INDEX REDUCTION FOR OPERATOR DIFFERENTIAL-ALGEBRAIC EQUATIONS IN ELASTODYNAMICS*

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ABSTRACT. In space semi-discretized equations of elastodynamics with weakly enforced Dirichlet boundary conditions lead to differential algebraic equations (DAE) of index 3. We rewrite the continuous model as operator DAE and present an index reduction technique on operator level. This means that a semi-discretization leads directly to an index-1 system.
We present existence results for the operator DAE with nonlinear damping term and show that the reformulated operator DAE is equivalent to the original equations of elastodynamics. Furthermore, we show that index reduction and semi-discretization in space commute.

Key words. elastodynamics, operator DAE, index reduction, Dirichlet boundary conditions

AMS subject classifications. 65J15, 65L80, 65M60

1. INTRODUCTION

Modeling mechanical systems leads to constrained systems of ordinary and partial differential equations [Sha97, Sim98, SGS06]. Even in the case of a single flexible body, the dynamics are constrained by Dirichlet boundary conditions, which can be incorporated by the Lagrange multiplier technique. A semi-discretization in space by finite elements then leads to a differential-algebraic equation (DAE) of index 3 [Sim06]. Here, we use the concept of the *differentiation index* which measures, loosely speaking, how far the DAE is apart from an ordinary differential equation (ODE).
It is well-known that Runge-Kutta methods as well as backward differentiation formulas cause difficulties while solving DAEs of index higher than one [KM06]. Also the Newmark method [New59], which is widely used in the simulation of structural dynamics, does not converge for the Lagrange multiplier. In this particular case, the multiplier equals the stress in normal direction at the boundary and thus, should also be computed accurately.
Several index reduction approaches were discussed in the last decades. A differentiation of the constraints leads to a loss of information which causes a violation of the constraints. Thus, stabilization methods are needed in this kind of approach

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[GGL85]. Other methods involve derivative arrays and projections which are not discussed here [HW96]. In many applications one takes advantage of a special structure which leads to index reduction methods with less effort. For mechanical systems this is the method of minimal extension [KM04, KM06]. With this approach, the index of the semi-discretized system can be reduced easily to one by an extension of the system combined with the introduction of dummy variables. For index-1 systems, implicit methods converge with the same order as for ODEs. If the DAE is given in a semi-explicit form, then even half-explicit methods work [BH93, LM11]. In contrast, the popular index reduction method by Gear et al. [GGL85] generates an index-2 formulation. This leads to satisfactory results for a single system but fails if we consider coupled systems as in multibody or even multi-physics dynamics [Ebe08]. As an example, consider the two DAEs of index 2,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}.$$

Coupling the two systems via $g = -\dot{x}_2$, we obtain a DAE of index 4 since the solution involves the third derivative of the right-hand side f. This example shows that the reduction to index 1 is necessary in order to allow automatic modeling. In particular for multi-physics systems, where different types of models are coupled, each module has to be utilizable as a black box. This makes the index-2 formulation unfeasible in this context.
In this paper, we aim to reformulate the constrained equations of elastodynamics such that a standard semi-discretization in space by finite elements leads to a DAE of index 1. For this index reduction on operator level, we use the ideas of minimal extension.

The paper is organized as follows. In Section 2 we discuss the equations of motion in elastodynamics with nonlinear damping term and its weak formulation. Dirichlet boundary conditions are included via a Lagrange multiplier method. The introduction of finite element spaces for the semi-discretization in space in subject of Section 3. We show that the resulting DAE is of index 3 and how to reduce the index by minimal extension. In Section 4 we formulate the equations of motion as operator DAE with generalized time derivatives and show the existence of a unique solution. The index reduction on operator level is then discussed in Section 5. Well-posedness is shown as well as the fact that a nonconforming semi-discretization in space results in an index-1 DAE. The paper ends with a conclusion about the order of index reduction and semi-discretization.

2. Equations of Motion

This section is devoted to the dynamics of elastic media with Dirichlet boundary conditions. First, we introduce the stationary partial differential equation of elasticity and its non-stationary counterpart. Second, we include Dirichlet boundary conditions in form of a constraint which leads to the index-3 structure. Finally, we discuss possible nonlinear damping terms.

2.1. **Principle of Virtual Work.** Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ such that the outer normal vector n exists almost everywhere on $\partial\Omega$. With $\Gamma_D \subseteq \partial\Omega$ we denote the Dirichlet boundary, $\Gamma_N = \partial\Omega \setminus \Gamma_D$ denotes the Neumann boundary. We neglect the pure Neumann problem, i. e., we assume that Γ_D has positive surface measure.
The equilibrium equations for elasticity are given by Cauchy's theorem [Cia88, chapter 2] and are concerned about the deformation of bodies under the influence of applied forces. This work deals with linear elasticity for homogeneous and isotropic materials. Thus, we assume linear material laws with constant Lamé parameters λ and μ which is justified in the case of small deformations.
By $u : \Omega \to \mathbb{R}^2$ we denote the displacement field. The linearized strain tensor $\epsilon(u) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is defined by

$$\epsilon(u) := 1/2\,\left(\nabla u + (\nabla u)^T\right).$$

In linear elasticity, the stress tensor $\sigma(u) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ depends linearly on the strain tensor (Hooke's law),

$$\sigma(u) := \lambda\,\mathrm{trace}\left(\epsilon(u)\right) I_2 + 2\mu\;\epsilon(u).$$

Therein, I_2 denotes the 2×2 identity matrix. The corresponding boundary value problem with Dirichlet data u_D and applied forces β and τ reads

$$-\mathrm{div}(\sigma(u)) = \beta \qquad \text{in } \Omega, \tag{2.1a}$$

$$u = u_D \qquad \text{on } \Gamma_D \subseteq \partial\Omega, \tag{2.1b}$$

$$\sigma(u) \cdot n = \tau \qquad \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D. \tag{2.1c}$$

Since equation (2.1a) is in divergence form, the variational form is achieved easily via integration by parts. The resulting equation is called the *principle of virtual work* (in the reference configuration). The variational form corresponds to the concept of weak solutions and Sobolev spaces. By $H^1(\Omega)$ we denote the space of square-integrable functions from Ω to $\mathbb{R}$ which have a square-integrable weak derivative. Furthermore, we define the Hilbert spaces

$$\begin{aligned}
\mathcal{V} &:= [H^1(\Omega)]^2,\\
\mathcal{V}_D &:= [H^1_D(\Omega)]^2 := \{v \in \mathcal{V} \mid v|_{\Gamma_D} = 0\}, \text{ and}\\
\mathcal{V}^* &:= \mathcal{L}(\mathcal{V}, \mathbb{R}) = \{f : \mathcal{V} \to \mathbb{R} \mid f \text{ is linear and continuous}\}.
\end{aligned}$$

With the scalar product for matrices, $A : B := \mathrm{trace}(AB^T) = \sum_{i,j} A_{ij}B_{ij}$, we define the symmetric bilinear form

$$a(u, v) := \int_\Omega \sigma(u) : \epsilon(v)\, dx.$$

By Korn's inequality [BS08, chapter 11.2], a is a coercive and bounded bilinear form on $\mathcal{V}_D$ if Γ_D has positive measure. Since $\epsilon(\cdot)$ vanishes for constant functions, a cannot be coercive on the entire space $\mathcal{V}$. The principle of virtual work for the stationary linear elasticity problem with zero Dirichlet boundary conditions reads

$$a(u, v) = (\beta, v)_{L^2(\Omega)} + (\tau, v)_{L^2(\Gamma_N)} \text{ for all } v \in \mathcal{V}_D.$$

Note that the Dirichlet boundary condition on u is taken into account by the fact that we search u in the space $\mathcal{V}_D$. The inclusion of inhomogeneous Dirichlet boundary conditions is part of the following subsection.
The non-stationary problem is called the weak formulation of elastodynamics. With constant density $\rho \in \mathbb{R}_{>0}$, we search for the time-dependent deformation with $u(t) \in \mathcal{V}_D$ for all $t \in [0,T]$ such that

$$(\rho\ddot{u}, v)_{L^2(\Omega)} + a(u,v) = (\beta, v)_{L^2(\Omega)} + (\tau, v)_{L^2(\Gamma_N)} \text{ for all } v \in \mathcal{V}_D,\ t \in [0,T]. \tag{2.2}$$

This problem requires initial conditions of the form

$$u(0) = g \in \mathcal{V}_D, \quad \dot{u}(0) = h \in \mathcal{V}_D.$$

Note that the applied forces β, τ may also be time-dependent. Equation (2.2) is formulated weakly in space but still in the classical form concerning the time variable. The fully weak formulation is given in Section 4 with the help of generalized time derivatives.

2.2. **Dirichlet Boundary Conditions.** In this section, we include inhomogeneous Dirichlet boundary conditions, i. e., we prescribe the deformation to be equal to a given function u_D on Γ_D. We follow the work of [Sim00] and include the boundary conditions with the help of Lagrange multipliers, which leads to a dynamic saddle point problem.
A different strategy is to include the boundary conditions in the right-hand side. This approach is often used to assume homogeneous boundary data but has the drawback that for computations one has to construct a function on Ω with the given Dirichlet data on Γ_D. A second disadvantage arises if the position of the Dirichlet boundary is time-dependent. In this case, the solution lies in different Hilbert spaces for each time step.
Since we work in the Sobolev space $H^1(\Omega)$, traces on Γ_D are well defined. For a definition of fractional order Sobolev spaces, such as $H^{1/2}(\Gamma_D)$, we refer to [AF03, chapter 7]. We denote the trace spaces by

$$\begin{aligned} \mathcal{Q}^* &:= [H^{1/2}(\Gamma_D)]^2 \subset [L^2(\Gamma_D)]^2 \text{ and} \\ \mathcal{Q} &:= \mathcal{Q}^{**} = \mathcal{L}(\mathcal{Q}^*, \mathbb{R}). \end{aligned}$$

By $\langle\cdot,\cdot\rangle_{\mathcal{Q},\mathcal{Q}^*}$ we denote the dual pairing which is densely defined for $\vartheta \in [L^2(\Gamma_D)]^2$ (since $[L^2(\Gamma_D)]^2 \overset{d}{\hookrightarrow} \mathcal{Q}$) and $q \in \mathcal{Q}^*$ by

$$\langle\vartheta, q\rangle_{\mathcal{Q},\mathcal{Q}^*} := \int_{\Gamma_D} \psi \cdot q \, dx. \tag{2.3}$$

The Dirichlet boundary condition in the classical form is given by $u(\cdot,t)|_{\Gamma_D} = u_D(\cdot,t)$ for all $t \in [0,T]$ and requires $u_D(\cdot,t) \in \mathcal{Q}^*$. In the weak form, this condition reads

$$\langle\vartheta, u(\cdot,t)\rangle_{\mathcal{Q},\mathcal{Q}^*} = \langle\vartheta, u_D(\cdot,t)\rangle_{\mathcal{Q},\mathcal{Q}^*} \text{ for all } \vartheta \in \mathcal{Q},\ t \in [0,T].$$

For this purpose, we introduce the bilinear form $b : \mathcal{V} \times \mathcal{Q} \to \mathbb{R}$, defined by

$$b(u,\vartheta) := \langle\vartheta, u\rangle_{\mathcal{Q},\mathcal{Q}^*} \ . \tag{2.4}$$

Note that this bilinear form is well-defined due to the trace theorem [Ste08, Theorem 2.21]. A subtle but important property of the bilinear form b is the inf-sup condition. Since b involves the boundary constraint, its analysis is the a main part of the existence theory of solutions below [Bra07, chapter 3].

Lemma 2.1 (Inf-sup condition)**.** *If Γ_D has positive measure, then the bilinear form b from* (2.4) *satisfies an* inf-sup condition*, i. e., there exists a positive constant β with*

$$\inf_{q\in\mathcal{Q}}\sup_{v\in\mathcal{V}}\frac{b(v,q)}{\|v\|_{\mathcal{V}}\|q\|_{\mathcal{Q}}}=\beta>0.$$

Proof. For details see [Ste08, Lemma 4.7]. □

For all $t\in[0,T]$, we define the linear functional $\mathcal{G}(t,\cdot)\in\mathcal{Q}^*$, which includes the Dirichlet data, by

$$\mathcal{G}(t,\vartheta):=\langle\vartheta,u_D(\cdot,t)\rangle_{\mathcal{Q},\mathcal{Q}^*}\ . \tag{2.5}$$

For the right-hand side with possible Neumann data, we introduce $\mathcal{F}(t,\cdot)\in\mathcal{V}^*$, defined by

$$\mathcal{F}(t,v):=\langle\beta(t),v\rangle_{\mathcal{V}^*,\mathcal{V}}+(\tau(t),v)_{L^2(\Gamma_N)}. \tag{2.6}$$

As a result, we obtain the dynamic elasticity problem with non-homogeneous Dirichlet conditions in form of a dynamic saddle point problem: determine $\big(u(t),\lambda(t)\big)\in\mathcal{V}\times\mathcal{Q}$ such that for all $t\in[0,T]$,

$$\begin{aligned}(\rho\ddot{u}(t),v)_{L^2(\Omega)}+a(u(t),v)+b(v,\lambda(t))&=\langle\mathcal{F}(t),v\rangle_{\mathcal{V}^*,\mathcal{V}}\qquad\text{for all } v\in\mathcal{V},\\ b(u(t),\vartheta)&=\langle\vartheta,\mathcal{G}(t)\rangle_{\mathcal{Q},\mathcal{Q}^*}\qquad\text{for all } \vartheta\in\mathcal{Q}.\end{aligned}$$

Remark 2.1. Note that since we search for $u(t)\in\mathcal{V}$, we also use $\mathcal{V}$ as test space. In the homogeneous case (2.2) we only had to test with functions in $\mathcal{V}_D$.

2.3. **Damping.** In many applications, one considers viscous damping [Hug87, chapter 7.2] and in particular Rayleigh damping [CP03, chapter 12] which is a generalization of the mass proportional and stiffness proportional damping. This combines frequency dependent and independent damping and is widespread in modeling internal structural damping.

Let $\zeta_1,\zeta_2\geq 0$ be two real parameters. The first parameter ζ_1 regularizes the frequency dependent damping and corresponds to a generalization of Hooke's law. Thus, the stress tensor does not depend linearly on the strain tensor anymore,

$$\begin{aligned}\sigma_D(u)&:=\lambda\,\mathrm{trace}\,\big(\epsilon(u+\zeta_1\dot{u})\big)+2\mu\ \epsilon(u+\zeta_1\dot{u})\\ &=\sigma(u)+\zeta_1\sigma(\dot{u}).\end{aligned}$$

Since the damping is linearly proportional to the response frequencies, stiffness proportional damping acts stronger on the higher modes of the structure. Although this is a common approach, it has no physical justification [Wil98, chapter 19].

The frequency independent damping is parametrized by ζ_2 and includes the additional term $\zeta_2(\rho\dot{u},v)_{L^2(\Omega)}$. Both damping parts can be combined in a bilinear form

$$d(\dot{u},v):=\zeta_1 a(\dot{u},v)+\zeta_2(\rho\dot{u},v)_{L^2(\Omega)}.$$

We consider a more general case with possibly nonlinear damping. Let $d : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ be a map, linear only in the second component, with constants $d_1, d_2 > 0$ and $d_0 \geq 0$ such that for all $u, v, w \in \mathcal{V}$ it holds that

(Lipschitz continuity) $\qquad d(u, w) - d(v, w) \leq d_2 \|u - v\|_{\mathcal{V}} \|w\|_{\mathcal{V}},$

(Strong monotonicity) $\qquad d_1 \|u - v\|^2_{\mathcal{V}} - d_0 \|u - v\|^2_{\mathcal{H}} \leq d(u, u - v) - d(v, u - v).$

In either case, the dynamic saddle point problem with dissipation has the form: determine $\big(u(t), \lambda(t)\big) \in \mathcal{V} \times \mathcal{Q}$ such that for all $t \in [0, T]$,

$$\begin{aligned} (\rho\ddot{u}, v)_{L^2(\Omega)} + d(\dot{u}, v) + a(u, v) + b(v, \lambda) &= \langle \mathcal{F}, v\rangle \qquad \text{for all } v \in \mathcal{V}, \\ b(u, \vartheta) &= \langle \mathcal{G}, \vartheta\rangle \qquad \text{for all } \vartheta \in \mathcal{Q}. \end{aligned} \tag{2.7}$$

3. Semi-Discretized Equations

To describe the system, we follow the method of lines approach in which system (2.7) is semi-descretized in space first. For the discretization, we use piecewise linear and globally continuous finite elements and show that this results in an index-3 DAE. A suitable method to reduce the index of this system is given by minimal extension [KM04, KM06].

3.1. **Ansatz Spaces.**

The conforming finite element ansatz requires finite dimensional subspaces of $\mathcal{V}$ and $\mathcal{Q}$. Let $\mathcal{T}$ be a regular triangulation in the sense of [Cia78], i. e., we exclude hanging nodes. For each node of the triangulation we define the standard nodal basis function (hat-function) which has the value one at this node and vanishes at any other node; see [Bra07, chapter II].
With the space of piecewise linear functions $\mathcal{P}(\mathcal{T})$, we define the finite dimensional space

$$\mathcal{S}_h := [\mathcal{P}(\mathcal{T})]^2 \cap \mathcal{V} = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\} \subset \mathcal{V}.$$

Therein, $\{\varphi_i\}_{i=1,\ldots,n}$ denotes the set of basis functions of $\mathcal{S}_h$. Note that the dimension n equals twice the number of nodes in $\mathcal{T}$.

Remark 3.1. In the sequel we assume that the basis functions are ordered such that $\mathcal{S}_h \cap \mathcal{V}_D = \operatorname{span}\{\varphi_1, \ldots, \varphi_{n-m}\}$. This means that the functions $\{\varphi_i\}_{i=n-m+1,\ldots,n}$ correspond to the nodes at the Dirichlet boundary Γ_D.

As finite dimensional subspace of $\mathcal{Q}$ we choose the traces of $\mathcal{S}_h$. With the special order of the basis functions, this space is given by

$$\mathcal{Q}_h := \operatorname{span}\{\varphi_{n-m+1}|_{\Gamma_D}, \ldots, \varphi_n|_{\Gamma_D}\} \subset [L^2(\Gamma_D)]^2 \subset \mathcal{Q}.$$

Note that $\mathcal{Q}_h$ is the space of edgewise linear and globally continuous functions on Γ_D. The dimension of $\mathcal{Q}_h$ is denoted by $m < n$ and equals twice the number of nodes on Γ_D. We denote the basis of $\mathcal{Q}_h$ by

$$\psi_1 := \varphi_{n-m+1}|_{\Gamma_D}, \ldots, \psi_m := \varphi_n|_{\Gamma_D}.$$

With this finite element scheme, we approximate the displacement field u by some time-dependent discrete function $u_h(t) \in \mathcal{S}_h$ and the Lagrange multiplier λ by

$\lambda_h(t) \in \mathcal{Q}_h$. The approximations can be represented by their coefficient vectors,

$$u_h(x,t) := \sum_{i=1}^{n} q_i(t)\varphi_i(x), \qquad \lambda_h(x,t) := \sum_{i=1}^{m} \mu_i(t)\psi_i(x).$$

The approximations u_h and λ_h are defined as solution of the discrete variational formulation. Thus, for all $v_h \in \mathcal{S}_h$ and $\vartheta_h \in \mathcal{Q}_h$ they satisfy the equations,

$$\begin{aligned} (\rho\ddot{u}_h, v_h)_{L^2(\Omega)} + d(\dot{u}_h, v_h) + a(u_h, v_h) + b(v_h, \lambda_h) &= \langle \mathcal{F}, v_h \rangle, \\ b(u_h, \vartheta_h) &= \langle \mathcal{G}, \vartheta_h \rangle. \end{aligned} \tag{3.1}$$

Because of the finite dimensional setting, system (3.1) can be written as a quasi-linear DAE for the coefficient vectors. In the case of linear damping, let $M, D, K \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ denote the time-independent matrices with

$$M_{ij} := (\rho\varphi_i, \varphi_j)_{L^2(\Omega)}, \quad K_{ij} := a(\varphi_i, \varphi_j), \quad D_{ij} := d(\varphi_i, \varphi_j), \quad B_{ij} := b(\varphi_j, \psi_i).$$

According to Remark 3.1, the matrix B has the form $B = [\,0 \;\; B_2]$ with non-singular and symmetric matrix $B_2 \in \mathbb{R}^{m\times m}$. For nonlinear damping, the matrix D is replaced by an appropriate nonlinear function which we also denote by D. For the right-hand side, we introduce the time-dependent vectors $f(t) \in \mathbb{R}^n$ and $g(t) \in \mathbb{R}^m$ by

$$f_i(t) := \langle \mathcal{F}(t), \varphi_i \rangle, \qquad g_i(t) := \langle \mathcal{G}(t), \psi_i \rangle.$$

The semi-discrete problem (3.1) is equivalent to the following differential algebraic equation for the coefficient vectors $q = [q_i]$ and $\mu = [\mu_i]$,

$$M\ddot{q}(t) + D(\dot{q}(t)) + Kq(t) + B^T\mu(t) = f(t), \tag{3.2a}$$

$$Bq(t) = g(t). \tag{3.2b}$$

3.2. **Index-3 DAE.** In this subsection we analyse the index of the DAE (3.2). As for the continuous problem, the inf-sup condition plays a crucial role. In general, proving the discrete inf-sup condition can cause difficulties. However, since we use for $\mathcal{Q}_h$ the traces of $\mathcal{S}_h$, the inf-sup condition can be shown easily for example by the Fortin criterion [Ste08, chapter 8.4]. To show that the DAE (3.2) is of index 3, we use the full rank property of the matrix B, which is equivalent to the discrete inf-sup condition.

The index equals the number of needed differentiation steps to obtain a unique continuously differentiable solution; see [KM06, chapter 3.3]. In our case, a double differentiation of the algebraic constraint (3.2b) gives $B\ddot{q} = \ddot{g}$. Along with (3.2a), this results in

$$\begin{bmatrix} M & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \mu \end{bmatrix} = \begin{bmatrix} f - D(\dot{q}) - Kq \\ \ddot{g} \end{bmatrix}.$$

If the matrix on the left-hand side is invertible, the DAE (3.2) is of index 3. In this case, we can solve for $\ddot{q}$ and μ but need one additional differentiation step in order to obtain that μ is continuously differentiable. Note that M is the mass matrix for given density $\rho > 0$ and thus positive definite. Due to the saddle point structure, the full rank property of B finally implies that the matrix on the left-hand side is non-singular.

3.3. **Index Reduction by Minimal Extension.** We summarize the minimal extension procedure for system (3.2). For details we refer to [KM04]. With $p := \dot{q}$, we write the DAE (3.2) as a first-order system. Because of the special order of the basis functions, it is useful to split the variables q and p into

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

with $p_1, q_1 \in \mathbb{R}^{n-m}$ and $p_2, q_2 \in \mathbb{R}^m$. The structure of B then implies $Bq = B_2 q_2$. The derivatives of the constraint (3.2b) read

$$B_2 p_2 = \dot{g}, \qquad B_2 \dot{p}_2 = \ddot{g}.$$

We add these two constraints and note that this results in a number of redundant equations. To obtain an equal number of equations and unknowns, we omit the equation $\dot{q}_2 = p_2$ and add a dummy variable $\tilde{p}_2 := \dot{p}_2$. Written as second order system, we obtain

$$\begin{aligned} M \begin{bmatrix} \ddot{q}_1(t) \\ \tilde{p}_2(t) \end{bmatrix} + D\left(\begin{bmatrix} \dot{q}_1(t) \\ p_2(t) \end{bmatrix} \right) + K \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \mu(t) &= f(t), \\ B_2 q_2(t) &= g(t), \\ B_2 p_2(t) &= \dot{g}(t), \\ B_2 \tilde{p}_2(t) &= \ddot{g}(t). \end{aligned} \tag{3.3}$$

We show that the DAE (3.3) has index 1. Since B_2 is invertible, we can solve directly for q_2, p_2, and $\tilde{p}_2$ without any differentiation steps. The remaining equation consists of a quasi-linear ODE for q_1 and an algebraic equation for μ. Thus, one differentiation step suffices to obtain a continuously differentiable solution.
Furthermore, any solution (q, μ) of (3.2) solves the index-1 formulation in the sense that $(q_1, q_2, \dot{q}_2, \ddot{q}_2, \mu)$ is a solution of (3.3).

4. Operator Formulation

The goal of this section is to reformulate the saddle point problem (2.7) with the help of operators. This leads to operator differential-algebraic equations, i. e., DAEs with operators. In addition, time derivatives will be used in a generalized meaning. A basic tool in functional analysis when dealing with operator differential equations are Gelfand triples. They are introduced in the following subsection.

4.1. **Gelfand Triples and Generalized Time Derivatives.** In this subsection we introduce the notion of Gelfand triples and discuss the role of time derivatives in this setting. All definitions and remarks in this subsection are based on Emmrich [Emm04] and Zeidler [Zei90].

Definition 4.1 (Gelfand Triple)**.** Let V be a real, separable and reflexive Banach space with dual space V^* and H a real, separable Hilbert space. If V is densely, continuously embedded in H, then the spaces V, H, V^* form a Gelfand triple,

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*.$$

Remark 4.1. The equivalence of H and H^* is given by the Riesz representation theorem. The continuous embedding $V \hookrightarrow H$ implies the existence of a constant $c_{HV} > 0$ with $\|v\|_H \leq c_{HV}\|v\|_V$. The embedding $H^* \hookrightarrow V^*$ in Definition 4.1 is justified by the fact that for $f \in H^*$, we have that

$$\|f\|_{V^*} = \sup_{v\in V} \frac{\langle f, v\rangle}{\|v\|_V} \leq \sup_{v\in H} \frac{\langle f, v\rangle}{\|v\|_V} \leq \frac{1}{c_{HV}} \sup_{v\in H} \frac{\langle f, v\rangle}{\|v\|_H} = \frac{1}{c_{HV}} \|f\|_{H^*}.$$

Remark 4.2. The embedding $H \subseteq V^*$ is meant in the following way. If $(\cdot,\cdot)_H$ denotes the inner product in H, it can be seen as a function on $H \times V$. The duality pairing $\langle\cdot,\cdot\rangle_{V^*,V}$ is then a continuous extension to $V^* \times V$. This means for all $f \in H \cong H^*$ and $v \in V \subseteq H$, we have

$$\langle f, v\rangle_{V^*,V} = (f, v)_H.$$

To shorten future notation, we introduce the Hilbert space $\mathcal{H} := [L^2(\Omega)]^2$.

Example 4.1. The Gelfand triple suitable for the analysis of the homogeneous Dirichlet problem is

$$\mathcal{V}_D \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} \mathcal{V}_D^*.$$

In the case $\Gamma_D = \partial\Omega$ the dual space is given by $\mathcal{V}_D^* = [H^{-1}(\Omega)]^2$; see [Bra07, chapter III.3]. For the non-homogeneous case we consider the triple $\mathcal{V}, \mathcal{H}, V^*$.

Example 4.2. To involve the boundary conditions, we have introduced the extension of the L^2-inner product on Γ_D in (2.3). This extension is nothing else than the embedding given by the Gelfand triple

$$\mathcal{Q}^* \overset{d}{\hookrightarrow} [L^2(\Gamma_D)]^2 \overset{d}{\hookrightarrow} \mathcal{Q}.$$

In what follows, we also work with L^2-spaces for the time variable. Over and above, time derivatives are interpreted in the generalized sense [Zei90]. This means that for $u, v \in L^1((0,T), V)$, we write $v = \dot{u}$ if for all $\Phi \in C_0^\infty(0,T)$,

$$\int_0^T u(t)\dot{\Phi}(t)\ dt = -\int_0^T v(t)\Phi(t)\ dt.$$

Bounded functionals $f \in V^*$ satisfy the equality $\frac{\mathrm{d}}{\mathrm{dt}}\langle f, u(t)\rangle = \langle f, v(t)\rangle$. In the remaining part of this subsection, we introduce common function spaces. Let V, H, V^* be a Gelfand triple. Then, we define the space

$$W(0,T) = \{u \in L^2((0,T), V) |\ \dot{u} \in L^2((0,T), V^*)\}$$

with the norm

$$\|u\|_{W(0,T)} = \left(\|u\|^2_{L^2((0,T),V)} + \|\dot{u}\|^2_{L^2((0,T),V^*)}\right)^{1/2}, \qquad \|u\|^2_{L^2((0,T),V)} = \int_0^T \|u\|_V^2\ dx.$$

The space $W(0,T)$ is a Hilbert space if V is a Hilbert space. Furthermore, there exists a continuous embedding $W(0,T) \hookrightarrow C([0,T], H)$, [Emm04, Theorem 8.1.9]. We write $\mathcal{W}(0,T)$ if we work with the Gelfand triple $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$. With Dirichlet conditions, i. e., in terms of the triple $\mathcal{V}_D, \mathcal{H}, \mathcal{V}_D^*$, we write $\mathcal{W}_D(0,T)$.

4.2. **Definition of Operators.** In order to reformulate the saddle point problem as operator equations in $\mathcal{V}^*$ and $\mathcal{Q}^*$, we need to define several operators.

Mass operator: We define the linear operator

$$\mathcal{M} : \mathcal{V}^* \to \mathcal{V}^* \tag{4.1}$$

by $\mathcal{M}u = \rho u$ with constant density $\rho > 0$. Since $\mathcal{M}$ is just a multiplication by a constant, the operator is symmetric and bijective. In the case $u \in \mathcal{H}$, we know by Remark 4.2 that $\langle \mathcal{M}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} = (\rho u, v)_{L^2(\Omega)}$ for all $v \in \mathcal{V}$.

Stiffness operator: For the stiffness part, we define the linear operator

$$\mathcal{K} : \mathcal{V} \to \mathcal{V}^* \tag{4.2}$$

by $\langle \mathcal{K}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} := a(u, v)$. Since the bilinear form a is symmetric, $\mathcal{K}$ is self-adjoint. Non-negativity follows from $a(v, v) \geq c\|\varepsilon(v)\|^2 \geq 0$ for some positive constant $c \in \mathbb{R}$ which depends on the Lamé constants of the material. On the subspace $\mathcal{V}_D$, the operator $\mathcal{K}$ is also positive definite.

Damping operator: Damping is described by the nonlinear operator

$$\mathcal{D} : \mathcal{V} \to \mathcal{V}^* \tag{4.3}$$

and is defined via $\langle \mathcal{D}u, v\rangle_{\mathcal{V}^*,\mathcal{V}} := d(u, v)$. The requested properties of d from Section 2.3 with constants d_0, d_1, d_2 imply that $\mathcal{D}$ is Lipschitz continuous,

$$\|\mathcal{D}u - \mathcal{D}v\|_{\mathcal{V}^*} \leq d_2\|u - v\|_{\mathcal{V}},$$

and $\mathcal{D} + d_0\,\mathrm{id}$ is strongly monotone,

$$d_1\|u - v\|_{\mathcal{V}}^2 \leq \big\langle (\mathcal{D} + d_0\,\mathrm{id})u - (\mathcal{D} + d_0\,\mathrm{id})v, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}} .$$

Note that the identity map id is used as an inclusion map from $\mathcal{V}$ to $\mathcal{V}^*$ in terms of the Gelfand triple $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$, i. e., $\langle \mathrm{id}\, u, v\rangle_{\mathcal{V}^*,\mathcal{V}} = (u, v)_{\mathcal{H}}$.

Trace operator: To include Dirichlet boundary conditions, we introduce the trace operator

$$\mathcal{B} : \mathcal{V} \to \mathcal{Q}^* \tag{4.4}$$

by $\langle \vartheta, \mathcal{B}u\rangle_{\mathcal{Q},\mathcal{Q}^*} := b(u, \vartheta)$ for $\vartheta \in \mathcal{Q}$. Its dual operator

$$\mathcal{B}^* : \mathcal{Q} \to \mathcal{V}^* \tag{4.5}$$

is defined via $\langle \mathcal{B}^*\vartheta, u\rangle_{\mathcal{V}^*,\mathcal{V}} := \langle \vartheta, \mathcal{B}u\rangle_{\mathcal{Q},\mathcal{Q}^*} = b(u, \vartheta)$. The properties of $\mathcal{B}$ and $\mathcal{B}^*$ result from the properties of b and are subject of the following lemma.

Lemma 4.1 (Properties of $\mathcal{B}$ and $\mathcal{B}^*$)**.** *Let Γ_D have positive measure and consider the orthogonal decomposition $\mathcal{V} = \mathcal{V}_D \oplus \mathcal{V}_D^\perp$ with respect to the inner product of $\mathcal{V}$. With the inf-sup constant β from Lemma 2.1, the following assertions hold,*

a) *$\mathcal{B}$ vanishes on $\mathcal{V}_D$,*
b) *$\mathcal{B}$ restricted to $\mathcal{V}_D^\perp$ is an isomorphism,*
c) *$\mathcal{B}^* : \mathcal{Q} \to (\mathcal{V}_D^\perp)^*$ defines an isomorphism,*
d) *$\beta\|v\|_{\mathcal{V}} \leq \|\mathcal{B}v\|_{\mathcal{Q}^*}$ for all $v \in \mathcal{V}_D^\perp$,*
e) *$\frac{d}{dt}(\mathcal{B}u) = \mathcal{B}\dot{u}$ for all $u \in H^1((0, T), \mathcal{V})$.*

Proof. a) Consider an arbitrary $v \in \mathcal{V}_D$ and $q \in [L^2(\Gamma_D)]^2 \overset{d}{\hookrightarrow} \mathcal{Q}$. Since v vanishes on Γ_D,

$$\langle q, \mathcal{B}v\rangle_{\mathcal{Q},\mathcal{Q}^*} = b(v,q) = \int_{\Gamma_D} v \cdot q \, dx = 0.$$

For the general case consider $q \in \mathcal{Q}$. Because of the dense embedding, there exists a sequence $\{q_n\} \subset [L^2(\Gamma_D)]^2$ with $q_n \to q$ in $\mathcal{Q}$ as $n \to \infty$. Thus,

$$\langle q, \mathcal{B}v\rangle_{\mathcal{Q},\mathcal{Q}^*} = \lim_{n\to\infty} \langle q_n, \mathcal{B}v\rangle_{\mathcal{Q},\mathcal{Q}^*} = \lim_{n\to\infty} 0 = 0.$$

b) The assertion follows with the help of [Bra07, chapter III, Theorem 3.6]. It requires the continuity of b (which follows by the trace theorem [Ste08, Theorem 2.21]), the inf-sup condition from Lemma 2.1, and a non-degeneration condition,

$$\forall v \in \mathcal{V}_D^\perp, v \neq 0, \quad \exists q \in \mathcal{Q}: \; b(v,q) \neq 0.$$

To show the latter, assume there exists a $v \in \mathcal{V}_D^\perp$ such that $b(v,q)$ vanishes for all $q \in \mathcal{Q}$. Thus, v has trace zero on Γ_D and hence $v \in \mathcal{V}_D \cap \mathcal{V}_D^\perp = \{0\}$.
c) The space $(\mathcal{V}_D^\perp)^*$ contains all functionals in $\mathcal{V}^*$ which vanish on $\mathcal{V}_D$. The isomorphy of $\mathcal{B}^* : \mathcal{Q}^* \to (\mathcal{V}_D^\perp)^*$ is equivalent to part b); see [Bra07, chapter III, Lemma 4.2].
d) Since $\mathcal{B}^*$ is an isomorphism, b also fulfills an inf-sup condition of the form

$$\inf_{v\in\mathcal{V}_D^\perp} \sup_{q\in\mathcal{Q}} \frac{b(v,q)}{\|v\|_\mathcal{V}\|q\|_\mathcal{Q}} = \beta > 0.$$

Thus, for all $v \in \mathcal{V}_D^\perp$ the following chain of inequalities holds,

$$\beta\|v\|_\mathcal{V} \leq \sup_{q\in\mathcal{Q}} \frac{b(v,q)}{\|q\|_\mathcal{Q}} \leq \sup_{q\in\mathcal{Q}} \frac{\|\mathcal{B}v\|_{\mathcal{Q}^*}\|q\|_\mathcal{Q}}{\|q\|_\mathcal{Q}} = \|\mathcal{B}v\|_{\mathcal{Q}^*} \; .$$

e) For every $q \in \mathcal{Q}$, $\mathcal{B}^*q$ is a bounded functional and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle q, \mathcal{B}u(t)\rangle_{\mathcal{Q},\mathcal{Q}^*} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{B}^*q, u(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \mathcal{B}^*q, \dot{u}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle q, \mathcal{B}\dot{u}(t)\rangle_{\mathcal{Q},\mathcal{Q}^*} \; . \qquad \square$$

4.3. **Operator DAE.** With the operators $\mathcal{M}, \mathcal{K}, \mathcal{D}, \mathcal{B}$, and $\mathcal{B}^*$ defined in (4.1)-(4.5) and linear functionals $\mathcal{F}$, $\mathcal{G}$ from (2.5)-(2.6), we are able to formulate the dynamic saddle point problem (2.7) as system of operator (differential-algebraic) equations. Assume that $u, \dot{u} \in L^2((0,T),\mathcal{V})$ with $\ddot{u} \in L^2((0,T),\mathcal{V}^*)$. At this point, we use the embedding given by the Gelfand triple $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$, i. e., in the case $\ddot{u}(t) \in \mathcal{H}$ we know from Remark 4.2 that

$$\langle \mathcal{M}\ddot{u}(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle \rho\ddot{u}(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} = (\rho\ddot{u}(t), v)_\mathcal{H}.$$

Otherwise, it is defined as the continuous extension of this map. The homogeneous Dirichlet case (2.2) in operator form reads: for given initial conditions determine a function $u \in L^2((0,T),\mathcal{V}_D)$ with $\dot{u} \in \mathcal{W}_D(0,T)$ such that

$$\mathcal{M}\ddot{u}(t) + \mathcal{D}\dot{u}(t) + \mathcal{K}u(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}_D^* \text{ for a. e. } t \in [0,T]. \tag{4.6}$$

The existence of a unique solution to this problem is subject of the next subsection. Let us consider the non-homogeneous case with Lagrange multiplier as in equation

(2.7). In the operator formulation, we search for $u \in L^2((0,T),\mathcal{V})$ with $\dot u \in \mathcal{W}(0,T)$ and $\lambda \in L^2((0,T),\mathcal{Q})$ such that

$$\begin{aligned} \mathcal{M}\ddot u(t) + \mathcal{D}\dot u(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) &= \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, && \text{(4.7a)} \\ \mathcal{B}u(t) &= \mathcal{G}(t) \quad \text{in } \mathcal{Q}^* && \text{(4.7b)} \end{aligned}$$

for a. e. $t \in [0,T]$ with initial conditions

$$\begin{aligned} u(0) &= g \in \mathcal{V}, && \text{(4.7c)} \\ \dot u(0) &= h \in \mathcal{H}. && \text{(4.7d)} \end{aligned}$$

Note that the initial conditions are well-defined due to the embedding $\mathcal{W}(0,T) \hookrightarrow C([0,T],\mathcal{H})$. Because of the constraint (4.7b), system (4.7) is an operator DAE.

4.4. **Existence Results.** In this subsection we analyse the existence of solutions for the operator DAE (4.7). We reduce the problem to the homogeneous Dirichlet case (4.6).

4.4.1. *Homogeneous Dirichlet Data.* Consider the Gelfand triple $\mathcal{V}_D, \mathcal{H}, \mathcal{V}_D^*$ and the operator ODE (4.6) with initial conditions

$$\begin{aligned} u(0) &= g \in \mathcal{V}_D, \\ \dot u(0) &= h \in \mathcal{H}. \end{aligned} \qquad \text{(4.8)}$$

The following theorem states sufficient conditions for the existence of a unique solution. Recall that the function space $\mathcal{W}_D(0,T)$ is based on $\mathcal{V}_D$, cf. Section 4.1.

Theorem 4.2. *Let $g \in \mathcal{V}_D$, $h \in \mathcal{H}$, and $\mathcal{F} \in L^2((0,T),\mathcal{V}^*)$. Then, there exists a unique solution $u \in C([0,T],\mathcal{V}_D)$ with $\dot u \in L^2((0,T),\mathcal{V}_D)$ of equation* (4.6) *with initial conditions* (4.8). *Furthermore, the time derivative satisfies $\dot u \in \mathcal{W}_D(0,T)$.*

Proof. Since $\mathcal{M}$ is just a multiplication by a constant, this theorem is a special case of a theorem in [GGZ74, chapter 7]. The proof makes use of Korn's inequality for the operator $\mathcal{K}$ as well as the given properties of $\mathcal{D}$ which imply that the operator $(\mathcal{D} + d_0 \,\mathrm{id})$ is continuous, monotone and coercive. The condition $h \in \mathcal{H}$ is justified by the fact $\dot u \in \mathcal{W}_D(0,T) \hookrightarrow C([0,T],\mathcal{H})$. □

Remark 4.3. Theorem 4.2 is not restricted to linear deformations. For details and assumptions on a possibly nonlinear elasticity operator, we refer to [GGZ74, chapter 7]. A nonlinear operator is necessary if we model large deformations, where the assumption that the stress depends linearly on the strain is not reasonable.

Remark 4.4. The existence result is also true for the damping-free case, i. e., $\mathcal{D} \equiv 0$; see [Zei90, chapter 24]. In the case where d is bilinear, defined on $\mathcal{H} \times \mathcal{H}$, and positive semidefinite, the existence result is stated in [HN98].

4.4.2. *Non-Homogeneous Dirichlet Data.* From Theorem 4.2 we obtain as corollary the existence of a unique solution for arbitrary Dirichlet boundary conditions, i. e., $u(t) = u_D(t)$ on Γ_D for given data u_D.

Theorem 4.3. *Let $u_D \in C([0,T],\mathcal{V})$ with $\dot{u}_D \in \mathcal{W}(0,T)$ be the given Dirichlet data on Γ_D. Furthermore, we assume that $g \in \mathcal{V}$ with $g = u_D(0)$ on Γ_D, $h \in \mathcal{H}$ and $\mathcal{F} \in L^2((0,T),\mathcal{V}^*)$. Then, there exists a unique solution $u \in C([0,T],\mathcal{V})$ with $\dot{u} \in L^2((0,T),\mathcal{V})$ to the problem*

$$\mathcal{M}\ddot{u}(t) + \mathcal{D}\dot{u}(t) + \mathcal{K}u(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}_D^* \tag{4.9}$$

with $u(t) = u_D(t)$ on Γ_D and initial conditions (4.7c),(4.7d)*. Furthermore, the time derivative satisfies $\dot{u} \in \mathcal{W}(0,T)$.*

Proof. Instead of finding $u \in C([0,T],\mathcal{V})$ as stated in the theorem, we consider the equivalent problem: find $w = u - u_D \in C([0,T],\mathcal{V}_D)$ with $\dot{w} \in L^2((0,T),\mathcal{V}_D)$ such that

$$\begin{aligned} \mathcal{M}\ddot{w} + \hat{\mathcal{D}}\dot{w} + \mathcal{K}w &= \mathcal{F} - \mathcal{M}\ddot{u}_D - \mathcal{K}u_D, \\ w(0) &= g - u_D(0) \in \mathcal{V}_D, \\ \dot{w}(0) &= h - \dot{u}_D(0) \in \mathcal{H}. \end{aligned} \tag{4.10}$$

Therein, $\hat{\mathcal{D}}$ denotes the operator defined by $\hat{\mathcal{D}}(\dot{w}) := \mathcal{D}(\dot{w} + \dot{u}_D)$. It is easy to see that $\hat{\mathcal{D}}$ is Lipschitz continuous and strongly monotone with the same constants as $\mathcal{D}$. Thus, we apply Theorem 4.2 to equation (4.10) which states the existence of a unique solution w and hence the unique solvability of the original problem (4.9). In addition, we obtain $\dot{w} \in \mathcal{W}_D(0,T)$ and thus $\dot{u} = \dot{w} + \dot{u}_D \in \mathcal{W}(0,T)$. □

4.4.3. *Lagrange Multipliers.* As mentioned above, we are interested in the formulation with Lagrange multipliers to involve the Dirichlet boundary conditions. Similar to [Sim00], we show that for a solution $u \in L^2((0,T),\mathcal{V})$ of the non-homogeneous operator ODE there exists a unique Lagrange multiplier $\lambda \in L^2((0,T),\mathcal{Q})$ such that the pair (u,λ) is a solution of the operator DAE (4.7).

Theorem 4.4. *Let $g, h, \mathcal{F}$, and u_D be as in Theorem 4.3 and $\mathcal{G} \in L^2((0,T),\mathcal{Q}^*)$ as defined in* (2.5)*. Furthermore, let $u \in C([0,T],\mathcal{V})$, with $\dot{u} \in \mathcal{W}(0,T)$, denote the unique solution from Theorem 4.3. Then, there exists a unique Lagrange multiplier $\lambda \in L^2((0,T),\mathcal{Q})$, such that (u,λ) is a solution of system* (4.7) *for a. e. $t \in [0,T]$.*

Proof. Note that u fulfills the desired Dirichlet boundary condition in the strong form and thus (4.7b). Since $\mathcal{B}^*\lambda$ vanishes on $\mathcal{V}_D$, the given solution u of (4.9) satisfies

$$\mathcal{M}\ddot{u}(t) + \mathcal{D}\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}_D^*.$$

In order to guarantee equation (4.7a) in $\mathcal{V}^*$, λ has to satisfy

$$\mathcal{B}^*\lambda(t) = \mathcal{F}(t) - \mathcal{M}\ddot{u}(t) - \mathcal{D}\dot{u}(t) - \mathcal{K}u(t) \quad \text{in } (\mathcal{V}_D^\perp)^*.$$

According to Lemma 4.1 c), the operator $\mathcal{B}^*$ defines an isomorphism. Thus, the equation has a unique solution $\lambda(t) \in \mathcal{Q}$ and the pair (u,λ) solves system (4.7). □

5. Index Reduction on Operator Level

In Section 3 we have shown that a semi-discretization of the operator DAE (4.7) leads to an index-3 DAE. This section is devoted to the reformulation of the operator DAE such that a semi-discretization leads to a DAE of index 1. Major part is the splitting of the space $\mathcal{V}$ into the trace-free space $\mathcal{V}_D$ and its orthogonal complement. As main theorem we state the well-posedness of the extended operator DAE.

5.1. **Extended First-Order System.** As a first step, we write the operator DAE as a first-order system. For this, we introduce a new variable $v \in H^1((0,T),\mathcal{V})$ and replace the appearance of $\dot{u}$ in the original system by v as well as $\ddot{u}$ by $\dot{v}$. Furthermore, we add the relation $\mathcal{M}\dot{u} = \mathcal{M}v$. The initial conditions then read

$$\begin{aligned} u(0) &= g \in \mathcal{V}, \\ v(0) &= h \in \mathcal{H}. \end{aligned} \tag{5.1}$$

As a second step, we add the first two time derivatives of the constraint to the system of operator equations. With Lemma 4.1 e), the additional equations are given by

$$\mathcal{B}v = \dot{\mathcal{G}} \quad \text{and} \quad \mathcal{B}\dot{v} = \ddot{\mathcal{G}}.$$

For this to make sense, we require $\mathcal{G} \in H^2((0,T),\mathcal{Q}^*)$. A sufficient condition is that the boundary values satisfy $u_D \in H^2((0,T),\mathcal{V})$. The extended first-order operator DAE then reads

$$\mathcal{M}\dot{u}(t) = \mathcal{M}v(t) \quad \text{in } \mathcal{V}^*, \tag{5.2a}$$

$$\mathcal{M}\dot{v}(t) + \mathcal{D}v(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, \tag{5.2b}$$

$$\mathcal{B}u(t) = \mathcal{G}(t) \quad \text{in } \mathcal{Q}^*, \tag{5.2c}$$

$$\mathcal{B}v(t) = \dot{\mathcal{G}}(t) \quad \text{in } \mathcal{Q}^*, \tag{5.2d}$$

$$\mathcal{B}\dot{v}(t) = \ddot{\mathcal{G}}(t) \quad \text{in } \mathcal{Q}^*. \tag{5.2e}$$

Despite we have added two constraints, the extended system (5.2) and the original system (4.7) are equivalent in the following sense. If (u, v, λ) solves (5.2), then (u, λ) solves (4.7). Conversely, if (u, λ) is a solution of (4.7) and $u \in H^2((0,T),\mathcal{V})$, $\mathcal{G} \in H^2((0,T),\mathcal{V})$, then $\mathcal{B}u = \mathcal{G}$ implies $\mathcal{B}\dot{u} = \dot{\mathcal{G}}$, $\mathcal{B}\ddot{u} = \ddot{\mathcal{G}}$ and thus $(u, \dot{u}, \lambda)$ solves the operator DAE (5.2).

5.2. **Index Reduction.** The reformulation of the operator DAE follows the ideas of minimal extension from [KM06]. The main point is the splitting of u into a boundary part and a part which vanishes at the boundary.

Definition 5.1. We define the orthogonal projection $\mathcal{P} : \mathcal{V} \to \mathcal{V}_D$ with respect to the inner product in $\mathcal{V}$. This leads to the unique decomposition of functions $v \in \mathcal{V}$ into

$$v = \mathcal{P}v + (\mathrm{id} - \mathcal{P})v =: v_1 + v_2, \quad v_1 \in \mathcal{V}_D,\ v_2 \in \mathcal{V}_D^\perp .$$

In what follows, $u = u_1 + u_2$ denotes the orthogonal decomposition as in Definition 5.1. Without further mention, we use $u_1 = \mathcal{P}u \in \mathcal{V}_D$ and $u_2 = (\mathrm{id} - \mathcal{P})u \in \mathcal{V}_D^\perp$.

Lemma 4.1 a) implies that $\mathcal{B}u = \mathcal{B}u_2$ and $\mathcal{B}v = \mathcal{B}v_2$. To obtain $\mathcal{B}\dot{v} = \mathcal{B}\dot{v}_2$, we need the following lemma.

Lemma 5.1. *Consider $u \in H^1((0,T),\mathcal{V})$ and the orthogonal decompositions*

$$u = u_1 + u_2, \qquad \dot{u} = v_1 + v_2,$$

with $u_1, v_1 \in L^2((0,T),\mathcal{V}_D)$ and $u_2, v_2 \in L^2((0,T),\mathcal{V}_D^\perp)$. Then, $\dot{u}_1 = v_1$ and $\dot{u}_2 = v_2$.

Proof. The essential observation is that $w \in H^1((0,T),\mathcal{V}) \cap L^2((0,T),\mathcal{V}_D)$ implies $\dot{w} \in L^2((0,T),\mathcal{V}_D)$. Owing to the orthogonality, $w(t) \in \mathcal{V}_D$ satisfies $0 = \big(w(t),\bar{v}\big)_\mathcal{V}$ for all $\bar{v} \in \mathcal{V}_D^\perp$. Thus, with the Riesz representative $\text{Riesz}(\bar{v}) \in \mathcal{V}^*$, we obtain

$$\big(\dot{w}(t),\bar{v}\big)_\mathcal{V} = \langle \text{Riesz}(\bar{v}), \dot{w}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \text{Riesz}(\bar{v}), w(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = \frac{\mathrm{d}}{\mathrm{d}t}\big(w(t),\bar{v}\big)_\mathcal{V} = 0.$$

Hence, $\dot{w}(t) \in \mathcal{V}_D$. Finally, we use the uniqueness of the orthogonal decomposition, $\dot{u}_1 + \dot{u}_2 = \dot{u} = v_1 + v_2$ which implies $\dot{u}_1 = v_1$ and $\dot{u}_2 = v_2$. □

According to Lemma 4.1 b), u_2 is already fixed by the equation $\mathcal{B}u_2 = \mathcal{G}$. Since v_2 is fixed by (5.2d), the equation $\dot{u}_2 = v_2$ is redundant. As a consequence, equation (5.2a) reduces to $\mathcal{M}\dot{u}_1 = \mathcal{M}v_1$. With the same argument, equation (5.2e) for $\dot{v}_2$ is redundant. Since $\dot{v}_2$ also appears in equation (5.2b), we introduce a dummy variable $\tilde{v}_2 \in L^2((0,T),\mathcal{V}_D^\perp)$ by $\tilde{v}_2 := \dot{v}_2$. The resulting system has the form

$$\begin{aligned}
\mathcal{M}\dot{u}_1 \qquad\qquad\qquad\qquad &= \mathcal{M}v_1 && \text{in } \mathcal{V}^*,\\
\mathcal{M}(\dot{v}_1 + \tilde{v}_2) + \mathcal{D}v + \mathcal{K}u + \mathcal{B}^*\lambda &= \mathcal{F} && \text{in } \mathcal{V}^*,\\
\mathcal{B}u_2 &= \mathcal{G} && \text{in } \mathcal{Q}^*,\\
\mathcal{B}v_2 &= \dot{\mathcal{G}} && \text{in } \mathcal{Q}^*,\\
\mathcal{B}\tilde{v}_2 &= \ddot{\mathcal{G}} && \text{in } \mathcal{Q}^*.
\end{aligned}$$

In order to obtain a second order operator DAE, we re-substitute v_1. Then, the reformulated problem reads: find $u_1 \in H^1((0,T),\mathcal{V}_D)$ with $\ddot{u}_1 \in L^2((0,T),\mathcal{V}^*)$ and $u_2, v_2, \tilde{v}_2 \in L^2((0,T),\mathcal{V}_D^\perp)$, $\lambda \in L^2((0,T),\mathcal{Q})$ such that

$$\mathcal{M}(\ddot{u}_1 + \tilde{v}_2) + \mathcal{D}(\dot{u}_1 + v_2) + \mathcal{K}(u_1 + u_2) + \mathcal{B}^*\lambda = \mathcal{F} \quad \text{in } \mathcal{V}^*, \tag{5.4a}$$

$$\mathcal{B}u_2 = \mathcal{G} \quad \text{in } \mathcal{Q}^*, \tag{5.4b}$$

$$\mathcal{B}v_2 = \dot{\mathcal{G}} \quad \text{in } \mathcal{Q}^*, \tag{5.4c}$$

$$\mathcal{B}\tilde{v}_2 = \ddot{\mathcal{G}} \quad \text{in } \mathcal{Q}^*, \tag{5.4d}$$

with initial conditions

$$u_1(0) = g, \qquad \dot{u}_1(0) = h. \tag{5.4e}$$

The justification of calling this procedure an index reduction on operator level is part of the next subsection. Note that we have only reformulated the system but did not change the underlying Gelfand triples $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$ and $\mathcal{Q}^*, [L^2(\Gamma_D)]^2, \mathcal{Q}$.

Theorem 5.2 (Well-posedness)**.** *Consider $\mathcal{F} \in L^2((0,T),\mathcal{V}^*)$ from (2.6), $\mathcal{G}$ as defined in (2.5) with Dirichlet data $u_D \in H^2((0,T),\mathcal{V})$, and initial data $g \in \mathcal{V}_D$, $h \in \mathcal{H}$. Then, problem (5.4) is well-posed in the following sense. First, there exists a unique solution $(u_1, u_2, v_2, \tilde{v}_2, \lambda)$ with $u_1 \in H^1((0,T),\mathcal{V}_D) \cap H^2((0,T),\mathcal{V}^*)$, $u_2, v_2, \tilde{v}_2 \in L^2((0,T),\mathcal{V}_D^\perp)$, and $\lambda \in L^2((0,T),\mathcal{Q})$. Second, the map*

$$\big(g, h, \mathcal{D}(0), \mathcal{F}, \mathcal{G}\big) \mapsto \big(u_1, u_2, v_2, \tilde{v}_2, \ddot{u}_1 + \mathcal{D}\dot{u}_1 + \mathcal{B}^*\lambda\big)$$

is a linear and continuous map of the form

$$\mathcal{V}_D \times \mathcal{H} \times \mathcal{V}^* \times L^2((0,T),\mathcal{V}^*) \times H^2((0,T),\mathcal{Q}^*) \to \\ C([0,T],\mathcal{V}) \cap C^1([0,T],\mathcal{H}) \times \big[L^2((0,T),\mathcal{V}_D^\perp)\big]^3 \times L^2((0,T),\mathcal{V}^*).$$

Proof. Note that $u_D \in H^2((0,T),\mathcal{V})$ and the trace theorem implies $\mathcal{G} \in H^2((0,T),\mathcal{Q}^*)$,

$$\langle q, \ddot{\mathcal{G}}(t)\rangle_{\mathcal{Q},\mathcal{Q}^*} := \langle q, \ddot{u}_D(t)\rangle_{\mathcal{Q},\mathcal{Q}^*}, \quad \|\ddot{\mathcal{G}}\|_{L^2((0,T),\mathcal{Q}^*)} \le c_{\mathrm{tr}}\|\ddot{u}_D\|_{L^2((0,T),\mathcal{V})}.$$

Uniqueness: Assume that $(u_1, u_2, v_2, \tilde{v}_2, \lambda)$ and $(U_1, U_2, V_2, \tilde{V}_2, \Lambda)$ are two solutions of problem (5.4). Equation (5.4b) provides $\mathcal{B}(u_2 - U_2) = 0$ in $\mathcal{Q}^*$. Using the isomorphism from Lemma 4.1 b), we obtain $u_2 = U_2$. With the same arguments, we achieve $v_2 = V_2$ and $\tilde{v}_2 = \tilde{V}_2$. With the differences $w := u_1 - U_1$ and $\mu := \lambda - \Lambda$, equation (5.4a) reads

$$\mathcal{M}\ddot{w} + \hat{\mathcal{D}}\dot{w} + \mathcal{K}w + \mathcal{B}^*\mu = 0 \text{ in } \mathcal{V}^*$$

with the operator $\hat{\mathcal{D}}(\dot{w}) := \mathcal{D}(\dot{w}+\dot{U}_1+v_2) - \mathcal{D}(\dot{U}_1+v_2)$ and initial conditions $w(0) = 0$ and $\dot{w}(0) = 0$. Obviously, $\hat{\mathcal{D}}$ is Lipschitz continuous and strongly monotone with the same constants as $\mathcal{D}$ such that the theorems of the previous section are applicable. Testing only with functions in $\mathcal{V}_D$, by Theorem 4.2 we obtain that $w = 0$. Since $\hat{\mathcal{D}}(0) = 0$, it remains the equation $b(\mu, v) = 0$ for all $v \in \mathcal{V}_D^\perp$ which implies $\mu = 0$.

Existence: Let (u, λ) denote the solution from Theorem 4.4 with initial data $u(0) = g + (\mathrm{id} - \mathcal{P})u_D(0)$ and $\dot{u}(0) = h + \mathcal{B}^{-1}\dot{\mathcal{G}}(0)$. Theorem 4.4 states that $u_1 := \mathcal{P}u$ satisfies $u_1 \in \mathcal{C}([0,T],\mathcal{V}_D)$, $\dot{u}_1 \in \mathcal{W}_D(0,T)$. With the help of Lemma 4.1, we define

$$u_2 := (\mathrm{id} - \mathcal{P})u = \mathcal{B}^{-1}\mathcal{G}, \quad v_2 := \dot{u}_2 = \mathcal{B}^{-1}\dot{\mathcal{G}}, \quad \tilde{v}_2 := \ddot{u}_2 = \mathcal{B}^{-1}\ddot{\mathcal{G}}.$$

The regularity of $\mathcal{G}$, namely $\mathcal{G}, \dot{\mathcal{G}}, \ddot{\mathcal{G}} \in L^2((0,T),\mathcal{Q}^*)$, implies $u_2, v_2, \tilde{v}_2 \in L^2((0,T),\mathcal{V}_D^\perp)$. Obviously, the tuple $(u_1, u_2, v_2, \tilde{v}_2, \lambda)$ satisfies equations (5.4a)-(5.4d). The initial conditions satisfy

$$u_1(0) = \mathcal{P}u(0) = \mathcal{P}g = g, \quad \dot{u}_1(0) = \dot{u}(0) - \dot{u}_2(0) = h + \mathcal{B}^{-1}\dot{\mathcal{G}}(0) - v_2(0) = h.$$

Continuous dependence on data: Lemma 4.1 d) implies the estimate

$$\|u_2(t)\|_{\mathcal{V}} \le \frac{1}{\beta}\|\mathcal{B}u_2(t)\|_{\mathcal{Q}^*} = \frac{1}{\beta}\|\mathcal{G}(t)\|_{\mathcal{Q}^*}\ .$$

Similar estimates for v_2 and $\tilde{v}_2$ result in

$$\|u_2\|^2_{L^2((0,T),\mathcal{V})} + \|v_2\|^2_{L^2((0,T),\mathcal{V})} + \|\tilde{v}_2\|^2_{L^2((0,T),\mathcal{V})} \le \frac{1}{\beta^2}\|\mathcal{G}\|^2_{H^2((0,T),\mathcal{Q}^*)}. \tag{5.5}$$

For an estimate of u_1, we test equation (5.4a) with $\dot{u}_1(t) \in \mathcal{V}_D$. We neglect to write the time-dependence in each step and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho\|\dot{u}_1\|^2_{\mathcal{H}} + a(u_1,u_1)\Big] + \big\langle \mathcal{D}(\dot{u}_1+v_2) - \mathcal{D}v_2, \dot{u}_1\big\rangle$$
$$= \langle \mathcal{F}, \dot{u}_1\rangle - (\rho\tilde{v}_2, \dot{u}_1)_{\mathcal{H}} - \big\langle \mathcal{D}v_2, \dot{u}_1\big\rangle - a(u_2, \dot{u}_1)$$
$$= \langle \mathcal{F}, \dot{u}_1\rangle - (\rho\tilde{v}_2, \dot{u}_1)_{\mathcal{H}} - \big\langle \mathcal{D}v_2 - \mathcal{D}(0), \dot{u}_1\big\rangle - \big\langle \mathcal{D}(0), \dot{u}_1\big\rangle - a(u_2, \dot{u}_1).$$

Recall that d_2 and a_2 denote the continuity constants of $\mathcal{D}$ and a, respectively, and d_0, d_1 the monotonicity constants of $\mathcal{D}$. With $\eta(t) := \rho\|\dot{u}_1(t)\|^2_{\mathcal{H}} + a(u_1(t), u_1(t))$, the strong monotonicity of $\mathcal{D}$, and the Cauchy-Schwarz inequality, we obtain the estimate

$$\begin{aligned}\dot{\eta} + 2d_1\|\dot{u}_1\|^2_{\mathcal{V}} - 2d_0\|\dot{u}_1\|^2_{\mathcal{H}} &\le \rho\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{u}_1\|^2_{\mathcal{H}} + \frac{\mathrm{d}}{\mathrm{d}t}a(u_1,u_1) + 2\big\langle \mathcal{D}(\dot{u}_1+v_2) - \mathcal{D}v_2, \dot{u}_1\big\rangle \\ &\le 2\|\mathcal{F}\|_{\mathcal{V}^*}\|\dot{u}_1\|_{\mathcal{V}} + 2\rho\|\tilde{v}_2\|_{\mathcal{H}}\|\dot{u}_1\|_{\mathcal{H}} + 2d_2\|v_2\|_{\mathcal{V}}\|\dot{u}_1\|_{\mathcal{V}} \\ &\qquad + 2\|\mathcal{D}(0)\|_{\mathcal{V}^*}\|\dot{u}_1\|_{\mathcal{V}} + 2a_2\|u_2\|_{\mathcal{V}}\|\dot{u}_1\|_{\mathcal{V}}\ .\end{aligned}$$

By Young's inequality $2ab \le a^2/c + cb^2$ [Eva98, appendix B] with appropriate choices of $c > 0$, we yield

$$\begin{aligned}\dot{\eta} + 2d_1\|\dot{u}_1\|^2_{\mathcal{V}} &\le \frac{2}{d_1}\|\mathcal{F}\|^2_{\mathcal{V}^*} + \frac{d_1}{2}\|\dot{u}_1\|^2_{\mathcal{V}} + \rho\|\tilde{v}_2\|^2_{\mathcal{H}} + (\rho+2d_0)\|\dot{u}_1\|^2_{\mathcal{H}} + \frac{2d_2^2}{d_1}\|v_2\|^2_{\mathcal{V}} \\ &\quad + \frac{d_1}{2}\|\dot{u}_1\|^2_{\mathcal{V}} + \frac{2}{d_1}\|\mathcal{D}(0)\|^2_{\mathcal{V}^*} + \frac{d_1}{2}\|\dot{u}_1\|^2_{\mathcal{V}} + \frac{2a_2^2}{d_1}\|u_2\|^2_{\mathcal{V}} + \frac{d_1}{2}\|\dot{u}_1\|^2_{\mathcal{V}} \\ &\le \frac{\rho+2d_0}{\rho}\eta + \frac{2}{d_1}\|\mathcal{F}\|^2_{\mathcal{V}^*} + \rho\|\tilde{v}_2\|^2_{\mathcal{H}} + \frac{2d_2^2}{d_1}\|v_2\|^2_{\mathcal{V}} + \frac{2}{d_1}\|\mathcal{D}(0)\|^2_{\mathcal{V}^*} \\ &\quad + \frac{2a_2^2}{d_1}\|u_2\|^2_{\mathcal{V}} + 2d_1\|\dot{u}_1\|^2_{\mathcal{V}}\ .\end{aligned}$$

Thus, there exists a generic constant c, such that

$$\dot{\eta}(t) \le (1 + 2d_0/\rho)\eta(t) + c\xi(t)$$

with

$$\xi(t) = \|\mathcal{F}(t)\|^2_{\mathcal{V}^*} + \|\tilde{v}_2(t)\|^2_{\mathcal{V}} + \|v_2(t)\|^2_{\mathcal{V}} + \|\mathcal{D}(0)\|^2_{\mathcal{V}^*} + \|u_2(t)\|^2_{\mathcal{V}}\ .$$

Thus, by the absolute continuity of η and Gronwall's lemma [Eva98, appendix B] we obtain that η is bounded by

$$\eta(t) \le (1 + 2d_0/\rho)e^t\Big(\eta(0) + c\int_0^t \xi(s)\ ds\Big).$$

The initial value of η is given by the initial values in (5.4e),

$$\eta(0) = \rho\|h\|^2_{\mathcal{H}} + a(g,g) \le \rho\|h\|^2_{\mathcal{H}} + a_2\|g\|^2_{\mathcal{V}}\ .$$

The integral term can be bounded with the help of (5.5). Therewith, the existence of a positive constant c follows such that for all $t \in [0,T]$,

$$\eta(t) \le c\big(\|g\|^2_{\mathcal{V}} + \|h\|^2_{\mathcal{H}} + \|\mathcal{D}(0)\|^2_{\mathcal{V}^*} + \|\mathcal{F}\|^2_{L^2((0,T),\mathcal{V}^*)} + \|\mathcal{G}\|^2_{H^2((0,T),\mathcal{Q}^*)}\big).$$

Since the right-hand side is independent of t, we can maximize over t and obtain bounds for u_1 and $\dot{u}_1$ in $C([0,T],\mathcal{V})$ and $C([0,T],\mathcal{H})$, respectively,

$$\|u_1\|_{C([0,T],\mathcal{V})}+\|\dot{u}_1\|_{C([0,T],\mathcal{H})} \\ \leq c\big(\|g\|_{\mathcal{V}} + \|h\|_{\mathcal{H}} + \|\mathcal{D}(0)\|_{\mathcal{V}^*} + \|\mathcal{F}\|_{L^2((0,T),\mathcal{V}^*)} + \|\mathcal{G}\|_{H^2((0,T),\mathcal{Q}^*)}\big).$$

It remains to bound $\rho\ddot{u}_1 + \mathcal{D}\dot{u}_1 + \mathcal{B}^*\lambda$ in $L^2((0,T),\mathcal{V}^*)$. By the definition of the $\mathcal{V}^*$-norm and equation (5.4a), we achieve

$$\begin{aligned}\|\rho\ddot{u}_1(t) + \mathcal{D}\dot{u}_1(t) + \mathcal{B}^*\lambda(t)\|_{\mathcal{V}^*} \\ &= \sup_{v\in\mathcal{V}} \frac{\langle\mathcal{M}\ddot{u}_1(t), v\rangle + \langle\mathcal{D}\dot{u}_1(t), v\rangle + \langle\mathcal{B}^*\lambda(t), v\rangle}{\|v\|_{\mathcal{V}}} \\ &= \sup_{v\in\mathcal{V}} \frac{\langle\mathcal{F}, v\rangle - \langle\mathcal{M}\tilde{v}_2, v\rangle - \langle\mathcal{K}(u_1+u_2), v\rangle + \langle\mathcal{D}\dot{u}_1 - \mathcal{D}(\dot{u}_1+v_2), v\rangle}{\|v\|_{\mathcal{V}}} \\ &\leq \|\mathcal{F}(t)\|_{\mathcal{V}^*} + \rho\|\tilde{v}_2(t)\|_{\mathcal{H}} + a_2\|u_1(t)+u_2(t)\|_{\mathcal{V}} + d_2\|v_2(t)\|_{\mathcal{V}}.\end{aligned}$$

Thus, by integration over $[0,T]$, Young's inequality, and the estimates for $u_1, u_2, v_2, \tilde{v}_2$ from above, we obtain a positive constant c with

$$\|\ddot{u}_1(t) + \mathcal{D}\dot{u}_1(t) + \mathcal{B}^*\lambda(t)\|_{L^2((0,T),\mathcal{V}^*)} \\ \leq c\big(\|g\|_{\mathcal{V}} + \|h\|_{\mathcal{H}} + \|\mathcal{D}(0)\|_{\mathcal{V}^*} + \|\mathcal{F}\|_{L^2((0,T),\mathcal{V}^*)} + \|\mathcal{G}\|_{H^2((0,T),\mathcal{Q}^*)}\big). \quad \square$$

5.3. **Nonconforming Semi-Discretization.** For a semi-discretization of system (5.4) we need to approximate the spaces $\mathcal{V}_D, \mathcal{V}_D^\perp$ and $\mathcal{Q}$ by finite dimensional spaces. In Section 3.1 we have introduced the spaces $\mathcal{S}_h$ and $\mathcal{Q}_h$ of piecewise linear and globally continuous functions as finite dimensional approximations of $\mathcal{V}$ and $\mathcal{Q}$, respectively. Since we have decomposed the space $\mathcal{V}$ into $\mathcal{V}_D$ and $\mathcal{V}_D^\perp$, we introduce the discrete spaces

$$\mathcal{S}_{h,D} := \mathcal{S}_h \cap \mathcal{V}_D = \operatorname{span}\{\varphi_1, \ldots, \varphi_{n-m}\} \quad \text{and} \quad \mathcal{S}_h^\perp := \operatorname{span}\{\varphi_{n-m+1}, \ldots, \varphi_n\}.$$

Note that $\mathcal{S}_h^\perp$ includes all basis functions which correspond to nodes at the Dirichlet boundary Γ_D but is not a subspace of $\mathcal{V}_D^\perp$. Thus, we commit a so-called *variational crime* and use a nonconforming finite element method for the discretization of $\mathcal{V}_D^\perp$ [Bra07, chapter III].

The discrete variational formulation of system (5.4) reads: determine $u_{1,h}(t) \in \mathcal{S}_{h,D}$, $u_{2,h}(t), v_{2,h}(t), \tilde{v}_{2,h}(t) \in \mathcal{S}_h^\perp$, and $\lambda_h(t) \in \mathcal{Q}_h$ such that for all $v_h \in \mathcal{S}_h$ and $\vartheta_h \in \mathcal{Q}_h$ the following equations hold,

$$\begin{aligned}(\rho(\ddot{u}_{1,h} + \tilde{v}_{2,h}), v_h)_{\mathcal{H}} + d(\dot{u}_{1,h} + v_{2,h}, v_h) + a(u_{1,h} + u_{2,h}, v_h) + b(v_h, \lambda_h) &= \langle\mathcal{F}, v_h\rangle, \\ b(u_{2,h}, \vartheta_h) &= \langle\mathcal{G}, \vartheta_h\rangle, \\ b(v_{2,h}, \vartheta_h) &= \langle\dot{\mathcal{G}}, \vartheta_h\rangle, \\ b(\tilde{v}_{2,h}, \vartheta_h) &= \langle\ddot{\mathcal{G}}, \vartheta_h\rangle.\end{aligned}$$

We represent the discrete solution by their coefficient vectors, i. e., $u_{1,h}(t)$ by $q_1(t) \in \mathbb{R}^{n-m}$, $u_{2,h}(t), v_{2,h}(t), \tilde{v}_{2,h}(t)$ by $q_2(t), p_2(t), \tilde{p}_2(t) \in \mathbb{R}^m$, and $\lambda_h(t)$ by $\mu(t) \in \mathbb{R}^m$.

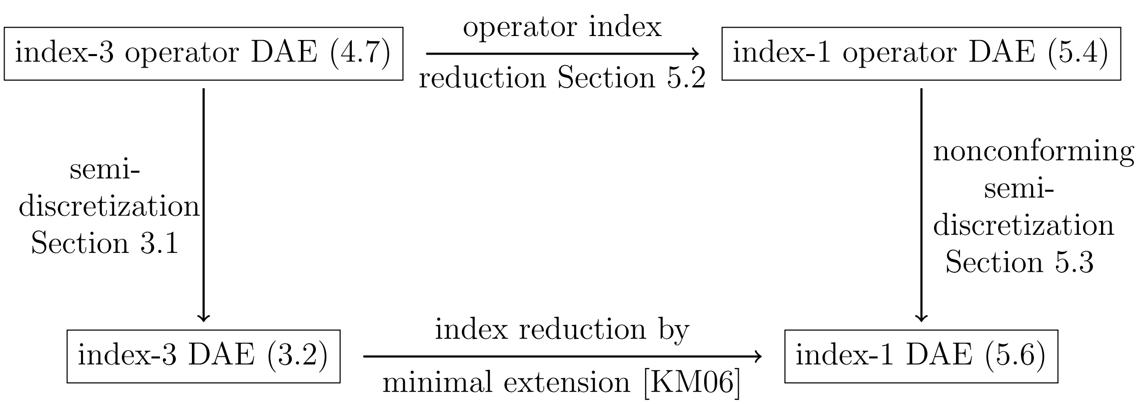
Then, with M, D, K, and B_2 from Section 3.1, the discrete variational formulation is equivalent to the DAE

$$
\begin{aligned}
M \begin{bmatrix} \ddot{q}_1 \\ \tilde{p}_2 \end{bmatrix} + D \left(\begin{bmatrix} \dot{q}_1 \\ p_2 \end{bmatrix} \right) + K \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \mu &= f, \\
B_2 q_2 &= g, \\
B_2 p_2 &= \dot{g}, \\
B_2 \tilde{p}_2 &= \ddot{g}.
\end{aligned} \tag{5.6}
$$

Note that this DAE equals the system in (3.3) and hence, is of index 1. Thus, the in Section 5.2 presented procedure is an index reduction on operator level.

6. Conclusion

We have reformulated the equations of elastodynamics with weakly enforced Dirichlet boundary constraint such that a semi-discretization in space by finite elements leads to a DAE of index 1. The fact that the DAEs (3.3) and (5.6) are equal shows that the order of semi-discretization and index reduction can be reversed. A summary of the paper is given in the following commutative diagram.



The commutativity of semi-discretization and index reduction opens up a new potential for adaptivity in space. The index-1 formulation of the operator DAE (5.4) allows to modify the finite element mesh within the simulation, without producing the necessity of another index reduction step afterwards. This provides a suitable formulation, especially for the simulation of multi-physics systems in which the elastic part would only be a single module in a large system.

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