# India and the Calculus of Trigonometric Functions 

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#### Abstract

We discuss the work of a brilliant line of Mathematicians who lived in central Kerala and starting with its founder Madhava ( 1350 CE) developed what can best be described as Calculus and applied it to a class of trigonometric functions. We explain, with the example of the expansion of the inverse tan function, how they handled integration. Further, they took forward the work of the fifth century mathematician Aryabahata ( 499 CE ), worked with differentials, and developed the expansion of the sine and the cosine functions. The work Yuktibhasa (circa 1500 CE ) which maybe described as the first textbook on Calculus, also describes in detail the evaluation of the area and volume of trigonometric functions as well as a variety of expansions for $p i$. Our treatment is pedagogical and we present exercise problems and invite the enterprising student to try their hands at approximations and integration $a$ la the Madhava way.


## I Introduction

Some six hundred years ago a cluster of temple villages, on the banks of the Nila (now called Bharathapuzha) river in central Kerala, was host to a brilliant line of mathematicians. Pre-eminent among them was the founder Madhava (1350 CE) who pioneered what came to be called Calculus. Little of what the prescient Madhava wrote has survived. A lineage of disciples not only kept this flame of calculus alive but developed and wrote about it. It is this writing which is available to us. We mention a few. Parameshvara, who was a direct disciple of Madhava, wrote profusely and spent 55 years examining the night sky and documenting eclipses. He along with his two sons Ravi and Damodara was a teacher to Nilakantha. Among the many books Nilakantha authored the Tantrasamgraha and the Bhasya (commentary) on a seminal text Aryabhatiya ( 499 CE) are notable. His student Jyeshthadeva is the one we owe a big debt to. He wrote the Yuktibhasa which Divakaran (see References) has designated as the "first text book on Calculus". The lineage continued till the 1800s and we refer the reader to Fig. 1 and its caption.

Madhava and his students developed for example expansions of trigonometric functions and their inverses. These expansions were developed a century or more later by European mathematicians using the Calculus of Newton and Leibniz. This fact was noted and reported by Charles M. Whish 11 . The Indian written tradition is largely word based. Results are mentioned and the derivations are omitted. The Aryabhatiya ( 499 CE ) with a little over a 100 cryptic, super-compressed verses of dense mathematics is a prime example. In the case of the Nila mathematicians however we are more

[^0]fortunate. We can, thanks particularly to the text Yuktibhasa, see the detailed reasoning although they are still word based. In what follows we shall provide a flavour of the methods used by these Indian mathematicians and some of their results. The exercises in the end will help you get a more hands on understanding.

## II Samskaram: Recursive Refining

As a methodology, recursion has been used extensively by Indian mathematicians. It would be best to explain the term Samskaram or recursive refining with a simple example. There is another name for it - Shudhikarna or Shudikarti. Consider the evaluation of $1 /(x-d)$ given the fact that we know $1 / x$. We write

$$
\begin{aligned}
\frac{1}{x-d} & =\frac{1}{x}-\left[\frac{1}{x}-\frac{1}{x-d}\right] \\
& =\frac{1}{x}+\frac{d}{x}\left(\frac{1}{x-d}\right)
\end{aligned}
$$

On the r.h.s. of the second step we have the "unknown term" $1 /(x-d)$. We replace it with the r.h.s. of step one,

$$
\begin{aligned}
\frac{1}{x-d} & =\frac{1}{x}+\frac{d}{x}\left[\frac{1}{x}-\left(\frac{1}{x}-\frac{1}{x-d}\right)\right] \\
& =\frac{1}{x}+\frac{d}{x}\left[\frac{1}{x}+\frac{d}{x}\left(\frac{1}{x-d}\right)\right]
\end{aligned}
$$

and continuing recursively one more step

$$
\begin{aligned}
\frac{1}{x-d} & =\frac{1}{x}+\frac{d}{x}\left[\frac{1}{x}+\frac{d}{x}\left(\frac{1}{x}-\left(\frac{1}{x}-\frac{1}{x-d}\right)\right)\right] \\
& =\frac{1}{x}+\frac{d}{x}\left[\frac{1}{x}+\frac{d}{x}\left(\frac{1}{x}+\frac{d}{x}\left(\frac{1}{x-d}\right)\right)\right]
\end{aligned}
$$

At this point we could drop the "d" on the extreme r.h.s to get

$$
\frac{1}{x-d} \approx \frac{1}{x}+\frac{d}{x^{2}}+\frac{d^{2}}{x^{3}}+\frac{d^{3}}{x^{4}}
$$

or "refine" our evaluation further namely

$$
\begin{equation*}
\frac{1}{x-d}=\frac{1}{x}+\frac{1}{x} \sum_{n=1}^{M}\left(\frac{d}{x}\right)^{n} \tag{1}
\end{equation*}
$$

This method of iterative refining or recursive refining is called samskaram. One of the earliest usages of this method was for obtaining the square root of a number and is in the Bakshali manuscript found near Peshawar and dated most probably 300-500 CE. The exercise at the end will give you a better feeling for this. We pause to note the following:

## Madhava School



Figure 1: Members of the Madhava school. Nilakantha's year of birth 1444 is firmly established. The other dates are tentative with some uncertainty ( $\pm 5$ years). Except for Ravi and Damodara who were sons of Parameshvara the other members are not direct descendants. The pioneering scholar of Indian mathematics Sarasvati Amma has designated this lineage as the Aryabhata School.

1. The method is not the same as the familiar Taylor expansion. In fact when used for the cosine series it yields

$$
\cos (\theta+\delta) \approx \cos (\theta)-\delta \sin (\theta)-\delta^{2} / 2 \cos (\theta)+\delta^{3} / 8 \sin (\theta)+. .
$$

the last term should have $\delta^{3} / 6$ and is erroneously given by samskaram. The Madhava schoo $2^{2}$ seemed to be aware of this and go on to derive the correct expansion (see Sec.IV).
2. For $x$ and $d$ positive and $d<x$ the series is convergent. The specific example cited by Nilakantha is $x=4$ and $d=1$.
3. The example above demonstrates a comfort level with infinite series. The l.h.s. is an unknown finite number and the r.h.s. is an infinite series which equals this number. This may not seem like an issue except when one views it from the perspective of Madhava. One is confronting for the first time an unknown irrational number $\pi$ and one claims to represent it with an infinite series - something that Madhava did. We shall see more of this later (Sec. III).

## III Samkalitam - Integration

## III. 1 Introduction

The term Samkalitam means sum. In our case it is a special sum, one whose limit is an integral. The term will become clearer as we proceed in this section.

A crowning achievement of Madhava is the series representation of the angle $\theta$ in terms of $t=\tan (\theta)$ for $\theta \leq \pi / 4$.

$$
\begin{equation*}
\theta=t-\frac{t^{3}}{3}+\frac{t^{5}}{5}-\ldots \quad(t=\tan (\theta)) \tag{2}
\end{equation*}
$$

We recognize this as the "Gregory-Leibniz" series and is one of the results which surprised Charles Whish since it preceded European calculus by more than two centuries (see footnote Sec. I). We stress that it was meticulously derived and not just, to use a cricketing terminology, a "one-off lucky strike". We shall derive this and thus get a taste of how the Nila lineage handled integration. Our demonstration proceeds in two stages. The first is the geometric part where Madhava, by intricate reasoning, established the expression (Sec. III.2)

$$
\begin{equation*}
\delta \theta=\frac{\delta \tan (\theta)}{1+\tan ^{2}(\theta)} \tag{3}
\end{equation*}
$$

The next two sections deal with the Samkalitam (integration) of the above expression (Sec III. 2 and III.3). The entire procedure is a first in the history of Calculus.

[^1]

Figure 2: A quadrant of the unit circle $O X Y$ is circumscribed by a unit square $O X W Z$. The points $A_{n-1}$ and $A_{n}$ are very close to each other but the distance between them is magnified for ease of view. The arc PQ subtends an angle $\delta \theta_{n}=\theta_{n}-\theta_{n-1}$ at $O$. $P R$ and $A_{n-1} B$ are perpendiculars on $O A_{n}$ and these will be needed only for one of the Exercises.

## III. 2 The Geometric Part

Madhava obtained the relation between the angle $\theta$ and the tangent $t(=\tan (\theta))$. Figure 2 depicts the quadrant of a unit circle. The circumscribing unit square is $O X W Y$. We denote the angle $X O A_{n-1}$ as $\theta_{n-1}$ and the angle $X O A_{n}$ as $\theta_{n}$. The line $X W$ is divided into a large number $N$ of equal segments with $X=A_{0}, A_{1}, A_{2}, . . A_{n-1}, A_{n}, \ldots A_{N}=$ $W$. The linear segment $A_{n-1} A_{n}=1 / N$ is small and equal to the increment $\delta t$ in the tangent. The corresponding increment in the arc of the unit circle is $P Q$ and equal to $\theta_{n}-\theta_{n-1}$. To repeat

$$
\begin{align*}
\delta t & =A_{n-1} A_{n}=1 / N  \tag{4}\\
\delta \theta_{n} & =\theta_{n}-\theta_{n-1} \tag{5}
\end{align*}
$$

Also from the right angle triangle $O X A_{n}$,

$$
\begin{align*}
O A_{n}^{2} & =O X^{2}+X A_{n}^{2} \\
& =1+\left(\frac{n}{N}\right)^{2} \tag{6}
\end{align*}
$$

Through an elaborate series of arguments based on similar triangles and the smallness of $\delta \theta_{n}$ Madhava shows that

$$
\begin{align*}
\sin \left(\delta \theta_{n}\right) & =\frac{\delta t}{O A_{n}^{2}} \\
& =\frac{1}{N\left(1+(n / N)^{2}\right)} \tag{7}
\end{align*}
$$

The Yuktibhasa asks us to think of $N$ as very large; it uses the word pararddham or $10^{17}$ and mentions that this is notional and to conceive of even larger numbers! The angle $\delta \theta_{n}$ and the $\delta t$ are, in its own words "shunyaprayam" meaning of the nature of zero (and not "shunyam" or zero). In other words this is the "infinitesimal" of Calculus. To drive home the point the text also refers to it as "anuprayam" or atomic. Thus

$$
\begin{equation*}
\delta \theta_{n}=\frac{1}{N\left(1+(n / N)^{2}\right)} \tag{8}
\end{equation*}
$$

Which is easily recognizable as Eq.(3). The next step is to integrate the expression Eq.(8). We do it in two stages.

## III. 3 Samkalitam of the first few terms

We can easily sum the l.h.s. of Eq.(8)

$$
\begin{align*}
\sum_{n=1}^{N} \delta \theta_{n} & =\left(\theta_{1}-\theta_{0}\right)+\left(\theta_{2}-\theta_{1}\right)+\ldots\left(\theta_{N}-\theta_{N-1}\right) \\
& =\pi / 4 \tag{9}
\end{align*}
$$

We employ Eq. (1) with $x=1$ and $d=-(n / N)^{2}$ to expand the r.h.s. of Eq.(8)) to obtain an infinite series

$$
\begin{align*}
\pi / 4 & \approx \frac{1}{N} \sum_{n=1}^{N}\left[1-\frac{n^{2}}{N^{2}}+\frac{n^{4}}{N^{4}}-. .-\right]  \tag{10}\\
& \approx I_{N}(0)-I_{N}(2)+I_{N}(4)-I_{N}(6)+\ldots \tag{11}
\end{align*}
$$

with

$$
\begin{equation*}
I_{N}(k) \approx \frac{1}{N} \sum_{n=1}^{N}\left[\frac{n^{k}}{N^{k}}\right] \tag{12}
\end{equation*}
$$

The first three terms as displayed above are easily summed.

$$
\begin{array}{r}
I_{N}(0)=\frac{1}{N} \sum_{n=1}^{N}[1]=1 \\
I_{N}(2)=\frac{1}{N} \sum_{n=1}^{N}\left[\frac{n^{2}}{N^{2}}\right]=\frac{N(N+1)(2 N+1)}{6 N^{3}} \\
I_{N}(4)=\frac{1}{N} \sum_{n=1}^{N}\left[\frac{n^{4}}{N^{4}}\right]=\frac{N(N+1)(2 N+1)\left(3 N^{2}+3 N-1\right)}{30 N^{5}}
\end{array}
$$

We next take the limit $N \rightarrow \infty$ and denote the limiting quantities by $J$. On inspection

$$
\begin{array}{r}
J_{0}=\lim _{N \rightarrow \infty} I_{N}(0)=1 \\
J_{2}=\lim _{N \rightarrow \infty} I_{N}(2)=1 / 3 \\
J_{4}=\lim _{N \rightarrow \infty} I_{N}(4)=1 / 5
\end{array}
$$

There are analytic expressions for $I_{N}(6)$ and $I_{N}(8)$ but higher orders would require a knowledge and manipulation of the Bernoulli numbers. Did Madhava and his disciples handle the higher orders, e.g. $J_{20}$ for example? The answer is they did so by induction and resorting to the asymptotic limit of large $N$. Their method is described in the Appendix and represents yet another example of their mathematical acumen. Explicitly they obtain

$$
\begin{equation*}
J_{k}=\frac{1}{k+1} \tag{13}
\end{equation*}
$$

Summarising, we have

$$
\begin{align*}
\frac{\pi}{4} & =1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \\
& =\sum_{k=0,2,4, \ldots}^{\infty}(-1)^{k / 2} \frac{1}{k+1} \tag{14}
\end{align*}
$$

We close this section with a few pertinent remarks.

1. We can connect the above with the Calculus we have been taught. For example in Eq. (12) above take

$$
1 / N \rightarrow d t ; \quad \sum_{1}^{N} \rightarrow \int_{0}^{1} ; \quad(n / N)^{k} \rightarrow t^{k}
$$

so

$$
\begin{equation*}
\frac{1}{N} \sum_{n=1}^{N}\left[\frac{n^{k}}{N^{k}}\right] \rightarrow \int_{0}^{1} t^{k} d t \tag{15}
\end{equation*}
$$

2. A key difference with the Calculus we are used to is that in the Madhava scheme we do the summation first and then take the limit $N$ going to infinity. In the former we take the limit first to get the differential $d t$ and then perform the integration. Since the derivatives of the powers of $t^{k}$ are known the fundamental theorem of Calculus $\sqrt[3]{ }$ is invoked to mechanically write down the result $1 /(k+1)$. In our case the integral is performed by first principles. There are some advantages to the Madhava scheme as we shall see. One of them, namely the interchange of summations, is mentioned above. The fundamental theorem follows trivially in our summation procedure. An example is Eq.(9) where the summation is replaced by the end points $(\pi / 4-0)$. We will point this out with another example in the next section.
3. We can take $\delta t=\tan (\theta) / N$ instead of $\delta t=1 / N$ resulting in the general series

$$
\begin{equation*}
\theta=1-\frac{\tan ^{3}(\theta)}{3}+\frac{\tan ^{5}(\theta)}{5}-\frac{\tan ^{7}(\theta)}{7}+\ldots \tag{16}
\end{equation*}
$$

We can legitimately use the term "function" here. It describes the dependence of a real quantity $\theta$ on another real variable $\tan \theta$. In other words we have a functional expression for every value of the variable.

[^2]4. Prior to Madhava we had an approximate value for $\pi$, namely $22 / 7$ or as given by Aryabhata, namely $62832 / 20000=3.1416$. Aryabhata is clear that this value is assana i.e. proximate ${ }^{4}$. meaning that it is close to but not quite $\pi$. The point to appreciate is that instead of another proximate value, Madhava has an exact infinite series for $\pi$. This suggests the irrationality of $\pi$ but Madhava does not clearly say so. The series in Eq.(14) is a slowly converging one and a number of methods were proposed to develop rapidly convergent series The world record up to the 1800s was held by Shankara Varman (see Fig. 1) of the Madhava school with $\pi$ up to 18 decimal places.

## IV Differentials; Sine and Cosine Expansions

In this Section we initially dwell on the work of Aryabhata and then see how it led the Madhava school to the develop the notion of the differential and the expansions of the Sine and Cosine series.

## IV. 1 The Aryabhata Connection

One can discern a continuity in Indian mathematics, howsoever tenuous, from preVedic times ( $<1000$ BCE) up and until 1800s. A striking example is the influence of the Aryabhatiya ( 499 CE ) on the Madhava school (1350 CE).

The Aryabhatiya has some 121 verses out of which 33 verses belong to the mathematics section (Ganitapada). Aryabhata defines, for the first time in the history of mathematics, the sine function. It is the half-chord $A P$ of the unit circle in Fig. 3.

$$
\begin{aligned}
\sin (\theta) & =\frac{A P}{O A} \\
& =A P \quad(O A=1)
\end{aligned}
$$

The circle maybe large or small; correspondingly $A P$ and $O A$ maybe large of small, but the l.h.s. is a function of $\theta$ and is invariant. With this, Aryabhata endowed circle geometry with metrical properties. This alone may qualify him as the founder of trigonometry. But he did more.

He knew that the difference in the sines of two angles $\phi+\delta \phi$ and $\phi-\delta \phi$ is proportional to the cosine of the mean angle $\phi$,

$$
\begin{equation*}
\sin (\phi+\delta \phi)-\sin (\phi-\delta \phi)=2 \sin (\delta \phi) \cos (\phi) \tag{17}
\end{equation*}
$$

and the difference in the cones of two angles $\phi+\delta \phi$ and $\phi-\delta \phi$ is proportional to the sine of the mean angle $\phi$.

$$
\begin{equation*}
\cos (\phi+\delta \phi)-\cos (\phi-\delta \phi)=-2 \sin (\delta \phi) \sin (\phi) \tag{18}
\end{equation*}
$$

Further he states the second sine difference. According to his commentators this is done by (once again) ingenuous arguments based on similar triangles. We shall not go over it here since our concern is different here. But Aryabhata did more.

[^3]

Figure 3: The quadrant of the unit circle. The half chord $A P$ is $\sin (\theta)$ as defined by Aryabhata. See text for comments.

He then obtained the values of the sines at fixed angles between 0 and $\pi / 2$ thus generating the sine table for $\pi / 48=3.75$ degrees, 7.5 degrees ... 90 degrees. This table has been used by Indian astronomers (and astrologers) in some form or another since 499 CE up to the present. We shall see this in the next section. To give you an idea every time you use the calculator, or look up a table, to search for the values of the sines and trigonometric functions, you may silently thank Aryabhata for showing the way. But Aryabhata did more.

## IV. 2 Finite Difference Calculus

Let us take $\delta \phi=\epsilon$ where $\epsilon$ is small but not infinitesimal (not "shunyaprayam"). We take $\phi=n \epsilon$ where $n$ is a positive integer from 1 to $N$. To fix our ideas $\epsilon=\pi / 48=$ $3.75^{\circ}=3438^{\prime}$. Employing the sine and cosine difference formulae from the previous section we define differences

$$
\begin{align*}
\delta s_{n} & =s_{n+1}-s_{n}=2 s_{1 / 2} c_{n+1 / 2}  \tag{19}\\
\delta c_{n} & =c_{n}-c_{n-1}=-2 s_{1 / 2} s_{n+1 / 2} \tag{20}
\end{align*}
$$

where the symbol $s_{n}$ stands for $\sin (n \epsilon), c_{n}$ stands for $\cos (n \epsilon)$ and $s_{1 / 2}$ for $\sin \epsilon / 2$. Note that these are essentially the same as Eqs. (17) and (18), e.g. $\sin (\phi+\delta \phi)-$ $\sin (\phi)=2 \sin (\delta \phi / 2) \cos (\phi+\delta \phi / 2))$. The above is a pair of coupled equations and it was Aryabhata's insight to take the second difference, namely

$$
\begin{align*}
\delta^{2} s_{n} & =\delta s_{n}-\delta s_{n-1}=2 s_{1 / 2}\left(c_{n+1 / 2}-c_{n-1 / 2}\right) \\
& =-4 s_{1 / 2}^{2} s_{n} \quad \text { using Eq. (20) } \tag{21}
\end{align*}
$$

Thus the second difference of the sines is proportional to the sine itself. Similarly the second difference in the cosine is also proportional to the cosine.

$$
\begin{align*}
\delta^{2} c_{n} & =\delta c_{n}-\delta c_{n-1}=2 s_{1 / 2}\left(s_{n+1 / 2}-s_{n-1 / 2}\right) \\
& =-4 s_{1 / 2}^{2} c_{n} \quad \text { using Eq. (19) } \tag{22}
\end{align*}
$$

The next step is to represent the r.h.s in terms of a recursion. We observe $s_{n}$ on the r.h.s. of Eq.(21) may be written as $s_{n}=s_{n}-s_{n-1}+s_{n-1}-s_{n-2}+s_{n-2}-\ldots=$

$$
\begin{equation*}
\delta s_{n}-\delta s_{n-1}=-4 s_{1 / 2}^{2} \sum_{m=1}^{n-1} \delta s_{m} \tag{23}
\end{equation*}
$$

A few remarks are in order here.

1. The above work is that of Aryabhata and he takes $\epsilon=\pi / 48$. He also approximates $s_{1}=\sin (\epsilon) \approx \epsilon$. Using it he derived the sines of angles from $3.75{ }^{0}$ to $90^{0}$ in 24 equi-spaced steps. We will suggest a simple problem along these lines in the Exercise.
2. Of more relevance is the fact that the above is the algorithm we would currently use to obtain derivatives numerically. We know that sine $\left(37^{0}\right)$ is close to 0.6 and $\operatorname{sine}\left(30^{\circ}\right)$ is 0.5 . Thus the difference in angle is $7^{0}$ which in radians is 0.122 . Thus derivative of sine of the median angle $33.5^{0}$ is from Eq. (19) is

$$
\delta \sin (\phi) / \delta \phi=(0.6-0.5) / 0.122=.82
$$

Looking up the sine table or the calculator yields $\cos (33.5)=0.83$. Similarly Eq. (21) yields the second derivative namely

$$
\delta^{2} \sin (\phi) / \delta^{2} \phi \approx-\sin (\phi)
$$

The above are now called central difference approximations to the derivative and the second derivative. Aryabhata does not mention the term finite difference calculus (let alone calculus). But similar methods are now used to numerically solve our differential equations. That includes Newton's II Law and the famous Schrodinger equation of quantum mechanics both of which are second order differential equations.

## IV. 3 The Trigonometric Series

Aryabhata took $\delta \phi=\pi / 48\left(=3.75^{\circ}=3438^{\prime}\right)$. But this is not sacrosanct and he states that $\delta \phi$ could be "chindyat yateshtani" meaning as small as you like. Or as large. Brahmagupta ( 600 CE ) for example took it to be $\pi / 12$ to generate the sine series using Aryabhata's formulae from the previous section. To his credit he also had robust methods to evaluate the sine for intermediate values. The Madhava school looked in the other direction. They took it to be very small.

We rewrite Eqs. (17) and (18) with $\delta \phi$ replaced by $\delta \phi / 2$.

$$
\begin{align*}
\delta \sin (\phi) & =\sin (\phi+\delta \phi / 2)-\sin (\phi-\delta \phi / 2) \\
& =2 \sin (\delta \phi / 2) \cos (\phi)  \tag{24}\\
\delta \cos (\phi) & =\cos (\phi+\delta \phi)-\cos (\phi-\delta \phi) \\
& =-2 \sin (\delta \phi / 2) \sin (\phi) \tag{25}
\end{align*}
$$

Like in the previous section we take $\delta \phi / 2=\phi / N$ where once again $N$ is unimaginably large, larger than paraddham or $10^{17}$ ! Then $\delta \phi / 2$ is sunyaprayam or "of the nature of zero" i.e., an infinitesimal. We can then take replace $2 \sin (\delta \phi / 2)$ by $\delta \phi$ to yield

$$
\begin{align*}
\delta \sin (\phi) & =\delta \phi \cos (\phi)  \tag{26}\\
\delta \cos (\phi) & =-\delta \phi \sin (\phi) \tag{27}
\end{align*}
$$

The Madhava school leaves the above equations in the differential form. They do not explicitly write the derivative. But nothing prevents us from doing so.

$$
\begin{align*}
\lim _{\delta \phi \rightarrow 0} \frac{\delta \sin (\phi)}{\delta \phi} & =\frac{d \sin (\phi)}{d \phi} \\
& =\cos (\phi) \tag{28}
\end{align*}
$$

Similarly

$$
\begin{equation*}
\frac{d \cos (\phi)}{d \phi}=-\sin (\phi) \tag{29}
\end{equation*}
$$

Taking inspiration from the previous section we may take one more derivative to obtain e.g. $\delta^{2} \sin \phi$ and $\delta^{2} \cos \phi$

$$
\begin{align*}
& \frac{d^{2} \sin (\phi)}{d \phi^{2}}=-\sin (\phi)  \tag{30}\\
& \frac{d^{2} \cos (\phi)}{d \phi^{2}}=-\cos (\phi) \tag{31}
\end{align*}
$$

Madhava and his disciples worked in the discrete domain leaving the limiting procedure for the end. We shall illustrate how they proceeded using the modern notation so familiar to us. Integrating once from zero to $\theta$

$$
\begin{aligned}
\left.\frac{d \sin (\phi)}{d \phi}\right|_{\theta}-\left.\frac{d \sin (\phi)}{d \phi}\right|_{0} & =-\int_{0}^{\theta} \sin (\phi) d \phi \quad \text { Or } \\
\left.\frac{d \sin (\phi)}{d \phi}\right|_{\theta} & =1-\int_{0}^{\theta} \sin (\phi) d \phi
\end{aligned}
$$

Note that we have used the fundamental theorem of calculus. We repeat this procedure once more to obtain

$$
\begin{equation*}
\sin (\theta)=\theta-\int_{0}^{\theta} d \phi \int_{0}^{\phi} \sin (\xi) d \xi \tag{32}
\end{equation*}
$$

The above expression can be readily subjected to recursive refining ("samskaram"). To start with we have

$$
\sin (\theta)_{1}=\theta
$$

Next we replace the sine in the second term on the r.h.s. of Eq.(32) by $\sin (\xi)=\xi$. Hence for the second recursion

$$
\begin{aligned}
\sin (\theta)_{2} & =\theta-\int_{0}^{\theta} d \phi \int_{0}^{\phi} \xi d \xi \\
& =\theta-\int_{0}^{\theta} \phi^{2} / 2 d \phi \\
& =\theta-\theta^{3} / 3!
\end{aligned}
$$

It is easy to see the trend. Next we replace the $\sin (\xi)$ in Eq. (32) by $\xi-\xi^{3} / 3$ !. This yields

$$
\sin (\theta)_{3}=\theta-\theta^{3} / 3!+\theta^{5} / 5!
$$

The entire sine series is thus obtained

$$
\begin{equation*}
\sin (\theta)=\sum_{k=1,3,5 \ldots}(-1)^{(k-1) / 2} \frac{\theta^{k}}{k!} \tag{33}
\end{equation*}
$$

One may similarly obtain the cosine series

$$
\begin{equation*}
\cos (\theta)=\sum_{k=0,2,4 \ldots}(-1)^{k / 2} \frac{\theta^{k}}{k!} \tag{34}
\end{equation*}
$$

We have carried out the expansion using the method of "Samskaram" but interpolated with the more familiar integration and appeal to the fundamental theorem of calculus. In what one may describe as a tour de force, the book Yuktibhasa carries out the entire exercise in the discrete formalism describing it in minute detail in the Malayalam languag ${ }^{5}$. We have forgone this. The Appendix which pertains to the previous section will give one a feeling for how this discrete formalism works.

## V Discussion

Among the other accomplishments of the Madhava school we mention two. They derived the formulae for the area and the volume of the sphere by methods of calculus. Secondly they realized that the $\pi$ series (Eq.14) has slow convergence. So they reformulated the series in multiple ways. A proof of their ingenuity is the calculation of the value of $\pi$ up to 18 decimal places by Sankara Varman (1800s, see Fig.1). At that time this was a world record. We shall not describe these accomplishments here.

From time to time one one hears of the Calculus accomplishments of Indian mathematicians predating Madhava. While describing the motion of celestial bodies, Bhaskara II (1100CE) has used terms like tatkalika (at that instant). He also mentions the stationarity property of elliptical orbits at the apogee and perigee. It is a creditable example of theoretical insight based on observation. A close reading reveals that he is still thinking in terms of small and finite differences in angles (not time). Similar claims have been made on behalf of Nilakantha (see Fig. 1) with his refined astronomical model. This is as per the Aryabhata framework described in Sec.IV.2. Quantities are small, but there is no infinitesimal ("shunyaprayam") and the careful treatment it requires. The area and volume of the sphere are also mentioned but the methods by which they are arrived at are unclear. There is a parallel in European mathematical history. Archimedes arrived at the value of $\pi$ by the method of exhaustion long ago. Both Descartes and Fermat had discussed "derivatives" in terms of the slope. Fermat even mentions points in the function $f(x)$ where a small change in $x$ has "almost no effect" on $\mathrm{f}(\mathrm{x})$. One can describe the work of these illustrious mathematicians as pre-Calculus or proto-Calculus at best.

Some shortcomings are apparent in the work of the Madhava school. They were sensitive to the convergence of infinite series. But they did not prove the convergence, absolute or conditional. Neither did Newton. An explicit derivative notation or its interpretation in terms of slope is not present. In a sense the Madhava school's

[^4]treatment of differentials is closer to Leibniz than Newton. One can sense their reluctance to divide "zero by zero". There is no treatment of conic sections (hyperbola for instance). Further, how would one accomplish the integration of say $\sqrt{x}$ in Samlkalitam? Or of exponential and logarithmic functions? The answer to these questions is that perhaps it is possible. It is for students, particularly Indian students, to push forward the Madhava program and to explore its advantages and limitations. All told it is a beautiful approach to Calculus. The Madhava school was in decline by the time serious European science and mathematics came to India and perhaps did not have a chance to address these concerns.

To sum up, the Madhava school had a consistent formalism using methods that can be identified as methods of Calculus and they applied it successfully to a class of functions, namely, trigonometric functions. The approach is refreshingly different from the European. To quote the Fields Medallist David Mumford, "It seems fair to me to compare [Madhava] with Newton and Leibniz" ${ }^{6}$

Acknowledgement: The author would place on record the many useful discussions he had with Prof. P. P. Divakaran.

## A Derivation of $J_{k}=1 /(k+1)$ (Eq. (13))

We define a related quantity $S_{N}(k)$

$$
\begin{equation*}
S_{N}(k)=N^{k+1} I_{N}(k)=\sum_{n=1}^{N} n^{k} \tag{35}
\end{equation*}
$$

The proof proceeds in two parts. We first obtain a recursion relation for $S_{N}(k)$. Next we take the asymptotic limit of large $N$ to obtain an explicit expression for $S_{N}(k)$.

To obtain the recursion relation we first note that

$$
\begin{align*}
N S_{N}(k-1)-S_{N}(k) & =\sum_{n=1}^{N}(N-n) n^{k-1}  \tag{36}\\
& =\sum_{n=1}^{N-1} \sum_{j=1}^{n} j^{k-1} \tag{37}
\end{align*}
$$

Thus

$$
\begin{equation*}
S_{N}(k)=N S_{N}(k-1)-\sum_{n}^{N-1} S_{n}(k-1) \tag{38}
\end{equation*}
$$

It is not easy to see how the single summation of Eq. (36) is reordered to the double summation of Eq. (37). One way to see this is to take some concrete values say $N=5$ and $k=3$ and convince oneself thereby. Divakaran's book (see References) takes the continuum limit of this and uses integration by parts to prove it. He also points out that this is a special case of the Abel re-summation formula and it is remarkable that Yuktibhasa had discovered and deployed it. Using the definition (Eq. (35)) we can arrive at the recursion relation given by Eq. (38) from Eq. (37).

[^5]The next step is to solve for $S_{N}(k)$ in the large $N$ limit. We begin by noting from Eq. (35) that $S_{N}(0)=N$. Hence Eq. (38) yields

$$
\begin{align*}
S_{N}(1) & =N^{2}-\sum_{n}^{N-1} S_{n}(0) \\
& =N^{2}-(1+2+3+\ldots(N-1)) \\
& =N^{2}-N(N-1) / 2 \\
& =N^{2} / 2 \tag{39}
\end{align*}
$$

where we take the large N limit in the last step. Eq. (39) suggests that

$$
\begin{equation*}
S_{N}(k)=N^{k+1} /(k+1) \tag{40}
\end{equation*}
$$

We then use mathematical induction. We shall establish that $S_{N}(k+1)=N^{k+2} /(k+2)$ using the recursion formula Eq. (38). Note

$$
\begin{align*}
S_{N}(k+1) & =N S_{N}(k)-\sum_{n}^{N-1} S_{n}(k) \\
& =N^{k+2} /(k+1)-\sum_{n}^{N-1} n^{k+1} /(k+1) \\
& =\frac{N^{k+2}}{k+1}-\frac{S_{N-1}(k+1)}{k+1} \tag{41}
\end{align*}
$$

where the last step follows from the definition of $S_{N}(k)$ in Eq. (35). Since N is large we take $S_{N-1}(k+1) \approx S_{N}(k+1)$. This then establishes our required relation

$$
\begin{equation*}
S_{N}(k+1)=\frac{N^{k+2}}{k+2} \tag{42}
\end{equation*}
$$

We now note that we have proved this asymptotically. Hence we replace $I_{N}(k)$ in Eq. (35) by the $N \rightarrow \infty$ limiting expression $J_{k}$,

$$
\begin{equation*}
J_{k+1}=\frac{1}{k+2} \tag{43}
\end{equation*}
$$

and similarly $J_{k}=1 /(k+1)$. We note that the derivation holds for all positive integers $k$ and not just for even integers that were required in Sec. III.

## EXERCISES

1. Samskaram for square root of a positive number $n$. Take a perfect square $m^{2}$ less than but closest to $n$. Define a correction $n=m^{2}+r$. We can rewrite this a

$$
n=(m+r / 2 m)^{2}-(r / 2 m)^{2} .
$$

For the first iteration drop the $(r / 2 m)^{2}$ term on the r.h.s. so

$$
\sqrt{n}_{1}=m+r / 2 m .
$$

Continue this iteration and show that

$$
\sqrt{n}_{2}=m+r / 2 m-(r / 2 m)^{2} \frac{1}{2(m+r / 2 m)} .
$$

Numerically compute for $n=95$ and $m=9$.
2. An alternate Samskaram for the square root of the positive number $n$. Take

$$
\sqrt{n}_{1}=\frac{1}{2}\left(m+\frac{n}{m}\right) .
$$

Obtain the next iteration. Once again numerically compute for the $n=95$ and $m=9$. Which of the two methods yields a closer value for $\sqrt{5}$ ?
3. Samskaram method for the cosine interpolation formula of Madhava. Consider the two trigonometric identities:

$$
\begin{align*}
\cos (\theta+\delta) & =\cos (\theta)-2 \sin (\delta / 2) \sin (\theta+\delta / 2)  \tag{44}\\
\sin (\theta+\delta) & =\sin (\theta)+2 \sin (\delta / 2) \cos (\theta+\delta / 2) \tag{45}
\end{align*}
$$

We point out that the Nila lineage acknowledge that these identities were first mentioned in the Ganita section of Aryabhatiya (499 CE). Madhava then approximates $\sin (\delta / 2)$ by $\delta / 2$ for small $\delta$ to write

$$
\begin{equation*}
\cos (\theta+\delta)=\cos (\theta)-\delta \sin (\theta+\delta / 2) \tag{46}
\end{equation*}
$$

and next drops the $\delta / 2$ in the sine term on the r.h.s. to obtain the first step of the recursion

$$
\cos (\theta+\delta)_{1}=\cos (\theta)-\delta \sin (\theta)
$$

If on the other hand we had retained Eq.(46) and employed the exact sine formula Eq.(45) we would refine the recursion. Show that continuing one will obtain

$$
\cos (\theta+\delta)_{3}=\cos (\theta)-\delta \sin (\theta)-\frac{\delta^{2}}{2} \cos (\theta)+\frac{\delta^{3}}{8} \sin (\theta)
$$

This does not lead to the correct expansion for the cosine. The Nila mathematicians were well aware of the limitations of the above expansion. The point of this exercise is to point out that the method of Samskaram is often accompanied by other approximations (in this case the $\sin (\delta / 2) \approx \delta / 2$ ), whose reliability must be gauged independent of the iterative process.
4. The Indian mathematical tradition made judicious use of similar triangles. We can get a taste of this by establishing the important relation between the angle and its tangent (Eq.(77)). First prove that the triangles $O X A_{n}$ and $A_{n-1} A_{n} B$ are similar and further that triangles $O P R$ and $O A_{n-1} B$ are similar. Thus establish

$$
P R=\frac{A_{n} A_{n-1}}{O A_{n} O A_{n-1}}
$$

Note that $P R=\sin \left(\delta \theta_{n}\right)$ and $O A_{n} \approx O A_{n-1}$ while $A_{n} A_{n-1}=\delta t$.
5. We can generate the sine table as per Aryabhata's suggestion but not exactly using the same value for $\epsilon$ he used. We choose $\epsilon=\pi / 80 \approx 0.0393$ which is the same as $4.5^{0}$. We take $\sin (\epsilon) \approx \epsilon$ and $\sin (2 \epsilon) \approx 2 \epsilon$ ). If you have a simple calculator generate all values of sine from 2.25 to 18 degrees in equal steps using Eq. (23). Alternatively if you have a programmable calculator or a computer generate all values of sine from 2.25 to 90 degrees. Compare with the results your calculator will otherwise yield.
6. Using the method of Section IV. 3 generate the cosine expansion to arrive at Eq.(34).
7. Take the continuum limit of the summation formulae (Eq. (36) and Eq. (37)). To do this see Eq. (15). Prove that the two summations are indeed equal. [Note: This proof is due to Divakaran (see References).]

## FEW REFERENCES

These references are in English.

## Primary

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2. "Aryabhatiya", Kripashanker Shukla and K V Sarma, Vols. I and II, Indian National Science Academy Publication (1976). The Section IV. 1 and IV. 2 have been shaped by the critical discussions of certain verses from the Ganitapada by these authors.
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## Secondary:

1. "The Mathematics of India", P. P. Divakaran, Hindustan Book Agency (2018). The book has shaped this article in ways covert and overt. Highly recommended.
2. "The History of Hindu Mathematics's", Bibhutibhusan Datta and Avadhesh Narayan Singh, Vols. I and II, Asia Publishing House, Delhi (1935 and 1938). A pioneering book on Indian Mathematics written in Pre-Independence India.
3. "Geometry in Ancient and Medieval India", Sarasvati Amma, Motilal Banarsidas, Delhi (1999). It has detailed discussions worth looking at. It also uncovers the element of continuity in the Indian mathematical traditions from ancient times to the pre-British era.

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[^0]:    1 "On the Hindu Quadrature of the Circle, and the Infinite Series of the Proportion of the Circumference to the Diameter Exhibited in the Four Sastras, Tantra Sanghraham, Yucti Bhasha, Carana Padhati, and Sadratnamala", Charles M. Whish Transactions of the Royal Asiatic Society of Great Britain and Ireland, Vol. 3, pg. 509, (1834). Whish knew Shankara Varman (see Fig.1) personally.

[^1]:    ${ }^{2}$ To avoid confusion in what follows we shall attribute all results to the "Madhava school" and only occasionally to the illustrious "Nila lineage" or to a text such as Yuktibhasa. The members of the lineage themselves from time to time use the phrase "Madhavoditam" (so said Madhava) and once in a while invoke Aryabahata.

[^2]:    ${ }^{3}$ Namely the integral of the function $f$ over a fixed interval is the change in its anti-derivative F between the ends of the interval.

[^3]:    ${ }^{4}$ This word is to be distinguished from sthula or approximate. For example when theorists use the value of the mass of the electron to be $9.1 \times 10^{-31} \mathrm{~kg}$ it is sthula. When experimentalists carefully quote the value $9.10938 \times 10^{-31} \mathrm{~kg}$ it is assana meaning that it can be refined with more careful experimentation.

[^4]:    ${ }^{5}$ Most manuscripts by Indian mathematicians are in Sanskrit and in verse form respecting the norms of grammar and meter of Sanskrit

[^5]:    ${ }^{6}$ Notices of the American Mathematical Society, Vol. 57, page 385 (2010).

