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Index Number Concepts, Measures and Decompositions of Productivity Growth

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Abstract

This paper explores the definitions and properties of total factor productivity growth (TFPG) indexes, focusing especially on the Paasche, Laspeyres, Fisher, Törnqvist, and implicit Törnqvist ones. These indexes can be evaluated from observable price and quantity data, and certain of these are shown to be measures of TFPG concepts and theoretical indexes that have been proposed in the literature. The mathematical relationships between these and quantity aggregates, financial measures, and price and quantity indexes are explored. Decompositions of the productivity growth indexes are also given. The paper concludes with a brief overview of some limitations on our analysis.

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> "**Productivity** A ratio of output to input."
> (Atkinson, Banker, Kaplan and Young, 1995, *Management Accounting,* p. 514)

> "To profit from productivity improvement, management needs measurement procedures for monitoring productivity performance and identifying improvement opportunities."
> (David M. Miller, *Harvard Business Review,* May–June 1984, pp. 145–153)

This paper presents and considers a number of conceptual definitions of productivity growth, and measures and decompositions of these that are useful not only for economic analysts and those involved in the construction of official statistics but also for business managers.

Section 1 introduces four concepts and measures of total factor productivity growth (TFPG) for the simplistic case in which the index number problem is absent: the one input, one output case. In the 1-1 case, it is easily seen that the measures are all equal.

In the general N input, M output case, the quantities for the different inputs and outputs must be aggregated and this leads to the use of index number formulas. If price weights are used, then issues of price change must be dealt with too. In Section 2, we introduce

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the Paasche, Laspeyres, Fisher and Törnqvist TFPG formulas. We present these after first defining aggregates and quantity and price indexes that can be used to define the TFPG index formulas. The measures of the four concepts of TFPG that are defined for a general N input, M output production situation are shown to be equal for appropriate choices of functional forms for aggregating the quantity or value and price data.

Section 3 gives a way of decomposing any TFPG index in terms of input-output coefficients. The reciprocals of the input-output coefficients—the output-input coefficients—are the most widely used type of single factor productivity measure for shop floor operations management. The decomposition presented provides a framework for examining the effects on overall TFPG of changes in specific single factor productivity coefficients.

Two main approaches to choosing among the different proposed functional forms for the TFPG index are the axiomatic (or test) approach reviewed briefly in Section 4, and the exact index number approach introduced in Section 5 and then used in Section 6. The axiomatic approach makes use of lists of desired properties, termed axioms or tests, that are formalizations of common sense properties of good index numbers or generalizations of properties that hold for all proposed index number formulas in the simplistic 1-1 case. The exact index number approach is rooted in economic theory, and is sometimes referred to as an economic approach. Section 7 is devoted to yet another measure of productivity performance proposed by Diewert and Morrison and decompositions of that measure. Section 8 concludes.

1. Four Concepts of TFPG

It is convenient to introduce concepts and notation in the simplified context of a one input, one output production process. For $t = 0, 1, \ldots, T$, the quantity of the input used in period t is x_1^t with unit price w_1^t and the quantity of the output produced is y_1^t with unit price p_1^t.

The basic definition of total factor productivity (TFP) is the rate of transformation of total input into total output. Engineers and others interested in physical efficiency often focus on this transformation rate, which is measured by the ratio of output produced to input:

$$\text{TFP}^t \equiv \left(y_1^t / x_1^t\right) \equiv a^t. \tag{1}$$

The superscript on TFP indicating the time period will only be shown hereafter when this is not clear from the context. The parameter a^t defined along with TFP in (1) is an output-input coefficient. An output-input coefficient always involves just one output and one input, although coefficients of this sort are often defined and used in the context of multiple input, multiple output production situations.[1] Indeed, we show a new way of doing this in Section 3.

In this paper, we focus on total factor productivity growth rather than TFP.[2] A number of different ways have been proposed for conceptualizing TFPG, four of which are considered in this paper.[3] A first concept is simply the rate of growth for TFP. Measures of this first concept will be denoted by TFPG(1). For the 1-1 case, the growth in TFP between periods s and t is measured by the ratio of the output-input coefficients for periods t and s:[4]

$$\text{TFPG}(1) \equiv \left(y_1^t / x_1^t\right) / \left(y_1^s / x_1^s\right) = a^t / a^s. \tag{2}$$

A second concept of total factor productivity growth is the ratio of the rate of output growth to input growth. For the 1-1 case, the natural measure for this is

$$\text{TFPG}(2) \equiv \left(y_1^t/y_1^s\right)/\left(x_1^t/x_1^s\right). \tag{3}$$

The third and fourth concepts relate TFPG to the financial revenue and cost totals and to margins. For the 1-1 case, total revenue and cost are given for $t = 1, \ldots, T$ by

$$R^t \equiv p_1^t y_1^t \quad \text{and} \quad C^t \equiv w_1^t x_1^t. \tag{4}$$

Dividing the revenue ratio for periods t and s by the output price ratio for the two periods yields

$$(R^t/R^s)/(p^t/p^s) = \left(p_1^t y_1^t/p_1^s y_1^s\right)/\left(p_1^t/p_1^s\right) = y_1^t/y_1^s. \tag{5}$$

Similarly, when the cost ratio for periods t and s is divided by the input price ratio, we get

$$(C^t/C^s)/(w^t/w^s) = \left(w_1^t x_1^t/w_1^s x_1^s\right)/\left(w_1^t/w_1^s\right) = x_1^t/x_1^s. \tag{6}$$

The third concept is the growth rate for real revenue versus real cost. For the 1-1 case, this is measured by the rate of growth in the ratio of revenue to cost controlling for price change:

$$\text{TFPG}(3) \equiv \left[\frac{R^t/R^s}{p_1^t/p_1^s}\right] \Big/ \left[\frac{C^t/C^s}{w_1^t/w_1^s}\right] = \left(\frac{y_1^t}{y_1^s}\right) \Big/ \left(\frac{x_1^t}{x_1^s}\right). \tag{7}$$

Business managers are interested in ensuring that revenues exceed costs, which leads to an interest in margins. The period t margin, m^t, can be measured by

$$1 + m^t \equiv R^t/C^t, \tag{8}$$

and the measure for margin growth controlling for price change is defined for the 1-1 case as

$$\text{TFPG}(4) \equiv [(1+m^t)/(1+m^s)]\left[\left(w_1^t/w_1^s\right)/\left(p_1^t/p_1^s\right)\right]. \tag{9}$$

If we interpret the margin as a reward for managerial or entrepreneurial input, then TFPG(4) can be interpreted as the growth rate of the price of the inputs broadly defined so as to include managerial and entrepreneurial input divided by the growth rate of the price of the output good.

Using (8) to eliminate the margin growth rate on the right-hand side of (9) and comparing this with the expressions in (2), (3) and (7), we see that in the 1-1 case, the measures for the four concepts of total factor productivity growth are all equal.

2. The *N*-*M* Case

Most real production processes yield multiple outputs and virtually all use multiple inputs, leading to a need for index numbers to aggregate these outputs and inputs. We begin by defining quantity aggregates that are components of the Paasche, Laspeyres and Fisher Ideal quantity, price and TFPG indexes, and then give the formulas for these indexes. Törnqvist and implicit Törnqvist index numbers are also defined.

2.1. Price Weighted Quantity Aggregates

For a general N-input, M-output production process, the period t input and output price vectors are denoted by $w^t \equiv [w_1^t, \ldots, w_N^t]$ and $p^t \equiv [p_1^t, p_2^t, \ldots, p_M^t]$, while $x^t \equiv [x_1^t, \ldots, x_N^t]$ and $y^t \equiv [y_1^t, \ldots, y_M^t]$ denote the period t input and output quantity vectors.

Nominal total cost C^t and revenue R^t can be viewed as price weighted quantity aggregates of the micro level data for the individual transactions, and are defined as follows for periods t and s:

$$C^t \equiv \sum_{n=1}^{N} w_n^t x_n^t, \qquad R^t \equiv \sum_{m=1}^{M} p_m^t y_m^t, \tag{10}$$

$$C^s \equiv \sum_{n=1}^{N} w_n^s x_n^s \quad \text{and} \quad R^s \equiv \sum_{m=1}^{M} p_m^s y_m^s. \tag{11}$$

We also define four hypothetical quantity aggregates.[5] The first two result from evaluating period t quantities using period s price weights:

$$\sum_{n=1}^{N} w_n^s x_n^t \quad \text{and} \quad \sum_{m=1}^{M} p_m^s y_m^t \tag{12}$$

These aggregates are what the cost and revenue would have been if the period t inputs had been purchased and the period t outputs had been sold at period s prices. In contrast, the third and fourth aggregates are sums of period s quantities evaluated using period t prices:

$$\sum_{n=1}^{N} w_n^t x_n^s \quad \text{and} \quad \sum_{m=1}^{M} p_m^t y_m^s. \tag{13}$$

These are what the cost and revenue would have been if the period s inputs had been purchased and the period s outputs had been sold at period t prices.

The eight aggregates given in (10) through (13) are all that are needed to define the Paasche, Laspeyres and Fisher quantity, price, and TFPG indexes. Traditionally these were defined as weighted averages of quantity and price relatives.[6] The equivalent definitions presented here are more convenient for establishing that each of these TFPG indexes is a measure of all four of the different concepts of TFPG introduced in Section 1.

2.2. The Paasche, Laspeyres and Fisher Quantity and Price Indexes

The Paasche (1874), Laspeyres (1871), and Fisher (1922, p. 234) output quantity indexes can be defined as follows using the quantity aggregates given in (10)–(13):

$$Q_P \equiv \sum_{i=1}^{M} p_i^t y_i^t \bigg/ \sum_{j=1}^{M} p_j^t y_j^s, \tag{14}$$

$$Q_L \equiv \sum_{i=1}^{M} p_i^s y_i^t \bigg/ \sum_{j=1}^{M} p_j^s y_j^s, \tag{15}$$

and

$$Q_F \equiv (Q_P Q_L)^{(1/2)}. \tag{16}$$

Similarly, the Paasche, Laspeyres and Fisher input quantity indexes can be defined as:

$$Q_P^* \equiv \sum_{i=1}^{N} w_i^t x_i^t \bigg/ \sum_{j=1}^{N} w_j^t x_j^s, \tag{17}$$

$$Q_L^* \equiv \sum_{i=1}^{N} w_i^s x_i^t \bigg/ \sum_{j=1}^{N} w_j^s x_j^s, \tag{18}$$

and

$$Q_F^* \equiv (Q_P^* Q_L^*)^{(1/2)}. \tag{19}$$

Output and input quantity indexes are all that are needed to define a measure of the second concept of TFPG since an output quantity index is a measure of growth for total output and an input quantity index is a measure of growth for total input. For the Paasche, Laspeyres and Fisher indexes, we show in Section 2.3 that a measure of the first concept of TFPG can be defined by rearranging the aggregates defining the measure of the second concept. However, in order to specify measures of the third and fourth concepts for the multiple input, multiple output case, price indexes are needed too.

Price indexes can be constructed with any of the functional forms used for quantity indexes by reversing the roles of the prices and quantities in the quantity index. Thus the Paasche and Laspeyres output and input price indexes can be defined as:

$$P_P \equiv \sum_{i=1}^{M} p_i^t y_i^t \bigg/ \sum_{j=1}^{M} p_j^s y_j^t, \tag{20}$$

$$P_P^* \equiv \sum_{i=1}^{N} w_i^t x_i^t \bigg/ \sum_{j=1}^{N} w_j^s x_j^t, \tag{21}$$

$$P_L \equiv \sum_{i=1}^{M} p_i^t y_i^s \bigg/ \sum_{j=1}^{M} p_j^s y_j^s, \tag{22}$$

and

$$P_L^* \equiv \sum_{i=1}^{N} w_i^t x_i^s \Bigg/ \sum_{j=1}^{N} w_j^s x_j^s. \tag{23}$$

The Fisher output and input price indexes are given by

$$P_F \equiv (P_P P_L)^{(1/2)} \tag{24}$$

and

$$P_F^* \equiv (P_P^* P_L^*)^{(1/2)}. \tag{25}$$

A price index is the implicit counterpart of a quantity index if the product rule is satisfied. This rule requires that the product of the quantity and price indexes must equal the total cost ratio for input side indexes or the total revenue ratio for output side indexes.[7] Usually the implicit price index will not have the same functional form as the quantity index it is associated with. For example, the Paasche price index is the implicit counterpart of a Laspeyres quantity index, and the Laspeyres price index is the implicit counterpart of a Paasche quantity index. The Fisher indexes are unusual in that the Fisher price index satisfies the product test rule when paired with a Fisher quantity index.

In defining and proving equalities for the measures of the four concepts of TFPG for a general multiple input, multiple output production situation, we use the following implications of the product rule: for the Paasche, Laspeyres and Fisher indexes, on the input side we have

$$Q_P^* \times P_L^* = Q_L^* \times P_P^* = Q_F^* \times P_F^* = C^t/C^s, \tag{26a}$$

and on the output side we have

$$Q_P \times P_L = Q_L \times P_P = Q_F \times P_F = R^t/R^s. \tag{26b}$$

2.3. *TFPG Measures for the N-M Case*

The traditional definition of a total factor productivity growth index in the index number literature is as a ratio of output and input quantity indexes:

$$\text{TFPG} \equiv Q/Q^*. \tag{27}$$

Thus the Paasche, Laspeyres and Fisher TFPG indexes can be defined using the Paasche, Laspeyres and Fisher quantity indexes. Given a choice of *any one* of these three functional forms, we will prove that the corresponding multiple input, multiple output case measures are all equal for the four concepts of TFPG introduced in Section 1.

To establish these equalities, we use the product rule results to define Paasche, Laspeyres and Fisher TFPG(3) measures. Then we use the definitions of the components of the TFPG(3) measures to define and establish the equality of TFPG(2) and TFPG(1) measures.

The definitions and equalities for these measures are as follows:

$$\mathrm{TFPG}_P = \frac{Q_P}{Q_P^*} = \frac{(R^t/R^s)/P_L}{(C^t/C^s)/P_L^*} \equiv \mathrm{TFPG}(3)_P$$

using (27) and (26)

$$= \frac{\dfrac{\sum_{m=1}^{M} p_m^t y_m^t}{\sum_{m=1}^{M} p_m^s y_m^s} \dfrac{\sum_{m=1}^{M} p_m^s y_m^s}{\sum_{m=1}^{M} p_m^t y_m^s}}{\dfrac{\sum_{n=1}^{N} w_n^t x_n^t}{\sum_{n=1}^{N} w_n^s x_n^s} \dfrac{\sum_{n=1}^{N} w_n^s x_n^s}{\sum_{n=1}^{N} w_n^t x_n^s}} = \frac{\sum_{m=1}^{M} p_m^t y_m^t / \sum_{m=1}^{M} p_m^t y_m^s}{\sum_{n=1}^{N} w_n^t x_n^t / \sum_{n=1}^{N} w_n^t x_n^s} \equiv \mathrm{TFPG}(2)_P$$

using (10), (11) and also (22) and (23)

$$= \frac{\sum_{m=1}^{M} p_m^t y_m^t / \sum_{n=1}^{N} w_n^t x_n^t}{\sum_{m=1}^{M} p_m^t y_m^s / \sum_{n=1}^{N} w_n^t x_n^s} \equiv \mathrm{TFPG}(1)_P \tag{28}$$

$$\mathrm{TFPG}_L = \frac{Q_L}{Q_L^*} = \frac{(R^t/R^s)/P_P}{(C^t/C^s)/P_P^*} \equiv \mathrm{TFPG}(3)_L$$

using (27) and (26)

$$= \frac{\dfrac{\sum_{m=1}^{M} p_m^t y_m^t}{\sum_{m=1}^{M} p_m^s y_m^s} \dfrac{\sum_{m=1}^{M} p_m^s y_m^t}{\sum_{m=1}^{M} p_m^t y_m^t}}{\dfrac{\sum_{n=1}^{N} w_n^t x_n^t}{\sum_{n=1}^{N} w_n^s x_n^s} \dfrac{\sum_{n=1}^{N} w_n^s x_n^t}{\sum_{n=1}^{N} w_n^t x_n^t}} = \frac{\sum_{m=1}^{M} p_m^s y_m^t / \sum_{m=1}^{M} p_m^s y_m^s}{\sum_{n=1}^{N} w_n^s x_n^t / \sum_{n=1}^{N} w_n^s x_n^s} \equiv \mathrm{TFPG}(2)_L$$

using (10), (11) and also (20) and (21)

$$= \frac{\sum_{m=1}^{M} p_m^s y_m^t / \sum_{n=1}^{N} w_n^s x_n^t}{\sum_{m=1}^{M} p_m^s y_m^s / \sum_{n=1}^{N} w_n^s x_n^s} \equiv \mathrm{TFPG}(1)_L \tag{29}$$

$$\mathrm{TFPG}_F = \frac{Q_F}{Q_F^*} = \frac{(R^t/R^s)/P_F}{(C^t/C^s)/P_F^*} \equiv \mathrm{TFPG}(3)_F$$

using (27) and (26)

$$= \frac{\left[\left(\frac{R^t}{R^s}\right) P_L\right]^{1/2} \left[\left(\frac{R^t}{R^s}\right) P_P\right]^{1/2}}{\left[\left(\frac{C^t}{C^s}\right) P_L^*\right]^{1/2} \left[\left(\frac{C^t}{C^s}\right) P_P^*\right]^{1/2}} = \frac{\left[\dfrac{\sum_{m=1}^{M} p_m^t y_m^t}{\sum_{m=1}^{M} p_m^t y_m^s}\right]^{1/2} \left[\dfrac{\sum_{m=1}^{M} p_m^s y_m^t}{\sum_{m=1}^{M} p_m^s y_m^s}\right]^{1/2}}{\left[\dfrac{\sum_{n=1}^{N} w_n^t x_n^t}{\sum_{n=1}^{N} w_n^t x_n^s}\right]^{1/2} \left[\dfrac{\sum_{n=1}^{N} w_n^s x_n^t}{\sum_{n=1}^{N} w_n^s x_n^s}\right]^{1/2}} \equiv \mathrm{TFPG}(2)_F$$

using (16), (26), (10), (11), and (20)–(23)

$$= \frac{\left[\dfrac{\sum_{m=1}^{M} p_m^t y_m^t}{\sum_{n=1}^{N} w_n^t x_n^t}\right]^{1/2} \left[\dfrac{\sum_{m=1}^{M} p_m^s y_m^t}{\sum_{n=1}^{N} w_n^s x_n^t}\right]^{1/2}}{\left[\dfrac{\sum_{m=1}^{M} p_m^t y_m^s}{\sum_{n=1}^{N} w_n^t x_n^s}\right]^{1/2} \left[\dfrac{\sum_{m=1}^{M} p_m^t y_m^s}{\sum_{n=1}^{N} w_n^t x_n^s}\right]^{1/2}} \equiv \mathrm{TFPG}(1)_F \tag{30}$$

TFPG(4) is the rate of growth in the margin after controlling for price change. In the general N-M case, just as in the 1-1 one, the margin m^t is given for $t = 0, 1, \ldots, T$ by

$$1 + m^t \equiv R^t/C^t. \tag{31}$$

Depending on whether Laspeyres, Paasche or Fisher price indexes are used to deflate the cost and revenue components of the margin, the expressions for TFPG(3) given in (28), (29) and (30) can be rewritten as:

$$\text{TFPG}(4)_P \equiv [(1 + m^t)/(1 + m^s)][P_L^*/P_L], \tag{32}$$

$$\text{TFPG}(4)_L \equiv [(1 + m^t)/(1 + m^s)][P_P^*/P_P], \tag{33}$$

and

$$\text{TFPG}(4)_F \equiv [(1 + m^t)/(1 + m^s)][P_F^*/P_F]. \tag{34}$$

Notice that if the margins m^t are zero, regardless of the reasons, then each of these expressions for TFPG(4) reduces to the ratio of the input price index to the output price index.[8]

2.4. Other Index Number Formulas

Many other index number formulas have been proposed besides the Paasche, Laspeyres and Fisher.[9] Here we will use Q_G and P_G and Q_G^* and P_G^* to denote any given output and input quantity and price indexes that satisfy the product rule so that $Q_G P_G = (R^t/R^s)$ and $Q_G^* P_G^* = (C^t/C^s)$. From these product rule results and (31), it is easily seen that the following measures of concepts 2, 3 and 4 of TFPG are equal:

$$\begin{aligned} \frac{(R^t/R^s)/P_G}{(C^t/C^s)/P_G^*} &\equiv \text{TFPG}(3)_G \\ &= Q_G/Q_G^* \equiv \text{TFPG}(2)_G \\ &= [(1 + m^t)/(1 + m^s)][P^*/P] \equiv \text{TPFG}(4). \end{aligned} \tag{35}$$

But what about $\text{TFPG}(1)_G$? A measure of the growth in the rate of transformation of total input into total output ideally should be defined using measures of total output and total input that are comparable for periods s and t in the sense that the micro level quantities for both periods are aggregated using the same price weights.[10] The quantity aggregates that are the components of the Paasche, Laspeyres and Fisher TFPG(1) measures defined in the first line of (28), (29) and (30) satisfy this comparability over time ideal.[11] However, there are many other index number formulas for which it is not possible to define this sort of an ideal TFPG(1) measure that also equals the corresponding measures for the other three concepts of TFPG.

For any pair of quantity and price indexes satisfying the product test, from (35) and the product rule implications we see that the following expressions equal those defined in (35) for $\text{TFPG}(2)_G$, $\text{TFPG}(3)_G$ and $\text{TFPG}(4)_G$:

$$\frac{Q_G}{Q_G^*} = \frac{(R^t/R^s)/P}{(C^t/C^s)/P^*} = \frac{\sum_{m=1}^{M} \left(p_m^t/P_G\right) y_m^t / \sum_{m=1}^{M} p_m^s y_m^s}{\sum_{n=1}^{N} \left(w_n^t/P_G^*\right) x_n^t / \sum_{n=1}^{N} w_n^s x_n^s}. \tag{36}$$

In the last of these expressions, the price vectors (p^t/P_G) and (w^t/P_G^*) appearing in the period t output and input quantity aggregates are the period t prices expressed in period s dollars. If we choose this expression as the measure of $\text{TFPG}(1)_G$, then for any index number formulas other than the Paasche, Laspeyres or Fisher, this measure will not be ideal in the sense of using the same price weights to compare the period t and period s quantities. However, there is an approximate solution to this problem for indexes that satisfy the product rule and are also what is termed "superlative." This approximate solution makes use of the Fisher functional form: a functional form for which we have an ideal TFPG(1) measure, defined in (30).

Diewert coined the term superlative for an index number functional form that is "exact" in that it can be derived algebraically from a producer or consumer behavioral equation that satisfies the Diewert flexibility criterion. According to this criterion, an equation is flexible if it can provide a second order approximation to an arbitrary twice continuously differentiable linearly homogeneous function. Diewert (1976, 1978) and Hill (2000) established that all of the commonly used superlative index number formulas (including the Fisher, and also the Törnqvist and implicit Törnqvist functional forms introduced below) approximate each other to the second order when evaluated at an equal price and quantity point. This is a numerical analysis approximation result that does not rely on any assumptions of economic theory.

Because the Fisher quantity and price indexes also satisfy the product rule, we have $Q_G P_G = (R^t/R^s) = Q_F P_F$ and $Q_G^* P_G^* = (C^t/C^s) = Q_F^* P_F^*$, and dividing through by P_G and P_G^*, respectively, yields

$$\frac{Q_G}{Q_G^*} = \left[\frac{Q_F}{Q_F^*}\right]\left[\frac{P_F/P_G}{P_F^*/P_G^*}\right]. \tag{37}$$

From (37), (35) and (30) we see that if we define the measure for the first concept of TFPG as

$$\text{TPFG}(1)_G \equiv \text{TPFG}(1)_F \left[\frac{P_F/P_G}{P_F^*/P_G^*}\right], \tag{38}$$

this measure will equal $\text{TFPG}(2)_G$, $\text{TFPG}(3)_G$ and $\text{TFPG}(4)_G$ as defined in (35). However, this measure does not have the ideal comparability over time property. In this $\text{TFPG}(1)_G$ measure, the period t price vectors, p^t and w^t, of the $\text{TFPG}(1)_F$ component are replaced by $(p^t/(P_F/P_G))$ and $(w^t/(P_F^*/P_G^*))$. As a consequence, unless the given price indexes are Laspeyres or Paasche or Fisher ones, the period t and period s quantities compared by the measure will not be aggregated using the same price weights when there have been changes in relative prices. Nevertheless, from (38) and the Diewert (1976, 1978)/Hill (2000) approximation results for superlative index numbers, it follows that when the chosen quantity and price indexes are any of the commonly used superlative indexes such as the Törnqvist or implicit Törnqvist, then we can use the result that $\text{TFPG}(1)_G \cong \text{TFPG}(1)_F$.

2.5. The Törnqvist (or Translog) Indexes[12]

Törnqvist (1936) indexes are weighted geometric averages of growth rates for the microeconomic data (the quantity or price relatives). These indexes appear more complicated than the others defined so far, but are also widely used. It is the formula for the natural logarithm of a Törnqvist index that is usually shown. For the output quantity index, this is

$$\ell\text{n}\, Q_T = (1/2)\sum_{m=1}^{M}\left[\left(p_m^s y_m^s \Big/ \sum_{i=1}^{M} p_i^s y_i^s\right) + \left(p_m^t y_m^t \Big/ \sum_{j=1}^{M} p_j^t y_j^t\right)\right]\ell\text{n}\big(y_m^t / y_m^s\big). \quad (39)$$

The Törnqvist input quantity index Q_T^* is defined analogously, with input quantities and prices substituted for the output quantities and prices in (39).

Reversing the role of the prices and quantities in the formula for the Törnqvist output quantity index yields the Törnqvist output price index, P_T, defined by

$$\ell\text{n}\, P_T = (1/2)\sum_{m=1}^{M}\left[\left(p_m^s y_m^s \Big/ \sum_{i=1}^{M} p_i^s y_i^s\right) + \left(p_m^t y_m^t \Big/ \sum_{j=1}^{M} p_j^t y_j^t\right)\right]\ell\text{n}\big(p_m^t / p_m^s\big). \quad (40)$$

The input price index P_T^* is defined in a similar manner.

The implicit Törnqvist output quantity index, denoted by $Q_{\tilde{T}}$, is defined implicitly by[13] $(R^t/R^s)/P_T \equiv Q_{\tilde{T}}$, and the implicit Törnqvist input quantity index, $Q_{\tilde{T}}^*$, is defined analogously using the cost ratio and P_T^*. The implicit Törnqvist output price index, $P_{\tilde{T}}$, is given by $(R^t/R^s)/Q_T \equiv P_{\tilde{T}}$, and the implicit Törnqvist input price index, $P_{\tilde{T}}^*$, is defined analogously.

Using the Törnqvist quantity and the implicit Törnqvist price indexes, or the implicit Törnqvist quantity and the Törnqvist price indexes, measurement formulas for concepts 2–4 of TFPG can be specified as in (35) above. As already noted, when Törnqvist or implicit Törnqvist indexes are used, it is not possible to define a TFPG(1) measure that is ideal in the sense discussed in Section 2.4. However, these are superlative indexes for which the Section 2.4 approximation result applies; that is, we have $\text{TFPG}(1)_T \cong \text{TFPG}(1)_F$ and $\text{TFPG}(1)_{\tilde{T}} \cong \text{TFPG}(1)_F$. The essence of this result is that the Fisher TFPG(1) measure, which is ideal in the sense discussed in Section 2.4, should be a good approximate measure for the first concept of TFPG when the Törnqvist or the implicit Törnqvist indexes are selected.

3. An Input-Output Coefficient Decomposition of TFPG

> "[I]t is our hypothesis that non-financial and financial control systems serve very different roles in organizations. The non-financial systems are used for day-to-day operations control."
>
> (Armitage and Atkinson, 1990, *The Choice of Productivity Measures in Organizations: A Field Study of Practice,* p. 141)

As Armitage and Atkinson report, business surveys reveal that single factor productivity and efficiency measures are widely used for operations monitoring. Thus, it is important to relate the single factor measures to overall TFPG. Fortunately, the TFPG(3) form of a total factor productivity growth index can be conveniently decomposed in terms of input-output coefficients or their reciprocals.

As established in the previous section, any TFPG index can be expressed in the form

$$\text{TFPG} = \frac{[R^t/\tilde{P}]/[C^t/\tilde{P}^*]}{R^s/C^s} \tag{41}$$

where $\tilde{P}$ and $\tilde{P}^*$ are the implicit counterparts of Q and Q^*. For any given time period, the ratio of total nominal revenue to total nominal cost can be rewritten as

$$\begin{aligned} R^t/C^t &= \sum_{m=1}^{M} \left[p_m^t y_m^t \Big/ \left(\sum_{n=1}^{N} w_n^t x_n^t \right) \right] \\ &= \sum_{m=1}^{M} \left[1 \Big/ \left(\sum_{n=1}^{N} w_n^t x_n^t / p_m^t y_m^t \right) \right] \\ &= \sum_{m=1}^{M} \left[\sum_{n=1}^{N} \left(w_n^t / p_m^t \right) \left(x_n^t / y_m^t \right) \right]^{-1}. \end{aligned} \tag{42}$$

It is clear from (42) that, after controlling for price changes, improvements in any single factor efficiency measure, say (y_m^t/x_n^t), contribute to an improved revenue cost ratio, as might be expected.

Substitution of (42) into (41) yields:

$$\text{TFPG} = \frac{\sum_{m=1}^{M} \left\{ \sum_{n=1}^{N} \left[\left(w_n^t/P^* \right) / \left(p_m^t/P \right) \right] \left[x_n^t/y_m^t \right] \right\}^{-1}}{\sum_{m=1}^{M} \left\{ \sum_{n=1}^{N} \left[w_n^s/p_m^s \right] \left[x_n^s/y_m^s \right] \right\}^{-1}}. \tag{43}$$

This expression can be used to consider the relative effects on TFPG of alternative single factor productivity improvements, or of anticipated changes in any of the real price ratios for individual input/output combinations.

4. The Axiomatic (or Test) Approach to Choosing Among Alternative Index Number Formulas

Which functional form should be used for the price indexes in (41) or the quantity indexes in (27)? Historically, index number theorists have relied on what is called the axiomatic or test approach to address this choice problem. An overview of this approach is provided here, concentrating on the output side quantity and price indexes since the approach is the same on the input side.

As before, P denotes an output price index and Q denotes an output quantity index.

The axiomatic approach to the determination of the functional forms for P and Q works as follows. The starting point is a list of mathematical properties that the price index should satisfy based on a priori reasoning. These are the index number theory 'tests' or 'axioms.' Mathematical reasoning is applied to determine whether the a priori tests are mutually consistent and whether they uniquely determine, or usefully narrow the range of choices for, the functional form for the price index.[14] The product test rule is then applied to solve for the functional form of the quantity index.

As already noted in Section 2.2, on the output side the *Product Test* states that the product of the price and quantity indexes, P and Q, should equal the nominal revenue ratio for periods t and s:[15]

$$PQ = R^t/R^s. \tag{44}$$

If the functional form for the output price index P is given, then imposing the product rule means that the functional form for the output quantity index must be given by the expression

$$Q = (R^t/R^s)/P. \tag{45}$$

If the product test is imposed as a rule on both the output and input sides, then once the functional forms have been specified for the price indexes, the functional forms have also been determined for the quantity indexes and for the TFPG index.[16]

There are many other axiomatic tests that can be applied for choosing among the quantity and price index pairs that satisfy the product test. Since the indexes that make up these pairs are not determined independently, we only need consider axioms that apply to the price indexes, with the process of selection being the same for input indexes as for the output side ones considered here. We conclude this overview of the axiomatic approach by listing four of these tests.

The *Identity* or *Constant Prices Test* is[17]

$$P(p, p, y^s, y^t) = 1. \tag{46}$$

What this means is that if $p^s = p^t = p = (p_1, \dots, p_M)$ so that all prices are equal in the two periods, then the price index should be one regardless of the quantity values for periods s and t.

The *Constant Basket Test,* also called the *Constant Quantities Test,* is[18]

$$P(p^s, p^t, y, y) = \sum_{i=1}^{N} p_i^t y_i \Bigg/ \sum_{j=1}^{N} p_j^s y_j. \tag{47}$$

This test states that if the quantities are constant over the two periods s and t so that $y^s = y^t = y \equiv (y_1, \dots, y_M)$, then the level of prices in period t compared to period s should equal the value of the constant basket of quantities evaluated at the period t prices divided by the value of this same basket evaluated at the period s prices.

The *Proportionality in Period t Prices Test* is[19]

$$P(p^s, \lambda p^t, y^s, y^t) = \lambda P(p^s, p^t, y^s, y^t) \quad \text{for } \lambda > 0. \tag{48}$$

According to this test, if each of the elements of p^t is multiplied by the positive constant λ, then the level of prices in period t relative to period s will differ by the same multiplicative factor λ.

Our final example of a price index test is the *Time Reversal Test:*[20]

$$P(p^t, p^s, y^t, y^s) = 1/P(p^s, p^t, y^s, y^t). \tag{49}$$

If this test is satisfied, then when the prices and quantities for s and t are interchanged, the resulting price index will be the reciprocal of the original price index.

The Fisher price index P_F satisfies all four of these additional tests. The Paasche and Laspeyres indexes, P_P and P_L, fail the time reversal test (49). The Törnqvist index, P_T, fails the constant basket test (47), and the implicit Törnqvist index, $P_{\tilde{T}}$, fails the constant prices test (48). When a more extensive list of tests is compiled, the Fisher price index continues to satisfy more tests than other leading candidates.[21] However, the Paasche, Laspeyres, Törnqvist, and implicit Törnqvist indexes all show up quite well according to the axiomatic approach.

5. The Exact Index Number Approach

An alternative approach to the determination of the functional form for a measure of total factor productivity growth is to derive the index from a producer behavioral model. Diewert's (1976) exact index number approach systematizes the procedures for developing equivalencies between different proposed definitions for index numbers and optimizing models of economic behavior. Using these equivalencies, a choice can be made among alternative productivity index number formulas based on the preferred properties for a specified behavioral equation.

A firm's technology can be summarized by its period t production function f^t. Focusing on output 1, the period t production function can be represented for $t = 0, 1, \ldots, T$ as

$$y_1 = f^t(y_2, y_3, \ldots, y_M, x_1, x_2, \ldots, x_N), \tag{50}$$

which gives the amount of output 1 the firm can produce using the technology available in any given period t if it also produces y_2 units of output 2, y_3 units of output 3, ..., and y_M units of output M using x_1 units of input 1, x_2 units of input 2, ..., and x_N units of input N.

The production function f^t can be used to define the period t cost function, c^t, as follows:

$$\begin{aligned} &c^t(y_1, y_2, \ldots, y_M, w_1, w_2, \ldots, w_N) \\ &\quad \equiv \min_x \left\{ \sum_{n=1}^{N} w_n x_n : y_1 = f^t(y_2, \ldots, y_M, x_1, \ldots, x_N) \right\}. \end{aligned} \tag{51}$$

This cost function gives the minimum cost of producing the outputs $y_1, \ldots, y_M$ using the period t technology and input prices. Under the assumption of cost minimizing behavior, the observed period t cost of production, C^t, is equal to the minimum cost. Hence for $t = 0, 1, \ldots, T$, we have

$$C^t \equiv \sum_{n=1}^{N} w_n^t x_n^t = c^t\left(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\right). \tag{52}$$

We need some way of relating the cost functions for periods $t = 0, 1, \ldots, T$ to each other. A common way of doing this is to assume that

$$c^t\left(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\right) = (1/a^t)c\left(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\right), \tag{53}$$

where $a^t > 0$ denotes a period t relative efficiency parameter and c denotes an atemporal cost function. The normalization $a^0 \equiv 1$ is usually imposed. Given (53), a natural measure of productivity change for a productive unit in going from period s to t is the following ratio:

$$a^t/a^s. \tag{54}$$

If this ratio is greater than 1, then efficiency is said to have improved.

Taking the natural logarithm of both sides of (53), we have

$$\ell\text{n}\, c^t\big(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\big) = -\ell\text{n}\, a^t + \ell\text{n}\, c\big(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\big). \tag{55}$$

If a translog functional form is assumed for $\ell\text{n}\, c$, where c is the atemporal cost function on the right hand side of (55), then we have

$$\begin{aligned}
&\ell\text{n}\, c\big(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\big) \\
&\quad = b_0 + \sum_{m=1}^{M} b_m\, \ell\text{n}\, y_m^t + \sum_{n=1}^{N} c_n\, \ell\text{n}\, w_n^t + (1/2)\sum_{i=1}^{M}\sum_{j=1}^{M} d_{ij}\, \ell\text{n}\, y_i^t\, \ell\text{n}\, y_j^t \\
&\qquad + (1/2)\sum_{i=1}^{N}\sum_{j=1}^{N} f_{ij}\, \ell\text{n}\, w_i^t\, \ell\text{n}\, w_j^t + \sum_{m=1}^{M}\sum_{n=1}^{N} g_{mn}\, \ell\text{n}\, y_m^t\, \ell\text{n}\, w_n^t.
\end{aligned} \tag{56}$$

An advantage of the translog functional form is that it does not impose *a priori* restrictions on the admissible patterns of substitution between inputs and outputs.[22] However, this specification also has a large number of unknown parameters. There are $M + 1$ of the b_m parameters; N of the c_n parameters; $M(M+1)/2$ independent d_{ij} parameters after imposing the requirement that $d_{ij} = d_{ji}$ for $1 \leq i < j \leq M$; $N(N + 1)/2$ independent f_{ij} parameters even after also imposing the requirement that $f_{ij} = f_{ji}$ for $1 \leq i < j \leq N$; and MN of the g_{mn} parameters in the atemporal cost function (56), in addition to another T independent a^t parameters in (55). If homogeneity of degree one in the input prices is imposed as well for the cost function, then we have

$$\sum_{n=1}^{N} c_n = 1; \quad \sum_{j=1}^{N} f_{ij} = 0 \quad \text{for } i = 1, \ldots, N, \quad \text{and} \quad \sum_{n=1}^{N} g_{mn} = 0 \quad \text{for } m = 1, \ldots, M. \tag{57}$$

With these additional restrictions, the number of independent parameters is reduced to $T + M(M + 1)/2 + N(N + 1)/2 + MN$. This is still a larger number than the number of observations which is $T + 1$. Both econometric and index number methods require the imposition of further restrictions for evaluation of a measure based on (55) and (56).[23]

If the producer is minimizing costs, then the following demand relationships will hold:[24]

$$x_n^t = \partial c^t\big(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\big)/\partial w_n \tag{58}$$

for $n = 1, \ldots, N$ and $t = 0, 1, \ldots, T$. Since $\ell\text{n}\, c^t$ is a quadratic function in the variables

$$\ell\text{n}\, y_1,\ \ell\text{n}\, y_2, \ldots,\ \ell\text{n}\, y_M, \quad \ell\text{n}\, w_1, \ell\text{n}\, w_2, \ldots, \ell\text{n}\, w_N,$$

then Diewert's (1976, p. 119) logarithmic quadratic identity can be applied yielding:[25]

$$\begin{aligned}
\ell\mathrm{n}\, c^t - \ell\mathrm{n}\, c^s = {} & (1/2)\sum_{m=1}^{M}\left[y_m^t \frac{\partial\, \ell\mathrm{n}\, c^t}{\partial y_m}(y^t, w^t) + y_m^s \frac{\partial\, \ell\mathrm{n}\, c^s}{\partial y_m}(y^s, w^s)\right] \ell\mathrm{n}\big(y_m^t/y_m^s\big) \\
& + (1/2)\sum_{m=1}^{M}\left[w_m^t \frac{\partial\, \ell\mathrm{n}\, c^t}{\partial w_m}(y^t, w^t) + w_m^s \frac{\partial\, \ell\mathrm{n}\, c^s}{\partial w_m}(y^s, w^s)\right] \ell\mathrm{n}\big(w_m^t/w_m^s\big) \\
& + (1/2)\left[\frac{\partial\, \ell\mathrm{n}\, c^t}{\partial a^t}(y^t, w^t) + \frac{\partial\, \ell\mathrm{n}\, c^s}{\partial a^s}(y^s, w^s)\right] \ell\mathrm{n}(a^t/a^s) \qquad (59)
\end{aligned}$$

$$\begin{aligned}
= {} & (1/2)\sum_{m=1}^{M}\left[y_m^t \frac{\partial\, \ell\mathrm{n}\, c^t}{\partial y_m}(y^t, w^t) + y_m^s \frac{\partial\, \ell\mathrm{n}\, c^s}{\partial y_m}(y^s, w^s)\right] \ell\mathrm{n}\big(y_m^t/y_m^s\big) \\
& + (1/2)\sum_{n=1}^{N}\left[\big(w_n^t x_n^t/C^t\big) + \big(w_n^s x_n^s/C^s\big)\right] \ell\mathrm{n}\big(w_m^t/w_m^s\big) \\
& + (1/2)[-1 + (-1)]\, \ell\mathrm{n}(a^t/a^s). \qquad (60)
\end{aligned}$$

We can further simplify (60) by imposing the additional assumption of competitive profit maximizing behavior. More specifically, suppose we can assume that the observed period t outputs $y_1^t, \ldots, y_M^t$ solve the following profit maximization problem for $t = 0, 1, \ldots, T$:

$$\underset{y_1,\ldots,y_M}{\text{maximize}}\left\{\sum_{m=1}^{M} p_m^t y_m - c^t\big(y_1, \ldots, y_M, w_1^t, \ldots, w_N^t\big)\right\}. \qquad (61)$$

This leads to the usual price equals marginal cost relationships that result when competitive price taking behavior is assumed; i.e., we now have

$$p_m^t = \partial c^t\big(y_1^t, \ldots, y_M^t, w_1^t, \ldots, w_N^t\big)/\partial y_m, \quad m = 1, \ldots, M. \qquad (62)$$

Making use of the definition of total costs in (52) as well as (62), expression (60) can now be rewritten as:

$$\begin{aligned}
\ell\mathrm{n}(C^t/C^s) = {} & (1/2)\sum_{m=1}^{M}\left[\big(p_m^t y_m^t/C^t\big) + \big(p_m^s y_m^s/C^s\big)\right] \ell\mathrm{n}\big(y_m^t/y_m^s\big) \\
& + (1/2)\sum_{n=1}^{N}\left[\big(w_n^t x_n^t/C^t\big) + \big(w_n^s x_n^s/C^s\big)\right] \ell\mathrm{n}\big(w_m^t/w_m^s\big) - \ell\mathrm{n}(a^t/a^s). \qquad (63)
\end{aligned}$$

Costs can be observed in each period, as can output and input prices and quantities. Thus the only unknown in (63) is the productivity change measure. Solving (63) for this yields

$$a^t/a^s = \left\{ \prod_{m=1}^{M} \left(y_m^t/y_m^s\right)^{(1/2)[(p_m^t y_m^t/C^t)+(p_m^s y_m^s/C^s)]} \right\} \Big/ \tilde{Q}_T^*, \tag{64}$$

where $\tilde{Q}_T^*$ is the implicit Törnqvist input quantity index that is defined analogously to the implicit Törnqvist output quantity index introduced in Section 2.5.

Formula (64) can be simplified further using the assumption that the technology exhibits constant returns to scale. If costs grow proportionally with output, then the cost functions must be linearly homogeneous in the output quantities (Diewert 1974, pp. 134–137). In that case, with competitive profit maximizing behavior, revenues must equal costs in each period so that

$$c^t(y^t, w^t) = C^t = R^t. \tag{65}$$

Using (65), we can replace C^t and C^s in (64) by R^t and R^s, respectively, and (64) becomes

$$a^t/a^s = Q_T/\tilde{Q}_T^* \tag{66}$$

where Q_T is the Törnqvist output quantity index.

The hypothesis of constant returns to scale invoked in moving from expression (64) to (66) is very restrictive. However, if the underlying technology is subject to diminishing returns to scale (or equivalently, to increasing costs), we can convert the technology into an artificial one still subject to constant returns to scale by introducing an extra fixed input, x_{N+1} say, and setting this extra fixed input equal to one (that is, $x_{N=1}^t \equiv 1$ for each period t). The corresponding period t price for this input, w_{N+1}^t, is set equal to the firm's period t profits, $R^t - C^t$. With this extra factor, the period t cost is redefined to be the adjusted cost given for $t = 0, 1, \ldots, T$ by

$$C_A^t = C^t + w_{N+1}^t x_{N+1}^t = \sum_{n=1}^{N+1} w_n^t x_n^t = R^t. \tag{67}$$

The derivation can now be repeated using the adjusted cost C_A^t rather than the actual cost C^t. What results is the same productivity change formula except that $\tilde{Q}_T^*$ is now the implicit translog quantity index for $N + 1$ instead of N inputs.

The index number method for measuring the productivity change term a^t/a^s that is illustrated by the formula on the right-hand side of (64) or by (66) can be used even with thousands of outputs and inputs. In deriving these expressions using the exact index number approach, we did assume competitive (i.e., price taking) profit maximizing behavior. When the assumption of competitive profit maximizing behavior breaks down, the relations (62) are not valid. However, all of the formulas defined in Section 2 (and many others not discussed) can still be directly evaluated using the observed price and quantity data and the axiomatic approach can still be used to choose among alternative index number formulas. It is only the economic theory interpretation that is lost.

6. Production Function Based Measures

The exact index number approach, like the axiomatic approach to index number theory, can be helpful in choosing a particular index formula from the many that have been proposed. In addition, when a TFPG index can be related to a specific producer behavioral relationship, this may enable a deeper understanding of the meaning of the index and may also be helpful to managers concerned with finding operational ways to enhance total factor productivity.

6.1. Technical Progress and Returns to Scale in the Simple 1-1 Case

When we know the production functioning corresponding to a TFPG index, one of the things this knowledge enables is a decomposition of TFPG into a technical progress term, TP, and a returns to scale term, RS. One reason for interest in this decomposition is that the policy options for enhancing TFPG gains from technical progress versus returns to scale are often different.[26]

This decomposition of a TFPG index is simplest to understand in the one input, one output case. We assume knowledge of the period s and period t input and output quantities and the true period s and period t production functions. That is, we assume we have:

$$y_1^s = f^s\left(x_1^s\right) \tag{68}$$

and

$$y_1^t = f^t\left(x_1^t\right). \tag{69}$$

If technical progress is characterized as a shift in the production function, it is still necessary to specify the type of shift. Four of the various possible shift measures we could define make use of the actual output and input quantity data for periods s and t (i.e., x_1^s, x_1^t, y_1^s, y_1^t), and can be easily related to standard managerial accounting concepts. We focus on these four TP measures here.

Hypothetical quantities are needed to define these measures: two on the output side and two on the input side. The output side hypothetical quantities are

$$y_1^{s*} \equiv f^t\left(x_1^s\right) \tag{70}$$

and

$$y_1^{t*} \equiv f^s\left(x_1^t\right). \tag{71}$$

The quantity y_1^{s*} is the output that *could be* produced with the period s input quantity x_1^s using the period t production technology embodied in f^t, and the quantity y_1^{t*} is the output that *could be* produced with the period t input quantity x_1^t using the period s technology. The input side hypothetical quantities, x_1^{s*} and x_1^{t*}, are defined implicitly by

$$y_1^s = f^t\left(x_1^{s*}\right) \tag{72}$$

and

$$y_1^t = f^s\left(x_1^{t*}\right). \tag{73}$$

The quantity x_1^{s*} is the hypothetical amount of the input factor that would be required to produce the actual period s output, y_1^s, using the more recent period t technology. Hence x_1^{s*} will usually be less than x_1^s. The quantity x_1^{t*} is the input quantity that would be required to produce the period t output y_1^t using the older period s technology, so x_1^{t*} is expected to be larger than x_1^t.

The first two of the four technical progress indexes defined are output based:[27]

$$\text{TP}(1) \equiv y_1^{s*}/y_1^s = f^t\left(x_1^s\right)/f^s\left(x_1^s\right), \tag{74}$$

and

$$\text{TP}(2) \equiv y_1^t/y_1^{t*} = f^t\left(x_1^t\right)/f^s\left(x_1^t\right). \tag{75}$$

Each of these describes the percentage increase in output resulting solely from switching from the period s to the period t production technology. For TP(1), technical progress is measured with the input level fixed at x_1^s, whereas for TP(2) the input level is fixed at x_1^t. The other two indexes of technical progress defined here are input based:[28]

$$\text{TP}(3) \equiv x_1^s/x_1^{s*} \tag{76}$$

and

$$\text{TP}(4) \equiv x_1^{t*}/x_1^t. \tag{77}$$

Each of these gives the reciprocal of the percentage decrease in input usage resulting solely from switching from the period s to the period t production technology. For TP(3), technical progress is measured with the output level fixed at y_1^s whereas for TP(4) the output level is fixed at y_1^t.

Each of the technical progress measures defined above is related to TFPG as follows:

$$\text{TFPG} = \text{TP}(i)\,\text{RS}(i) \tag{78}$$

where for $i = 1, 2, 3, 4$ and depending on the selected TP measure, the RS term is

$$\text{RS}(1) \equiv \left[y_1^t/x_1^t\right]/\left[y_1^{s*}/x_1^s\right], \tag{79}$$

$$\text{RS}(2) \equiv \left[y_1^{t*}/x_1^t\right]/\left[y_1^s/x_1^s\right], \tag{80}$$

$$\text{RS}(3) \equiv \left[y_1^t/x_1^t\right]/\left[y_1^s/x_1^{s*}\right], \tag{81}$$

or

$$\text{RS}(4) \equiv \left[y_1^t/x_1^{t*}\right]/\left[y_1^s/x_1^s\right]. \tag{82}$$

In the TFPG decompositions in (78), the technical progress term, TP(i), corresponds to a *shift*.[29] The returns to scale term, RS(i), corresponds to a *movement along* a fixed production function due solely to changes in input quantity utilization. Each returns to scale measure will be greater than one if output divided by input increases as we move along the production surface.

For two periods, say $s = 0$ and $t = 1$, and with just one input factor and one output good, the four measures of TP defined in (74)–(77) and the four measures of returns to scale

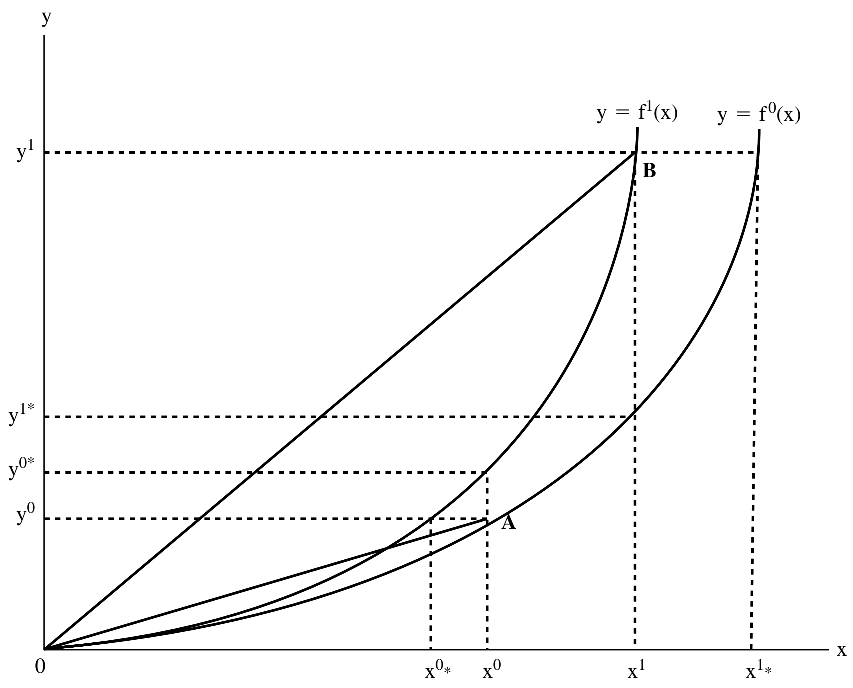

Figure 1. Production function based measures of technical progress.

defined in (79)–(82) can be illustrated graphically, as in Figure 1. (Here the subscript 1 is dropped for both the single input and the single output.)

The lower curved line is the graph of the period 0 production function; i.e., it is the set of points (x, y) such that $x \geq 0$ and $y = f^0(x)$. The higher curved line is the graph of the period 1 production function; i.e., it is the set of points (x, y) such that $x \geq 0$ and $y = f^1(x)$. The observed data points are A for period 0 with coordinates (x^0, y^0), and B for period 1 with coordinates (x^1, y^1).[30] Applying formula (2) from Section 1, for this example we have $\text{TFPG} = [y^1/x^1]/[y^0/x^0]$ which is the slope of the straight line OB divided by the slope of the straight line OA in Figure 1. The reader can use Figure 1 and the definitions provided above to verify that each of the four decompositions of TFPG given by (78) corresponds to a different combination of shifts of and movements along a production function that take us from observed point A to observed point B.[31]

Geometrically, each of the returns to scale measures is the ratio of two output-input coefficients, say $[y^j/x^j]$ divided by $[y^k/x^k]$ where the points (y^j, x^j) and (x^k, y^k) are on the *same* fixed production function with $x^j > x^k$. Thus, if $\text{RS}(i) = [y^j/x^j]/[y^k/x^k]$ is greater than 1, the production function exhibits increasing returns to scale, while if $\text{RS}(i) = 1$ we have constant returns to scale and if $\text{RS}(i) < 1$ we have decreasing returns. If the returns to scale are constant, then $\text{RS}(i) = 1$ and $\text{TP} = \text{TFPG}$.[32] Figure 1 illustrates that constant returns to scale assumption need not be assumed for evaluating the TFPG measures in Section 2.[33]

6.2. Malmquist Indexes

The fundamental insight of the exact index number approach is that, if the technology of a production unit can be represented in each time period by some known multiple input, multiple output production function, then quantity and TFPG indexes can be derived within an optimizing model of producer behavior. We do this here.

As in (50), we let y_1^t denote the amount of output 1 produced in period t for $t = 0, 1, \ldots, T$. Here we also let $\tilde{y}^t \equiv [y_2^t, y_3^t, \ldots, y_M^t]$ denote the vector of other (than good 1) outputs jointly produced in each period t using the vector of input quantities $x^t \equiv [x_1^t, x_2^t, \ldots, x_N^t]$. Thus, the production functions for output 1 in period s and period t can be represented compactly as:

$$y_1^s = f^s(\tilde{y}^s, x^s) \quad \text{and} \quad y_1^t = f^t(\tilde{y}^t, x^t). \tag{83}$$

In defining Malmquist output indexes, we begin by defining three alternative formulations on the output side.[34]

The first Malmquist output index, α^s, is defined as the number which satisfies

$$y_1^t/\alpha^s = f^s(\tilde{y}^t/\alpha^s, x^s). \tag{84}$$

This index α^s measures output growth using the comparison period s technology and input vector. Its value is the number which just deflates the period t vector of outputs, $y^t \equiv [y_1^t, y_2^t, \ldots, y_M^t]$, into an output vector y^t/α^s that can be produced with the period s vector of inputs, x^s, using the period s technology. Due to substitution, when the number of output goods, M, is greater than 1, then y^t/α^s will not usually be equal to the actual period s output vector, y^s. However, if there is only one output, equation (84) becomes $y_1^t/\alpha^s = f^s(x^s) = y_1^s$, and we have $\alpha^s = y_1^t/y_1^s$.

A second Malmquist output index, α^t, is defined as the number which satisfies

$$\alpha^t y_1^s = f^t(\alpha^t \tilde{y}^s, x^t). \tag{85}$$

The index α^t measures output growth using the current period t technology and input vector. It is the number that inflates the period s vector y^s into $\alpha^t y^s$, an output vector that can be produced with the period t input vector x^t using the period t technology. The vector $\alpha^t y^s$ will not usually be equal to y^t when there are multiple outputs. However, when $M = 1$, then (85) becomes $\alpha^t y_1^s = f^t(x^t) = y_1^t$ and $\alpha^t = y_1^t/y_1^s$ reduces to the (single) output growth rate.

When there is no reason to prefer α^s or α^t and no need to distinguish between them, we recommend defining the geometric mean of α^s and α^t as *the* Malmquist output quantity index:

$$\alpha \equiv [\alpha^s \alpha^t]^{1/2}. \tag{86}$$

When there are only two output goods, the Malmquist output indexes α^s and α^t can be illustrated as in Figure 2 for time periods $t = 1$ and $s = 0$. The lower curved line represents the set of outputs $\{(y_1, y_2) : y_1 = f^0(y_2, x^0)\}$ that can be produced with period 0 technology and inputs.

The higher curved line represents the set of outputs $\{(y_1, y_2) : y_1 = f^1(y_2, x^1)\}$ that can be produced with period 1 technology and inputs. The period 1 output possibilities set

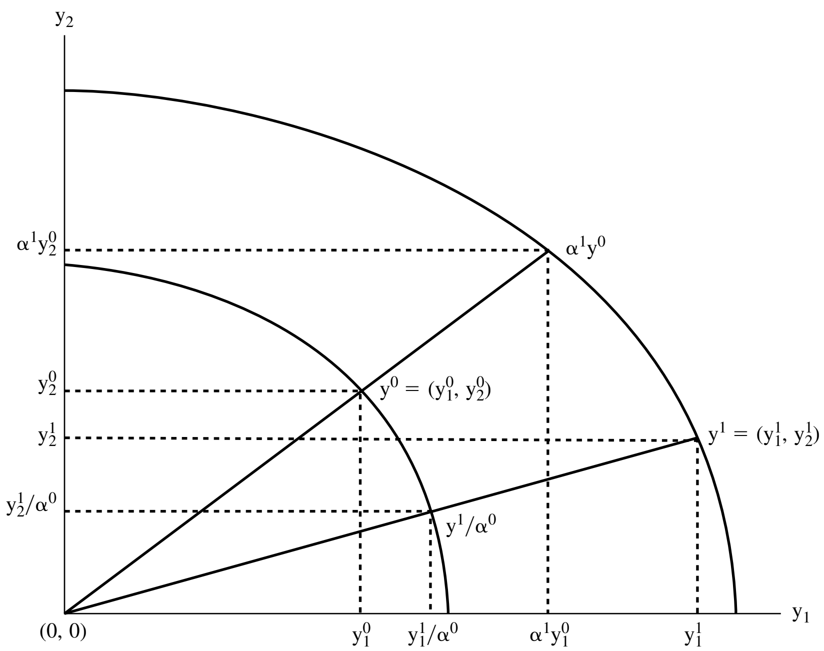

Figure 2. Alternative economic output indexes illustrated.

will generally be higher than the period 0 one for two reasons: (i) technical progress and (ii) input growth.[35] In Figure 2, the point $\alpha^1 y^0 = [\alpha^1 y_1^1, \alpha^1 y_2^1]$ is the straight line projection of the period 0 output vector $y^0 = [y_1^0, y_2^0]$ onto the period 1 output possibilities set, and $y^1/\alpha^0 = [y_1^1/\alpha^0, y_2^1/\alpha^0]$ is the straight line contraction of the output vector $y^1 = [y_1^1, y_2^1]$ onto the period 0 output possibilities set.

We now turn to the input side. A first Malmquist input index, β^s, is defined as follows:

$$y_1^s = f^s(\tilde{y}^s, x^t/\beta^s) \equiv f^s\left(y_2^s, \ldots, y_M^s, x_1^t/\beta^s, \ldots, x_N^t/\beta^s\right). \tag{87}$$

This index measures input growth using the period s technology and output vector. A second Malmquist input index, denoted by β^t, is the solution to the following equation

$$y_1^t = f^t(\tilde{y}^t, \beta^t x^s) \equiv f^t\left(y_2^t, \ldots, y_M^t, \beta^t x_1^s, \ldots, \beta^t x_N^s\right). \tag{88}$$

It measures input growth using the period t technology and output vector. When there is no particular reason to prefer either the index β^s to β^t and no need to distinguish between them, we recommend defining their geometric average as *the* Malmquist input quantity index:

$$\beta \equiv [\beta^s \beta^t]^{1/2}. \tag{89}$$

Figure 3 illustrates the Malmquist indexes β^s and β^t for the case where there are just two input goods and for the time periods $t = 1$ and $s = 0$.

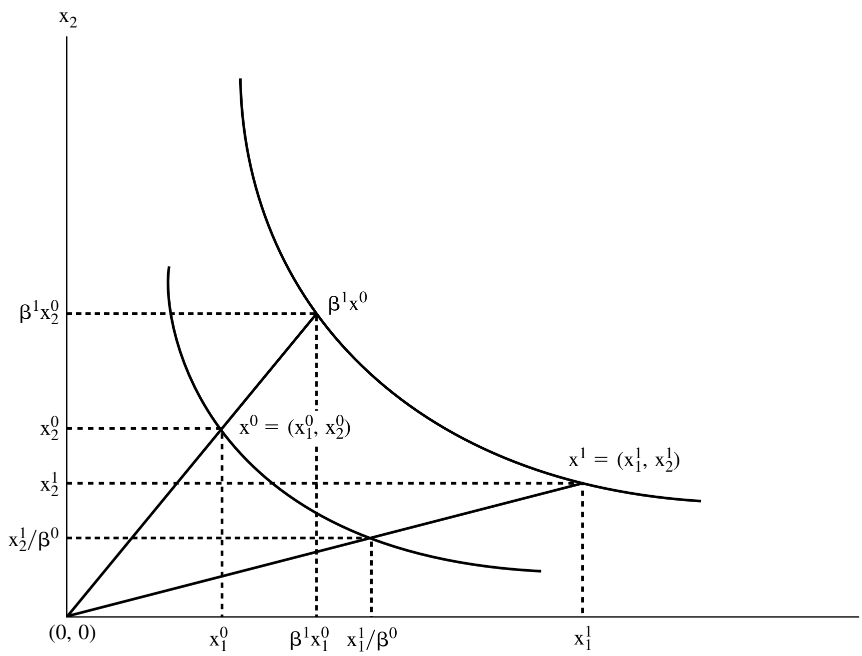

Figure 3. Alternative Malmquist input indexes illustrated.

The lower curved line in Figure 3 represents the set of inputs needed to produce the vector of outputs y^0 using period 0 technology. This is the set $\{(x_1, x_2) : y_1^0 = f^0(\tilde{y}^0, x_1, x_2)\}$. The higher curved line represents the set of inputs that are needed to produce the period 1 vector of outputs y^1 using period 1 technology. This is the set $\{(x_1, x_2) : y_1^1 = f^1(\tilde{y}^1, x_1, x_2)\}$.[36] The point $\beta^t x^0 = [\beta^t x_1^0, \beta^t x_2^0]$ is the straight line projection of the input vector $x^0 \equiv [x_1^0, x_2^0]$ onto the period 1 input requirements set. The point x^1/β^0 is the straight line contraction of the input vector $x^1 \equiv [x_1^1, x_2^1]$ onto the period 0 input requirements set.

Once theoretical Malmquist quantity indexes measuring the growth of total output and input have been defined, then a Malmquist TFPG index for the general N-M case can be defined too. Using the preferred Malmquist output and input quantity indexes given in (96) and (99), the definition we recommend for the Malmquist TFPG index is

$$\text{TFPG}_M \equiv \alpha/\beta. \tag{90}$$

In the 1-1 case, expression (90) is readily seen to reduce to the measure of TFPG in Section 1.

6.3. Direct Evaluation of Malmquist Indexes for the N-M Case

Recall that $\alpha \equiv [\alpha^s \alpha^t]^{1/2}$ and $\beta \equiv [\beta^s \beta^t]^{1/2}$ are the Malmquist output and input quantity indexes defined in (86) and (89). Using the exact index number approach, Caves, Christensen and Diewert (1982, pp. 1395–1401) give conditions under which the following equalities hold:

$$\alpha = Q_T \tag{91}$$

and

$$\beta = Q_T^*, \tag{92}$$

where Q_T is the Törnqvist output quantity index defined by (39) and Q_T^* is the Törnqvist input quantity index defined analogously. If the assumptions that justify (91) and (92) are satisfied, then we can evaluate the Malmquist measure TFPG_M defined in (90) by taking the ratio of the Törnqvist output and input indexes:

$$\text{TFGP}_M = \alpha/\beta = Q_T/Q_T^* \equiv \text{TFPG}_T. \tag{93}$$

The assumptions required to derive (91) and (92) are, roughly speaking: (i) price taking, revenue maximizing behavior, (ii) price taking, cost minimizing behavior, and (iii) a translog technology.[37] The practical importance of (93) is that the expression can be evaluated from observable prices and quantities without knowledge of the parameters of f^s and f^t. However, changes in the value of this index cannot by decomposed into technical progress and returns to scale components without knowing or estimating the production functions.

7. Diewert-Morrison Decompositions of an Alternative Productivity Measure

In Section 5, we used the period t production function f^t to define the period t cost function, c^t. Here we use the period t production function to define the period t (net) revenue function:

$$r^t(p, x) \equiv \max_y \{p \cdot y : y \equiv (y_1, y_2, \ldots, y_M); y_1 = f^t(y_2, \ldots, y_M; x)\}, \tag{94}$$

where $p \equiv (p_1, \ldots, p_M)$ is an output price vector that the producer faces and $x \equiv (x_1, \ldots, x_N)$ is the input vector.[38] Following Diewert and Morrison (1986), revenue functions for period t and the period s are used to define another family of theoretical productivity change indexes:

$$\text{RG}(p, x) \equiv r^t(p, x)/r^s(p, x). \tag{95}$$

From (94), it can be seen that $\text{RG}(p, x)$ is the ratio of the output that can be produced using the period t versus the period s technology with the reference input vector x and with both of the resulting output vectors evaluated using the reference price vector, p. This is a different approach to the problem of controlling for total factor input utilization in judging the success of the period t versus the period s production outcomes.

Two special cases of (95) are of interest:

$$\text{RG}^s \equiv \text{RG}(p^s, x^s) = r^t(p^s, x^s)/r^s(p^s, x^s)$$

and

$$\text{RG}^t \equiv \text{RG}(p^t, x^t) = r^t(p^t, x^t)/r^s(p^t, x^t). \tag{96}$$

RG^s is the theoretical productivity index obtained by choosing the reference vectors p and x to be the period s observed output price vector p^s and input quantity vector x^s. RG^t is the theoretical productivity index obtained by choosing the reference vectors to be the period t observed output price vector p^t and input quantity vector x^t.[39]

Under the assumption of revenue maximizing behavior in both periods, we can identify the denominator of RG^s and the numerator of RG^t. That is, we have:

$$p^t \cdot y^t = r^t(p^t, x^t) \quad \text{and} \quad p^s \cdot y^s = r^s(p^s, x^s). \tag{97}$$

Our problem in evaluating the theoretical productivity indexes RG^s and RG^t is that we cannot directly observe the hypothetical revenues, $r^t(p^s, x^s)$ and $r^s(p^t, x^t)$. The first of these is the hypothetical revenue that would result from using the period t technology with the period s input quantities and output prices. The second is the hypothetical revenue that would result from using the period s technology with the period t input quantities and output prices. However, these hypothetical revenue figures can be evaluated from observable data if we know that the period t revenue function has the following translog functional form:

$$\begin{aligned} \ell\text{n}\, r^t(p, x) \equiv \alpha_0^t &+ \sum_{m=1}^{M} \alpha_m^t\, \ell\text{n}\, p_m + \sum_{n=1}^{N} \beta_n^t\, \ell\text{n}\, x_n + (1/2) \sum_{i=1}^{M} \sum_{j=1}^{M} \alpha_{ij}\, \ell\text{n}\, p_i\, \ell\text{n}\, p_j \\ &+ (1/2) \sum_{n=1}^{N} \sum_{j=1}^{N} \beta_{ij}\, \ell\text{n}\, x_i\, \ell\text{n}\, x_j + \sum_{m=1}^{M} \sum_{n=1}^{N} \gamma_{mn}\, \ell\text{n}\, p_m\, \ell\text{n}\, x_n, \end{aligned} \tag{98}$$

where $\alpha_{ij} = \alpha_{ji}$ and $\beta_{ij} = \beta_{ji}$ and the parameters satisfy various other restrictions to ensure that $r^t(p, x)$ is linearly homogeneous in the components of the price vector p.[40] Note that the coefficient vectors α_0^t, α_m^t and β_n^t can be different in each period t but the quadratic coefficients are assumed to be constant over the two periods.

Diewert and Morrison (1986; p. 663) show that under the above assumptions, the geometric mean of the two theoretical productivity indexes defined in (96) can be identified using the observable price and quantity data that pertain to the two periods; i.e., we have

$$[\text{RG}^s \text{RG}^t]^{1/2} = a/bc \tag{99}$$

where a, b and c are given by

$$a \equiv p^t \cdot y^t / p^s \cdot y^s, \tag{100}$$

$$\ell\text{n}\, b \equiv \sum_{m=1}^{M} (1/2) \left[\left(p_m^s y_m^s / p^s \cdot y^s \right) + \left(p_m^t y_m^t / p^t \cdot y^t \right) \right] \ell\text{n} \left(p_m^t / p_m^s \right), \tag{101}$$

and

$$\ell\text{n}\, c = \sum_{n=1}^{N} (1/2) \left[\left(w_n^s x_n^s / p^s \cdot y^s \right) + \left(w_n^t x_n^t / p^t \cdot y^t \right) \right] \ell\text{n} \left(x_n^t / x_n^s \right). \tag{102}$$

If we have constant returns to scale production functions f^s and f^t, then the value of outputs will equal the value of inputs in each period and we have

$$p^t \cdot y^t = w^t \cdot x^t. \tag{103}$$

The same result can be derived without the constant returns to scale assumption if we have a fixed factor that absorbs any pure profits or losses, with this fixed factor defined as in (67).

Substituting (103) into (102), we see that expression c becomes the Törnqvist input index Q_T^*. By comparing (101) and (40), we see also that b is the Törnqvist output price index P_T. Thus a/b is an implicit Törnqvist output quantity index.

If (103) holds, then we have the following decomposition for the geometric mean of the product of the theoretical productivity change indexes defined in (103):

$$[\mathrm{RG}^s \mathrm{RG}^t]^{1/2} = [p^t \cdot y^t / p^s \cdot y^s]/[P_T Q_T^*] \tag{104}$$

where P_T is the Törnqvist output price index and Q_T^* is the Törnqvist input quantity index, both of which are function of the prices and quantities in both periods t and s. Diewert and Morrison (1986) use the period t revenue functions to define two *theoretical output price effects* for $m = 1, \dots, M$ which show how revenues change as a single output price changes:

$$P_m^s \equiv r^s\left(p_1^s, \dots, p_{m-1}^s, p_m^t, p_{m+1}^s, \dots, p_M^s, x^s\right)\big/r^s(p^s, x^s), \tag{105}$$

and

$$P_m^t \equiv r^t(p^t, x^t)\big/r^t\left(p_1^t, \dots, p_{m-1}^t, p_m^s, p_{m+1}^t, \dots, p_M^t, x^t\right), \quad m = 1, \dots, M. \tag{106}$$

These theoretical indexes give the proportional changes in the value of output that would result if we changed the price of the mth output from its period s level p_m^s to its period t level p_m^t holding constant all other output prices and the input quantities at reference levels and using the same technology in both situations. For the theoretical index defined in (105), the reference output prices and input quantities and technology are the period s ones, whereas for the index defined in (106) they are the period t ones. Now define the theoretical output price effect b_m as the geometric mean of the two effects defined by (105) and (106):

$$b_m \equiv \left[P_m^s P_m^t\right]^{1/2}, \quad m = 1, \dots, M. \tag{107}$$

Diewert and Morrison (1986) and Kohli (1990) show that the b_m given by (107) can be evaluated by the following observable expression if conditions (97), (98) and (103) hold:

$$\ell\mathrm{n}\, b_m = (1/2)\left[\left(p_m^s y_m^s / p^s \cdot y^s\right) + \left(p_m^t y_m^t / p^t \cdot y^t\right)\right] \ell\mathrm{n}\left(p_m^t / p_m^s\right), \quad m = 1, \dots, M. \tag{108}$$

Comparing (101) with (108), it can be seen that we have the following decomposition for b:

$$b = \prod_{m=1}^{M} b_m = P_T. \tag{109}$$

Thus the overall Törnqvist output price index can be decomposed into a product of the individual output price effects, b_m.

Diewert and Morrison (1986) also use the period t revenue functions in order to define for $n = 1, \dots, N$ two *theoretical input quantity effects:*

$$Q_n^s \equiv r^s\left(p^s, x_1^s, \dots, x_{n-1}^s, x_n^t, x_{n+1}^s, \dots, x_N^s\right)\big/r^s(p^s, x^s), \tag{110}$$

and

$$Q_n^t \equiv r^t(p^t, x^t)/r^t\left(p^t, x_1^t, \ldots, x_{n-1}^t, x_n^s, x_{n+1}^t, \ldots, x_N^t\right). \tag{111}$$

These theoretical indexes give the proportional change in the value of output that would result from changing input n from its period s level x_n^s to it period t level x_n^t, holding constant all output prices and other input quantities at reference levels and using the same technology in both situations. For the theoretical index defined by (110), the reference output prices, input quantities and technology are the period s ones, whereas for the index in (111) they are the period t ones.

Now define the theoretical input quantity effect c_n as the geometric mean of the two effects defined by (100) and (101):

$$c_n \equiv \left[Q_n^s Q_n^t\right]^{1/2}, \quad n = 1, \ldots, N. \tag{112}$$

Diewert and Morrison (1986) show that the c_n defined by (112) can be evaluated by the following empirically observable expression provided that assumptions (97) and (98) hold:

$$\ell\text{n}\, c_n = (1/2)\left[\left(w_n^s x_n^s / p^s \cdot y^s\right) + \left(w_n^t x_n^t / p^t \cdot y^t\right)\right] \ell\text{n}\left(x_n^t / x_n^s\right) \tag{113}$$

$$\ell\text{n}\, c_n = (1/2)\left[\left(w_n^s x_n^s / w^s \cdot x^s\right) + \left(w_n^t x_n^t / w^t \cdot x^t\right)\right] \ell\text{n}\left(x_n^t / x_n^s\right). \tag{114}$$

The expression (114) follows from (113) provided that the assumptions (103) also hold. Comparing (113) with (112), it can be seen that we have the following decomposition for c:

$$c = \prod_{n=1}^{N} c_n \tag{115}$$

$$c = Q_T, \tag{116}$$

where (116) follows from (115) provided that the assumptions (103) also hold. Thus if assumptions (97), (98) and (93) hold, the overall Törnqvist input quantity index can be decomposed into a product of the individual input quantity effects, the c_n for $n = 1, \ldots, N$.

Having derived (109) and (115), we can substitute these decompositions into (99) and rearrange the terms to obtain the following very useful decomposition:

$$p^t \cdot y^t / p^s \cdot y^s = [\text{RG}^s \text{RG}^t]^{1/2} \prod_{m=1}^{M} b_m \prod_{n=1}^{N} c_n \tag{117}$$

This is a decomposition of the growth in the nominal value of output into the productivity growth term $[\text{RG}^s\text{RG}^t]^{1/2}$ times the product of the output price growth effects b_m times the product of the input quantity growth effects (the c_n). The effects on the right-hand side of expression (117) can be calculated using only the observable price and quantity data for the two periods.[41]

An interesting special case of (117) results when there is only one input in the x vector and it is fixed. Then the input growth effect c_1 is unity and variable inputs appear in the y vector with negative components. In this special case, the left-hand side of (117) becomes the pure profits ratio that is decomposed into a productivity effect times various price effects (the b_m).

8. Conclusions

Before summing up our findings, we must mention explicitly some limitations of our analysis. Direct evaluation of the measures presented in this paper depends on four problematic preconditions: (i) the list of inputs used and outputs produced must remain constant over the current period t and the comparison period s; (ii) quantity and either unit price or total value information must be available for each of the two periods for all inputs purchased and outputs sold; (iii) it must be possible to calculate user costs or rental prices in an unambiguous way for all capital inputs (i.e., for all durable inputs whose initial cost should be spread over the multi-period life of the good); and, (iv) for some sorts of studies it is important for the differences between the ex ante expected prices and the ex post realized prices to be negligible. We briefly discuss why each of these conditions is problematic below.

The first and second limitations are problematic because of data collection shortcomings and because of the ongoing appearance of new goods.[42] There are also issues having to do with the choice of outputs and inputs that should be included in an analysis since firm productivity is affected by factors external to the firms and unintended outputs such as pollutants.[43]

The third limitation of our analysis is problematic because of unresolved conceptual issues concerning the measurement of user costs for durable inputs. For instance, when a durable input is purchased, the purchase price should be spread over its useful lifetime. Cost accounting depreciation allowances attempt to do this, but the traditional accounting treatment of depreciation in an inflationary environment is unsatisfactory and there is disagreement on how this traditional practice should be altered. For instance, what interest rate should be used in determining the value of financial capital tied up in the ownership of durable goods? Should imputed equity interest costs be included too? There are also more basic unresolved conceptual problems associated with the measurement of capital inputs. For example, should the quantity of the capital services provided by a machine in each accounting period be treated as constant (that is, should it be measured as an average per unit time period) over the lifetime of the machine, or should the quantity be reduced each period by a deterioration factor to reflect the decline in efficiency of the machine? The first view leads to a gross capital services concept and the second to a net capital services concept. These two views can lead to significantly different measures of capital services input and, hence, to significantly different measures of productivity.[44]

The fourth limitation is problematic because during inflationary time periods substantial differences can develop between ex ante and ex post prices. Many capital inputs cannot be adjusted instantaneously (i.e., they cannot be bought or sold instantaneously); therefore, a cost minimizing producer forms a priori expectations about the purchase and disposal prices as well as future interest rates, depreciation rates, and tax rates in order to calculate the ex ante user cost of capital inputs. However, as researchers, we can only observe ex post prices, interest rates, depreciation rates, and tax rates; thus we can only calculate ex post user costs. If the expectations about future prices and rates are not realized, then the ex ante user costs—the prices which *should* appear in our cost functions and in the exact index number formulas—may differ significantly from the ex post user costs.

These limitations must be kept in mind in applying the methods reviewed here.

In the initial sections of this paper we developed and explored properties of the TFPG index number formulas that are most commonly used: the Paasche, Laspeyres, Fisher, and Törnqvist indexes. We also developed the mathematical relationships among the quantity, price and TFPG indexes. Methods for separating out different components of the growth in the total input, output, cost and revenue were made explicit, and then were utilized throughout the paper.

In Section 3, we showed how a TFPG index can be expressed as a function of single factor input-output, or output-input, coefficients. This decomposition is an algebraic result, not depending on any assumptions concerning the underlying technology or business practices. This decomposition provides a framework for relating widely used single factor efficiency and productivity measures to an overall measure of productivity performance.

Section 4 provided a brief review of the axiomatic approach to choosing a TFPG index formula. Application of this approach does not depend on assumptions about the nature of the underlying technology or producer behavior. However, when it does seem reasonable to assume specific sorts of optimizing behavior and the underlying technology can be modelled using a specific functional form, then the exact index number approach, introduced in Section 5, can be used as well for selecting a formula for the TFPG index. In Section 6 we reviewed results establishing that if the underlying technology can be represented by a translog production function and the producer is optimizing in certain ways, then the theoretical Malmquist TFPG index can be directly evaluated from observable price and quantity data using a Törnqvist type of index. Decompositions developed by Diewert and Morrison of a related measure of overall productive performance also apply in this case. These decompositions, presented in Section 7, allow an assessment of the contributions to an overall measure of productivity change that are due to quantity and price changes at the level of the individual output goods. The Diewert-Morrison decompositions, like the decomposition presented in Section 3, should be helpful in following through on the advice that Arnold Harberger (1998, p. 1) provides in his American Economics Association Presidential Address: that we should approach the measurement of productivity by trying to "think like an entrepreneur or a CEO, or a production manager."

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Notes

1. Empirical examples of this include Diewert and Nakamura (1999). The fact that output-input coefficients are pure quantity measures is something that many in engineering and the business world view as an important advantage.
2. Some use TFP to refer to total factor productivity growth as well as total factor productivity. Following Bernstein (1999), we use TFPG for total factor productivity growth to avoid the inevitable confusion that otherwise results.
3. See Diewert (2000) for additional ways of conceptualizing TFPG.
4. We refer to t and s as time periods, but the comparison situation could be for some other unit of production.
5. Formally, the first two of these can be shown to result from deflating the period t nominal cost and revenue by a Paasche price index. The second two result from deflating the period t nominal cost and revenue by a Laspeyres price index. See Horngren and Foster (1987, Chapter 24, Part One) or Kaplan and Atkinson (1989, Chapter 9) for examples of this accounting practice of controlling for price level change without explicit use of price indexes.
6. A quantity (price) relative for a good is the ratio of the quantity (price) for that good in a specified period t to the quantity (price) for that good in some comparison period s. One advantage of defining a quantity (or price) index as a weighted average of quantity (price) relatives is that the relatives are unit free, making it clear that this is an acceptable way of incorporating even goods (prices) for which there is no generally accepted unit of measure.
7. The implicit price (quantity) index corresponding to a given quantity (price) index can always be derived by imposing the product test and solving for the price (quantity) index that satisfies this rule. The product test is part of the axiomatic approach to the choice of an index number functional form that is reviewed in Section 4.
8. Jorgenson and Griliches (1967, p. 252) suggested this formula. One set of conditions under which the margins will be zero is perfect competition and a constant returns to scale technology.
9. See Diewert (1987, 1993c) and Fisher (1911, 1922).
10. This criterion is developed more fully in a different context by Emi Nakamura (2002).
11. The period t cost and revenue and the hypothetical aggregates of period s output and input quantities defined in expressions (10) and (13) are comparable in this sense because the quantities for periods s and t are evaluated using the same period t price vectors. Similarly, the period s cost and revenue and the hypothetical aggregates of period t output and input quantities defined in expressions (11) and (12) are comparable in this sense because the quantities of the output and input goods are evaluated using the same period s price vectors. These aggregates are what are used to define the Paasche, Laspeyres and Fisher measures given in (28), (29) and (30).
12. Törnqvist indexes are also known as translog indexes following Jorgenson and Nishimizu (1978) who introduced this terminology because Diewert (1976, p. 120) related Q_T^* to a translog production function. The exact index number approach used for relating specific quantity indexes to specific production functions is the topic of Section 5.
13. See Diewert (1992a, p. 181).
14. Contributors to this approach include Walsh (1901, 1921), Irving Fisher (1911, 1922), Eichhorn (1976), Eichhorn and Voeller (1976), Funke and Voeller (1978, 1979), Diewert (1976, 1987, 1988, 1992a, 1992b) and Balk (1995).
15. The product test was proposed by Irving Fisher (1911, p. 388) and named by Frisch (1930, p. 399).
16. Alternatively, an axiomatic approach can be applied directly to the formula choice problem for TFPG.
17. Laspeyres (1871, p. 308), Walsh (1901, p. 308) and Eichhorn and Voeller (1976, p. 24) proposed this test.
18. Walsh (1901, p. 540) was one of many researchers who proposed this test.
19. Walsh (1901, p. 385) and Eichhorn and Voeller (1976, p. 24) proposed this test.
20. This test was first informally proposed by Pierson (1896, p. 128) and was formalized by Walsh (1901, p. 368; 1921, p. 541) and Fisher (1922, p. 64).
21. See Diewert (1976, p. 131; 1992b) and also Funke and Voeller (1978, p. 180).
22. Christensen, Jorgenson and Lau (1971) introduced the translog functional form for a single output technology. Burgess (1974) and Diewert (1974, p. 139) defined this for the multiple output case. Instead of specifying the firm's cost function, some researchers have specified functional forms for the firm's production function

(e.g., Diewert, 1976, p. 127; 1980, p. 488) or the firm's revenue or profit function (e.g., Diewert, 1980, p. 493; 1988) or for the firm's distance function (e.g., Caves, Christensen and Diewert, 1982, p. 1404).

23. On the econometric estimation of cost functions using more flexible functional forms that permit theoretically plausible types of substitution, see Berndt (1991) and also Diewert and Wales (1992) and the references therein.
24. This follows by applying a theoretical result due initially to Hotelling (1932, p. 594) and Shephard (1953, p. 11).
25. Expression (60) follows from (59) by applying the Hotelling-Shephard relations (58) for t and s.
26. This decomposition has no implications for the choice of a measurement formula for TFPG. The new parameters introduced in making this decomposition cancel out in the product representation of TFPG. That is, TFPG includes the effects of *both* technical progress (a shift in the production function) and returns to scale (a movement along a production function). We are assuming here that producers are on their production frontier in each period; i.e., that they are technically efficient. A more complete analysis might allow for technical inefficiency as well. Favorable or adverse changes in environmental factors facing the firm going from period s to t are regarded as shifts in the production function.
27. TP(1) and TP(2) are the one input, one output special cases of Caves, Christensen and Diewert's (1982, p. 1402) output based 'productivity' indexes.
28. TP(3) and TP(4) are the one input, one output special cases of the input based 'productivity' indexes proposed by Caves, Christensen and Diewert (1982, p. 1407).
29. This shift can be conceptualized as either a move from one production function to another, or equivalently as a change in the location and perhaps the shape of the original production function.
30. In Figure 1, note that if the production function shifts were measured in absolute terms as differences in the direction of the y axis, then these shifts would be given by $y^{0^*} - y^0$ (at point A) and $y^1 - y^{1^*}$ (at point B). If the shifts were measured in absolute terms as differences in the direction of the x axis, then the shifts would be given by $x^0 - x^{0^*}$ (at point A) and $x^{1^*} - x^1$ (at point B). An advantage of measuring TP (and TFPG) using ratios is that the relative measures are invariant to changes in the units of measurement whereas the differences are not.
31. For firms in a regulated industry, returns to scale will generally be greater than one, since increasing returns to scale in production is often the reason for regulation in the first place.
32. Solow's (1957, p. 313) Chart I is similar in concept, but for the simpler case of constant returns to scale.
33. See, for example, Basu and Fernald (1997) and Nakajima, Nakamura and Yoshioka (1998) for evidence of nonconstant returns to scale and references to other research on this subject.
34. These output indexes correspond to the two Malmquist output indexes defined in Caves, Christensen and Diewert (1982, p. 1400). Similar indexes were proposed by Malmquist (1953) for the consumer context, and by Moorsteen (1961) and Hicks (1961; 1981, pp. 192 and 256) for the producer context. See also Balk (1998, Chapter 4).
35. However, if there were technical regress (so that production became less efficient in period 1 compared to period 0) or if the utilization of inputs declined going from period 0 to period 1, then the period 1 output production possibilities set could lie below the period 0 output production possibilities set.
36. If technical progress were sufficiently positive or if output growth between the two periods were sufficiently negative, then the period 1 input requirements set could lie *below* the period 0 one instead of above.
37. An intuitive explanation for the remarkable equalities in (91) and (92) rests on the following fact: if $f(z)$ is a quadratic function, then $f(z^t) - f(z^r) = (1/2)[\nabla f(z^t) + \nabla f(z^r)]^T[z^t - z^r]$. This result follows from applying Diewert's (1976, p. 118) Quadratic Approximation Lemma. Under the assumption of optimizing behavior on the part of the producer, the vectors of first order partial derivatives, $\nabla f(z^t)$ and $\nabla f(z^r)$, will be equal to or proportional to the observed prices. Thus the right-hand side of this identity becomes observable without estimation. In actual applications, we assume that various transformations of f are quadratic and apply the resulting identity.
38. If y_m is positive (negative), then the net output m is an output (input). We assume that all output prices p_m are positive. We assume that all input quantities x_n are positive and if the net input n is an input (output), then w_n is positive (negative).
39. This approach can be viewed as an extension to the general N-M case of the methodology used in defining the output based measures of technical progress given in (74) and (75).
40. These conditions can be found in Diewert (1974; p. 139).

41. Kohli (1990) and Fox and Kohli (1998) use (134) to examine the factors behind the growth in the nominal GDP of several countries. See Morrison and Diewert (1990) for decompositions for other functional forms.
42. For other issues and references on new goods and the measurement of productivity see Diewert (1987, 1995, 1998a, 1998b, 1999), Diewert and Fox (1999), Diewert and Smith (1994), Wolfson (1999), and Baldwin, Despres, Nakamura and Nakamura (1997) as well as other papers and the Introduction in Bresnahan and Gordon (1997).
43. Studies emphasizing externalities like R&D spillovers are Bernstein (1996, 1998), Berndt and Fuss (1986), and Gera, Wu and Lee (1999).
44. For discussions on the measurement problems associated with capital, see Jorgenson (1963, 1980, 1995a, 1995b), Jorgenson and Griliches (1967, pp. 254–260), Morrison (1986, 1988), Diewert (1980, pp. 470–486; 1992a), Diewert and Lawrence (1999), and the references in those papers.

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