

$$\mathbf{\Sigma} = \int_0^1 \mathbf{\Omega}_i \mathrm{d}i = \mathbf{\Omega}\text{diag}(\lambda) \tag{19}$$

Because assets are independent, $\boldsymbol{U}$ is also diagonal. We focus on the case that all model parameters are commonly known among investors except $\boldsymbol{U}$. We assume that for each asset n, a subset of uninformed investors do not know u_n. We say that such a group of investors are subject to *model uncertainty* (or are *ambiguous*) about asset n. Any investor i who is ambiguous about asset n will have her own subjective prior belief about u_n with the support $\left(0, \bar{u}_n^i\right)$, where $\bar{u}_n^i > 0$. So we allow different investors who are ambiguous about a particular asset n to have different supports of their beliefs about u_n. Let $\mathcal{U}_i$ be the set of all possible subjective beliefs of investor i about $\boldsymbol{U}$, let $\boldsymbol{U}_i$ be a typical element in $\mathcal{U}_i$, and let $\underline{\boldsymbol{U}}_i$ be the lower bound of $\mathcal{U}_i$.

Let the measure of the group of investors who are ambiguous about n be $1 - q_n \in (0, 1 - \lambda_n)$. Let $\boldsymbol{Q}$ be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being q_n. For simplicity, we assume that an investor who is ambiguous about asset n is also uninformed about asset n. However, an investor who is uninformed about asset n may know u_n and so is not ambiguous about asset n.

Each investor i's decision problem is then

$$\max_{d_i, D_i} \inf_{\boldsymbol{U}_i \in \mathcal{U}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{20}$$

B.1 Equilibrium Asset Holdings

Since the composition of *VWMP* is commonly known, even if no fund is available that offers *VWMP*, an investor can attain the same effective risky asset holdings by directly trading individual assets only. So an investor's optimal risky asset holdings are the same regardless of whether this passive fund is available. As a result, in order to analyze investors' equilibrium asset holdings when *VWMP* is publicly offered, we can first consider an economy without *VWMP*.

Investor i is risk averse, so she only holds a non-zero position of asset n, if her subjective belief of asset n's payoff has a finite variance, conditional on her information. When investor i is uninformed about asset n, however, she has neither prior information nor private information about the payoff of asset n. Hence, she estimates the payoff based on only the price, which partially aggregates informed investors' private information. Since the precision of asset n's random supply, u_n, is strictly positive (no matter how small it is), if investor i knows u_n, her belief of asset n's payoff has a finite variance and, therefore, she will hold a non-zero position of asset n.

On the other hand, if investor i is subject to model uncertainty about asset n, she does not know the precision of asset n's random supply. By assumption, investor i's subjective prior about u_n has the support $\left(0, \overline{u}_n^i\right)$. Since all random variables in the model are normally distributed, observing the asset price does not change the support of investor i's belief about u_n, although investor i may extract some information about u_n from asset n's price. Hence, the worst-case scenario is independent of asset n's price. Specifically, when the precision of the random supply is arbitrarily close to zero, price becomes almost uninformative. So as investor i considers the worst-case scenario in making the investment decision, she focuses on the possibility that the true u_n is very close to 0. For any non-zero position of asset n, as the price becomes almost uninformative, the payoff variance conditional upon investor i's information diverges to infinity. So holding a non-zero position is extremely risky in the worst-case scenario. To avoid this risk, investor i optimally chooses a zero position. Lemma 1 below summarizes the argument above.

Lemma 1. *An investor i who is ambiguous about asset n optimally holds a zero position in it.*

Proof of Lemma 1:

Because investor i is ambiguous about asset n, by assumption, she does not have private signal about asset n's payoff; that is, $\kappa_i = 0$. Hence, investor i's only information about the distribution of asset n's payoff is its price, which may partially aggregate informed investors' private signals. Suppose the uninformed investors' aggregate demand for asset n is $(1-\lambda_n)D(p_n)$. Since uninformed investors do not observe u_n, $D(p_n)$ is not a function of u_n.

Given any P and any $u_n \in \left(0, \overline{u}_n^i\right)$, we derive investor i's expected utility conditional on P_n as follows. Suppose asset n's pricing function in a linear equilibrium is

$$f_n = a + bp_n + cz_n,$$

where a, b, and c are undetermined parameters. Since informed investors know u_n, they can extract information from the price without any ambiguity. Therefore, any informed investor j's demand is

$$D_j = \rho\left[\kappa_n s_j + \frac{u_n}{c^2}a + \frac{u_n}{c^2}(b-r)p_n - r\kappa_n p_n\right].$$

Then, the informed investors' aggregate demand will be

$$\lambda_n\rho\left[\kappa_n f_n + \frac{u_n}{c^2}a + \frac{u_n}{c^2}(b-r)p_n - r\kappa_n p_n\right].$$

Then, the market clearing condition implies that

$$\lambda_n\rho\left[\kappa_n f_n + \frac{u_n}{c^2}a + \frac{u_n}{c^2}(b-r)p_n - r\kappa_n p_n\right] + (1-\lambda_n)D(P_n) = w_n + z_n.$$

Matching the coefficient of the market clearing condition and the pricing function, we have

$$\begin{aligned}
a &= \frac{w_n}{\lambda_n\kappa_n\rho} - \frac{u_n}{c^2\kappa_n}a \\
bp_n &= -\frac{(1-\lambda_n)D(p_n)}{\lambda_n\kappa_n\rho} - \frac{u_n}{c^2\kappa_n}(b-r)p_n + rp_n \\
c &= \frac{1}{\lambda_n\kappa_n\rho}
\end{aligned}$$

Therefore, for any given $u_n \in \left(0, \overline{u}_n^i\right)$, conditional on the price P_n, $|\mathbb{E}(f_n - rp_n|p_n)| < +\infty$. On the other hand, the variance of asset n's payoff conditional on p_n is

$$\mathbb{V}\left(f_n|p_n\right) = c^2 u_n^{-1},$$

which diverges to $+\infty$ as u_n goes to 0. Hence, any non-zero position D_i of asset n brings investor i a utility

$$-\exp\left(-\frac{1}{\rho}w_i r p_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(f_n - rp_n|p_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(f_n|p_n\right)\right], \tag{21}$$

which goes to $-\infty$ as u_n goes to 0. Therefore, if investor i is ambiguous about asset n, investor i will hold a zero position of asset n.

Q.E.D.

We now analyze the investors' equilibrium asset holdings. The model is similar to the rational expectations equilibrium model with multiple risky assets (Admati 1985). The key difference is that for each asset n, there are $1 - q_n$ measure investors who will hold a zero position (by Lemma 1). Proposition 5 below characterizes a linear rational expectations equilibrium. Recall that since the composition of *VWMP* is commonly known, whether or not such a portfolio is offered by the passive fund, investors will have the same effective risky asset holdings.

Proposition 5. *In the model where the passive fund offers VWMP (formally, $X = W$), there exists a linear equilibrium with the pricing function*

$$P = \boldsymbol{B}_V^{-1}\left[F - A_V - \boldsymbol{C}_V Z\right], \tag{22}$$

where

$$A_V = \frac{1}{\rho}\left[\rho^2(\boldsymbol{\Sigma}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{\Sigma}) + \boldsymbol{\Sigma}\right]^{-1} W \tag{23}$$

$$\boldsymbol{B}_V = r\boldsymbol{I} \tag{24}$$

$$\boldsymbol{C}_V = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}. \tag{25}$$

Any investor i's effective risky asset holding is

$$d_i X + D_i = \lim_{\boldsymbol{U}_i \to \underline{\boldsymbol{U}}_i}\left[\boldsymbol{I} + \frac{1}{\rho^2}\left(\boldsymbol{\Sigma}\boldsymbol{U}_i\right)^{-1}\right]^{-1} W + \rho\boldsymbol{\Omega}_i\left(S_i - rP\right) \tag{26}$$

Proof of Proposition 5:

We assume that the pricing function can be written as

$$F = A_V + B_V P + C_V Z,$$

where B_V is nonsingular. Then, conditional on the asset prices P, an investor i's updated belief about asset payoffs is

$$F|P \sim \mathcal{N}\left(A_V + \boldsymbol{B}_V P, \boldsymbol{C}_V' \boldsymbol{U}_i^{-1} \boldsymbol{C}_V\right),$$

where $\boldsymbol{U}_i \in \mathcal{U}_i$

Then, each investor i's optimal asset holdings are

$$d_i X + D_i = \rho \left\{ (\boldsymbol{C}_V \boldsymbol{U}_i^{-1} \boldsymbol{C}_V')^{-1} (\boldsymbol{B}_V - r\boldsymbol{I}) - r\boldsymbol{\Omega}_i \right\} P + \rho \boldsymbol{\Omega}_i S_i + \rho [(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} A_V]. \quad (27)$$

It follows from Lemma 1 that if investor i is ambiguous about asset n, her holding of asset n is zero. Then, aggregating all investors' effective asset holdings and applying the market clearing condition yield

$$\begin{aligned} D &= \rho \left[(\boldsymbol{C}_V')^{-1} \boldsymbol{QU} (\boldsymbol{C}_V)^{-1} A_V \right] + \rho \boldsymbol{\Sigma} F + \rho \left[(\boldsymbol{C}_V')^{-1} \boldsymbol{QU} (\boldsymbol{C}_V)^{-1} (\boldsymbol{B}_V - r\boldsymbol{I} - r\boldsymbol{\Sigma}) \right] P \\ &= W + Z. \end{aligned}$$

Therefore, by matching coefficients, we have

$$\begin{aligned} \boldsymbol{C}_V &= \frac{1}{\rho} \boldsymbol{\Sigma}^{-1} \\ \boldsymbol{B}_V &= r\boldsymbol{I} \\ A_V &= \frac{1}{\rho} \left[\rho^2 \boldsymbol{\Sigma} \boldsymbol{QU} \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \right]^{-1} W. \end{aligned}$$

Substituting these parameters into the pricing function and the individual investor's asset holding function, we get Proposition 5.

Q.E.D.

Equations (22) and (23) show that for each risky asset n, the measure of investors who know the precision of its random supply affects its equilibrium price. In particular, $\boldsymbol{QU}$ is the matrix of the average precisions of asset random supplies in the investors' subjective "worst-case scenarios," which positively affect the asset prices. Hence, *ceteris* *paribus*, if $q_k > q_n$, the equilibrium price of asset k is greater than that of asset n. Intuitively, when $q_k > q_n$, on average the subjective worst case for k is not as bad as for n, so the demand function for asset k is higher than the demand function for asset n. So when both assets have the same supply, asset k's price is higher than that of asset n.

Equation (26) characterizes investor i's effective risky asset holdings in equilibrium. For each asset n, if investor i knows the precision of its random supply, she will hold a position based on the equilibrium price. Formally, in such a case, the n^{th} diagonal entry of $\underline{\boldsymbol{U}}_i$ is $u_n > 0$; hence, the first term in equation (26) is positive. Furthermore, if such an investor receives a private signal about asset n's payoff, the n^{th} diagonal entry of $\boldsymbol{\Omega}_i$ is $\kappa_n > 0$, and so the second term is also positive.

At the other extreme, if investor i is ambiguous about asset n, both the n^{th} diagonal entry of $\underline{\boldsymbol{U}}_i$ and the n^{th} diagonal entry of $\boldsymbol{\Omega}_i$ are zero, implying that investor i holds a zero position of asset n. Therefore, even with a passive fund offering *VWMP*, ambiguity averse investors will not participate in some assets' markets. As a result, *VWMP* is not effective in encouraging ambiguity averse investors to hold better-diversified portfolios.

B.2 Asset Pricing with *VWMP*

We next analyze asset risk premia to see if *VWMP* is, as in the CAPM, the relevant pricing portfolio. Given any realized equilibrium price P, the volatility of asset payoffs derives from the supply shock only. Let $\text{diag}(P)$ be an $N \times N$ diagonal matrix, whose n^{th} diagonal element is the n^{th} element of the vector P. Generically, as no asset has a zero price, $\text{diag}(P)$ is invertible. Then, by the definition of $\text{diag}(P)$,

$$\text{diag}(P)^{-1}P = \mathbb{1}, \tag{28}$$

where $\mathbb{1} = (1, 1, \ldots, 1)'$. From equation (22), we can calculate the difference between individual assets' expected rates of return and the riskfree rate as

$$\text{diag}(P)^{-1}\mathbb{E}(F) - r\mathbb{1} = \text{diag}(P)^{-1}A_V, \tag{29}$$

where A_V is characterized in equation (23), and the expectation is taken conditional on the asset prices.

Since the pricing portfolio is *VWMP*, we calculate the market capitalization weights for individual assets as

$$\omega_V = \frac{\text{diag}(P)W}{P'W}. \tag{30}$$

Then, the variance of *VWMP* is

$$Var(R_V) = \left(\frac{1}{P'W}\right)^2 W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W, \tag{31}$$

and the covariances between individual assets and *VWMP* are

$$Cov(R, R_V) = \frac{1}{P'W}\text{diag}(P)^{-1}\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V. \tag{32}$$

Here, $\boldsymbol{C}_V$ is characterized in equation (25).

Therefore, when the pricing portfolio is *VWMP*, the assets' betas are

$$\beta_V = \frac{Cov(R, R_V)}{Var(R_V)} = P'W\frac{\text{diag}(P)^{-1}\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}, \tag{33}$$

and their alphas are

$$\alpha_V = \text{diag}(P)^{-1}A_V - \beta_V\left[\mathbb{E}(R_V) - r\right] = \text{diag}(P)^{-1}\left[\boldsymbol{I} - \frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}\right]A_V. \tag{34}$$

Simple algebra verifies that *VWMP* does not successfully price assets in the capital market.

Proposition 6. *With VWMP being the pricing portfolio, the assets' alphas are not equal to zero.*

Proof of Proposition 6:

From equation (34), individual assets' alphas are

$$\alpha_V = \text{diag}(P)^{-1}\left[\boldsymbol{I} - \frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}\right]A_V.$$

Because $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W} \neq \boldsymbol{I}$, $\alpha_V = 0$ if and only if A_V is an eigenvector of $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ with the associated eigenvalue 1. Therefore, A_V should not be a function of $\boldsymbol{Q}$, since $\boldsymbol{I} - \frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ is not a function of $\boldsymbol{Q}$. However, it follows from Proposition 5 that $A_V = \frac{1}{\rho}\left[\rho^2\boldsymbol{\Sigma}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\right]^{-1}W$, and so A_V does depend on $\boldsymbol{Q}$. Therefore, A_V is not an eigenvector of $\frac{\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V WW'}{W'\boldsymbol{C}_V\boldsymbol{U}^{-1}\boldsymbol{C}_V W}$ with the associated eigenvalue 1. Hence, $\alpha_V \neq 0$.

Q.E.D.

The non-zero alphas show that in contrast with the CAPM, in our model *VWMP* does not price assets correctly in the cross section. These alphas derive from both information asymmetry and ambiguity aversion. First, the traditional CAPM is based on homogeneous beliefs which are fully impounded in the market capitalization weights in *VWMP*. Hence, conditional on the asset prices, *VWMP* is mean-variance efficient. In contrast, in our setting, owing to information asymmetry, the average precision of private signals ($\boldsymbol{\Sigma}$) and the precision of random supply ($\boldsymbol{U}$) will determine the price signal distribution. This is directly implied by the equilibrium pricing function (equation (22)). Therefore, the weights in a portfolio that can price assets correctly must be functions of these two precisions. *VWMP*, however, has value weights, which are not functions of these two precisions, so it cannot be efficient conditional on asset prices in the financial market with information asymmetry. Hence, using *VWMP* as the pricing portfolio, the assets should have non-zero alphas.

Second, ambiguity aversion also affects appropriate index portfolio weights for asset pricing, which further contributes to nonzero alphas relative to *VWMP*. Intuitively, when more investors are ambiguous about an asset, its demand curve shifts leftward, leading to a lower price and a higher risk premium. We then refer to the increment in the asset's risk premium due to ambiguity aversion as the asset's *ambiguity premium*.

C The Information Separation Theorem

In this appendix, we prove a new separation theorem in an economy with asymmetric information but without model uncertainty.[18] Then the model is a traditional rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985). Proposition 7 characterizes a linear rational expectations equilibrium and shows investors' optimal risky assets holding when all parameters are common knowledge.

Proposition 7. *In the model whose parameters are all common knowledge among investors, there exists an equilibrium with the pricing function*

$$P = \boldsymbol{B}^{-1}\left[F - A - \boldsymbol{C}Z\right], \tag{35}$$

[18]The theorem we are about to state does not require the assumption of an uninformative prior. Since both the equilibrium pricing function and investors' equilibrium holdings are continuous in the prior precisions of assets' payoffs, substituting zero prior precision will lead to the separation theorem in the setting with a common uniform uninformed prior.

where

$$A = \frac{1}{\rho}\left[\rho^2(\mathbf{\Sigma U \Sigma}) + \mathbf{\Sigma}\right]^{-1} W \quad (36)$$

$$\mathbf{B} = r\mathbf{I} \quad (37)$$

$$\mathbf{C} = \frac{1}{\rho}\mathbf{\Sigma}^{-1}. \quad (38)$$

Any investor i's risky asset holding is

$$D_i = \left[\mathbf{I} + \frac{1}{\rho^2}(\mathbf{\Sigma U})^{-1}\right]^{-1} W + \rho\mathbf{\Omega}_i (S_i - rP). \quad (39)$$

Proof of Proposition 7:

Let's first prove a more general version of Proposition 7, when investors hold a common prior belief about F, $F \sim \mathcal{N}\left(\overline{F}, \mathbf{V}\right)$. As is standard in the literature of rational expectations equilibrium, we consider the linear pricing function

$$F = A + \mathbf{B}P + \mathbf{C}Z, \quad \text{with } C \text{ nonsingular.} \quad (40)$$

If and only if B is nonsingular, equation (40) can be rearranged to

$$P = -\mathbf{B}^{-1}A + \mathbf{B}^{-1}F - \mathbf{B}^{-1}\mathbf{C}Z, \quad (41)$$

which solves for prices. Recall that $S_i = F + \epsilon_i$, so conditional on F, P and S_i are independent. Therefore, we can write down assets' payoffs' posterior means and posterior variances conditional on all information that are available to investor i as follows.

First consider investor i's belief about F conditional on P. Conditional on P, F is normally distributed with mean $A + \mathbf{B}P$ and precision $[\mathbf{CU}^{-1}\mathbf{C}']^{-1}$. On the other hand, conditional on S_i, investor i's belief about F is also normally distributed, with mean S_i and precision $\mathbf{\Omega}_i$. Therefore, investor i's belief about F conditional on what the investor observes, P and S_i, is also normally distributed. The mean of the conditional distribution of F is the weighted average of the expectation conditional on the price P, the expectation conditional on investor i's private signal S_i, and the prior mean $\overline{F}$. Therefore, the conditional mean of F is

$$\left[(\mathbf{CU}^{-1}\mathbf{C}')^{-1} + \mathbf{\Omega}_i + \mathbf{V}^{-1}\right]^{-1}\left[(\mathbf{CU}^{-1}\mathbf{C}')^{-1}(A + \mathbf{B}P) + \mathbf{\Omega}_i S_i + \mathbf{V}^{-1}\overline{F}\right]. \quad (42)$$

The precision of the conditional distribution of F is

$$(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}. \tag{43}$$

Then, from any investor i's first order condition, investor i's demand is

$$\begin{aligned} D_i &= \rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right] \\ &\quad \left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]^{-1}\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - rP\right\} \\ &= \rho\left\{\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(A + \boldsymbol{B}P) + \boldsymbol{\Omega}_i S_i + \boldsymbol{V}^{-1}\overline{F}\right] - \left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1} + \boldsymbol{\Omega}_i + \boldsymbol{V}^{-1}\right]rP\right\} \\ &= \rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\boldsymbol{\Omega}_i - r\boldsymbol{V}^{-1}\right\}P \\ &\quad + \rho\boldsymbol{\Omega}_i S_i + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}]. \end{aligned} \tag{44}$$

Integrating across all investors' demands gives the aggregated demand as

$$\begin{aligned} \int_0^1 D_i \mathrm{d}i = &\rho\left\{(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\left(\int_0^1 \boldsymbol{\Omega}_i \mathrm{d}i\right) - r\boldsymbol{V}^{-1}\right\}P \\ &+ \rho\left(\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i\right) + \rho[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}]. \end{aligned} \tag{45}$$

By equation (1), we have $\int_0^1 \boldsymbol{\Omega}_i \mathrm{d}i = \boldsymbol{\Sigma}$. Also, note that

$$\int_0^1 \boldsymbol{\Omega}_i S_i \mathrm{d}i = \boldsymbol{\Sigma}F.$$

Therefore, from the market clearing condition, we have

$$\int_0^1 D_i \mathrm{d}i = Z + W. \tag{46}$$

In an equilibrium, both equation (40) and equation (46) hold simultaneously for any realized F and Z, therefore, by matching coefficients in these two equations, we have

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}A + \boldsymbol{V}^{-1}\overline{F}\right] - W = -\boldsymbol{C}^{-1}A \tag{47}$$

$$\rho\left[(\boldsymbol{C}\boldsymbol{U}^{-1}\boldsymbol{C}')^{-1}(\boldsymbol{B} - r\boldsymbol{I}) - r\boldsymbol{\Sigma} - r\boldsymbol{V}^{-1}\right] = -\boldsymbol{C}^{-1}\boldsymbol{B} \tag{48}$$

$$\rho\boldsymbol{\Sigma} = \boldsymbol{C}^{-1} \tag{49}$$

Therefore, from equation (49), we have

$$\boldsymbol{C} = \frac{1}{\rho}\boldsymbol{\Sigma}^{-1}$$

Obviously, $\boldsymbol{C}$ is positive definite and symmetric. Then from equation (47), we have

$$[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]A = \frac{1}{\rho}W - \boldsymbol{V}^{-1}\overline{F}.$$

Because both $(\boldsymbol{\Sigma U \Sigma})$ and $\boldsymbol{\Sigma}$ are both positive definite, we have

$$A = [\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]^{-1}\left(\frac{1}{\rho}W - \boldsymbol{V}^{-1}\overline{F}\right).$$

From equation (48), we have

$$[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}](\boldsymbol{B} - r\boldsymbol{I}) = r\boldsymbol{V}^{-1}.$$

Again, because $[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]$ is positive definite, we have

$$\boldsymbol{B} = r\boldsymbol{I} + r[\rho^2(\boldsymbol{\Sigma U \Sigma}) + \boldsymbol{\Sigma}]^{-1}\boldsymbol{V}^{-1}.$$

Obviously, $\boldsymbol{B}$ is invertible. By substituting A, $\boldsymbol{B}$, and $\boldsymbol{C}$ into equation (41), we solve the equilibrium pricing function.

Now, let's look at any investor i's holding. Substituting the coefficients into investor i's holding function (44), we have

$$D_i = \left(\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1}\right)^{-1}W + \rho\left[\boldsymbol{I} + \rho^2\boldsymbol{\Sigma U}\right]^{-1}\boldsymbol{V}^{-1}(\overline{F} - rP) + \rho\boldsymbol{\Omega}_i(S_i - rP).$$

Finally, because the pricing function P and any investor i's demand function D_i are continuous in $\boldsymbol{V}^{-1}$, we can substitute $\boldsymbol{V}^{-1} = 0$ to get Proposition 7.

Q.E.D.

Owing to supply shocks, asset prices are not fully revealing, so information asymmetry persists in equilibrium and different investors have different asset holdings. An investor's asset holding is the sum of two components. The first term in equation (39),

$$\left[\boldsymbol{I} + \frac{1}{\rho^2}(\boldsymbol{\Sigma U})^{-1}\right]^{-1}W$$

is just *RAMP*, which is deterministic.

The second component of any investor's risky asset holding, the second term in (39), is what we call information-based portfolio. This position, $\rho\boldsymbol{\Omega}_i(S_i - rP)$, consists of extra holdings in the securities about which the investor has information. Investor i

holds such an extra position of an asset n if and only if the n^{th} diagonal entry of $\mathbf{\Omega}_i$ is $\kappa_n > 0$. This suggests that any investor i holds direct positions of a risky asset because possessing an informative signal about such an asset reduces its conditional volatility (independent of the signal realization). Investor i's direct positions of a risky asset also come from her speculation, which is taken to exploit superior information. Different investors, even if they are informed about asset n, hold different speculative portfolios, because they receive heterogeneous private signals.

Crucially, in each investor's equilibrium asset holdings in equation (39), the two components depend upon investors' information sets in different ways. The first component, *RAMP*, is formed based only on the information that the investor gleans from asset prices; it is independent of the investor's private information. In contrast, the second component, the information-based portfolio, can be formed based only on the investor's own private information; it is independent of the information content of the market price. Since the supply shock precisions do not affect the distributions of the private signals, it follows that the information-based portfolio is *independent of the supply shock precisions*. The reason for this independence is that each individual investor is "small" and thus her trading cannot affect the asset prices and thus the price informativeness. [19] This independence implies the Information Separation Theorem, which is stated in Section 3.1 and repeated below.

Theorem (The Information Separation Theorem)**.** *When the characteristics of all assets are common knowledge, equilibrium portfolios have three components: a deterministic risk-adjusted market portfolio (RAMP); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.*

Theorem 1 indicates that any investor can form an optimal portfolio in separate steps: (1) buy one share of *RAMP*; (2) buy the information-based portfolio using only private information, not the information extracted from price; and (3) put any left-over funds into the riskfree asset. This separation theorem derives from market equilibrium as well as optimization considerations. This differs from those (non-informational) separation theorems in the literature that are based solely on individual optimization arguments.[20]

[19] Vives (2008) derives investors' equilibrium asset holdings in a single-asset environment with a normal prior and zero aggregate endowment. Therefore, his result cannot be directly used in our analysis when investors are ambiguity averse about some assets.

[20] It may seem puzzling that none of the three portfolio components depend on the information that an investor extracts from price. How then does this information enter into the investor's portfolio decision?

In our model, the fund can provide *RAMP* because it knows all the model parameters, and *RAMP* does not include any investor's private information. Meanwhile, the information-based portfolio is exactly the same as the direct holdings of the risky assets in Proposition 1. To form the information-based portfolio, an investor does not need to extract information from the equilibrium price: she can treat the equilibrium prices as given parameters, and solve for the information-based portfolio from her CARA utility maximization problem as in a partial equilibrium model.

The answer is that *RAMP* is optimal precisely because of the ability of investors to extract information from price. As mentioned before, *RAMP* is deterministic; it does not depend on the private signals. But the fact that *RAMP* is an optimal choice is true only because investors update their beliefs based on price. So the optimal portfolio choice is indeed influenced by such information extraction.

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