The Annals of Statistics
2003, Vol. 31, No. 3, 984–994

INDICATOR FUNCTION AND ITS APPLICATION IN TWO-LEVEL FACTORIAL DESIGNS

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A two-level factorial design can be uniquely represented by a polynomial indicator function. Therefore, properties of factorial designs can be studied through their indicator functions. This paper shows that the indicator function is an effective tool in studying two-level factorial designs. The indicator function is used to generalize the aberration criterion of a regular two-level fractional factorial design to all two-level factorial designs. An important identity of generalized aberration is proved. The connection between a uniformity measure and aberration is also extended to all two-level factorial designs.

1. Introduction. Two-level factorial designs are the most popular designs among experimenters. A 2^{s-k} fractional factorial design is said to be "regular" if it is generated by k generators. The structures of regular fractional factorial designs are described by group theory and well understood, as discussed in detail by Dey and Mukerjee (1999). Unfortunately, the same theory and mathematical tools cannot be applied to nonregular designs and there is no unified mathematical representation for those designs in general. A recent paper by Fontana, Pistone and Rogantin (2000) introduces indicator functions for studying fractional factorial designs (with no replicates). In this paper, indicator functions are extended to study general two-level factorial designs, with or without replicates, regular or nonregular. As will be demonstrated in this paper, they constitute a very effective tool for studying these designs.

This paper is organized as follows. In Section 2, the indicator function is defined for all two-level factorial designs. In Section 3, indicator functions are related to aberration of regular two-level factorial designs. Based on this relation, the definition of aberration is naturally generalized to all two-level factorial designs. In addition, an important property of the generalized criterion is also obtained. In Section 4, the indicator function is applied to extend the connection between aberration and a uniformity measure to all two-level factorial designs.

2. Indicator functions. Following the path-breaking paper of Pistone and Wynn (1996), which applies computational algebraic geometry methods to study statistics, Fontana, Pistone and Rogantin (2000) (henceforth FPR) introduce indicator functions as a tool to study fractional factorial designs. Let $\mathcal{D}$ be a 2^s

Received February 2001; revised June 2002.
AMS 2000 subject classification. 62K15.
Key words and phrases. Generalized aberration, uniform design, orthogonality, projection properties.

full factorial design with levels being -1 and 1. The design points of $\mathcal{D}$ are the solutions of the polynomial system $\{x_1^2 - 1 = 0, x_2^2 - 1 = 0, \dots, x_s^2 - 1 = 0\}$. A fractional factorial design $\mathcal{F}$ is a subset of $\mathcal{D}$. Notice that $\mathcal{F}$ could be any subset of $\mathcal{D}$ without any restriction on its run size. FPR define the *indicator function* of $\mathcal{F}$ as follows.

DEFINITION 1 (FPR). Let $\mathcal{D}$ be a 2^s design. The indicator function F of its fraction $\mathcal{F}$ is a function defined on $\mathcal{D}$ such that

$$F(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathcal{F}, \\ 0, & \text{if } \mathbf{x} \in \mathcal{D} - \mathcal{F}. \end{cases}$$

Using the theory of Gröbner basis and algebraic geometry, FPR show that each indicator function has a unique polynomial representation. A more elementary and constructive but less elegant approach is given below.

The existence of an indicator function in a polynomial form follows immediately from the following two lemmas. The proofs of the lemmas are straightforward and are omitted here.

LEMMA 1. *The indicator function of a single point* $\mathbf{a} = (a_1, a_2, \dots, a_s) \in \mathcal{D}$ *is*

$$F_{\mathbf{a}} = \frac{\prod_{i=1}^s (x_i + a_i)}{\prod_{i=1}^s 2a_i}. \tag{1}$$

LEMMA 2. *Let* $F_{\mathcal{A}}$ *and* $F_{\mathcal{B}}$ *be indicator functions of two disjoint designs* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively. The indicator function of design* $\mathcal{A} \cup \mathcal{B}$ *is then*

$$F_{\mathcal{A}\cup\mathcal{B}} = F_{\mathcal{A}} + F_{\mathcal{B}}. \tag{2}$$

The uniqueness of the polynomial form can be established as follows. Define contrasts $X_I(\mathbf{x}) = \prod_{i \in I} x_i$ on $\mathcal{D}$ for $I \in \mathcal{P}$, where $\mathcal{P}$ is the collection of all subsets of $\{1, 2, \dots, s\}$. It is well known in experimental design that $\{X_I, I \in \mathcal{P}\}$ forms an orthogonal basis of $\mathbb{R}^{2^s}$. Therefore, any function f on $\mathcal{D}$ is a unique linear combination of 2^s contrasts. Since each contrast is a monomial on $\mathcal{D}$, f has the polynomial form

$$f(\mathbf{x}) = \sum_{I \in \mathcal{P}} b_I X_I(\mathbf{x}). \tag{3}$$

Note that since $x_i^2 = 1$ on $\mathcal{D}$ for $i = 1, \dots, s$, the monomials in the polynomial form are "square free." Therefore, an indicator function has the unique square-free polynomial representation on $\mathcal{D}$.

The above definition, however, only deals with unreplicated designs. Designs with replicates are often seen in practice and are of theoretical interest as well. For example, Lin and Draper (1992) identify two nonisomorphic five-column projections of the twelve-run Plackett–Burman design. In one of the two projections, a design point repeats twice. In order to study these designs, the definition of indicator functions is generalized as follows.

DEFINITION 2. Let $\mathcal{D}$ be a 2^s design and $\mathcal{A}$ be a design such that $\forall \mathbf{a} \in \mathcal{A}$, $\mathbf{a} \in \mathcal{D}$ but $\mathbf{a}$ might be repeated in $\mathcal{A}$. The generalized indicator function of $\mathcal{A}$ can be obtained as

$$F(x_1, \dots, x_s) = \sum_{\mathbf{a} \in \mathcal{A}} F_{\mathbf{a}}(x_1, \dots, x_s),$$

where $F_{\mathbf{a}}$ is the indicator function of point $\mathbf{a}$.

By this definition, the value of the indicator function of a design $\mathcal{A}$ at point $\mathbf{a} \in \mathcal{A}$ is the number of appearances of $\mathbf{a}$ in $\mathcal{A}$. By the same argument as for the regular indicator function, a generalized indicator function has a unique polynomial representation as in (3). In the remainder of the paper, I will refer to a generalized indicator function as "indicator function." The new definition of indicator function allows one to study general two-level factorial designs, with or without replicates. *Note that under the new definition*, *Lemma* 2 *holds for any two designs* $\mathcal{A}$ *and* $\mathcal{B}$ *and the disjoint condition is no longer needed.*

One of the recent advances in experimental design is on the hidden projection properties of fractional factorial designs; see Wang and Wu (1995) and Cheng (1998). One might be interested in the projections of a design when one or more factors in the original design are no longer considered. The remainder of this section will show that indicator functions of projections can be easily obtained given the indicator function of the original design. Let $\mathcal{B}$ be a projection of a design $\mathcal{A}$ onto a subset of factors. It contains the same design points of $\mathcal{A}$ but each point only has coordinates of the factors that are projected onto. Therefore, projections often have replicated points. The following lemma is obtained directly from the definitions.

LEMMA 3. *Consider design* $\mathcal{A}$. *Without loss of generality*, *let* $\mathcal{B}$ *be its projection to* $\{x_{l+1}, \dots, x_s\}$ *and* $\mathcal{D}^l$ *be the* 2^l *design on* $\{x_1, \dots, x_l\}$. *The* (*generalized*) *indicator function of* $\mathcal{B}$ *is then*

$$F_{\mathcal{B}}(x_{l+1}, \dots, x_s) = \sum_{(x_1, \dots, x_l) \in \mathcal{D}^l} F_{\mathcal{A}}(x_1, \dots, x_l, x_{l+1}, \dots, x_s). \tag{4}$$

PROOF. For $\mathbf{a}_2 \notin \mathcal{B}$, by definition, $F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2) = 0$ for all $\mathbf{a}_1 \in \mathcal{D}^l$. Hence $F_{\mathcal{B}}(\mathbf{a}_2) = 0$. For $\mathbf{a}_2 \in \mathcal{B}$, $F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2)$ is the number of appearances of $(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathcal{A}$. Hence, $\sum_{\mathbf{a}_1 \in \mathcal{D}^l} F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2)$ is the number of appearances of $\mathbf{a}_2$ in $\mathcal{B}$. □

The following theorem is the same as Proposition 8.2 in FPR, who present it without introducing the generalized indicator functions.

THEOREM 1. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}^l$ *be as defined in Lemma* 3. *Let* $F_{\mathcal{A}} = \sum_{I\in\mathcal{P}} b_I X_I$ *be the indicator function of* $\mathcal{A}$. *Let* $\mathcal{P}_1$ *be the collection of all subsets of* $\{l+1,\dots,s\}$. *Then*

$$F_{\mathcal{B}}(x_{l+1},\dots,x_s) = 2^l \sum_{I\in\mathcal{P}_1} b_I X_I. \tag{5}$$

PROOF. Let $\mathcal{P}_2$ be the collection of all subsets of $\{1,\dots,l\}$. We have

$$F_{\mathcal{A}} = \sum_{I\in\mathcal{P}_1}\left(\sum_{J\in\mathcal{P}_2} b_{I\cup J} X_J\right) X_I. \tag{6}$$

Applying (6) to (4), we have

$$F_{\mathcal{B}}(x_1,\dots,x_l) = \sum_{I\in\mathcal{P}_1}\left(\sum_{J\in\mathcal{P}_2} b_{I\cup J}\left(\sum_{(x_1,\dots,x_l)\in\mathcal{D}^l} X_J\right)\right) X_I. \tag{7}$$

From the definition of X_J, it is easy to see that $\sum_{(x_1,\dots,x_l)\in\mathcal{D}^l} X_J = 0$ for all $J \in \mathcal{P}_2 - \{\varnothing\}$ and $\sum_{(x_1,\dots,x_l)\in\mathcal{D}^l} X_\varnothing = 2^l$. Equation (5) follows immediately. □

This section defines the indicator function which is unique for each factorial design. Therefore, it can be used to study the properties of a design. In the next section, a connection between the indicator function and aberration of a design is explored.

3. Orthogonality and aberration criterion. One important design property is orthogonality. A related criterion is the minimum aberration that was first proposed by Fries and Hunter (1980) for regular fractional factorial designs. It is discussed in detail in Wu and Hamada (2000) which also provides a catalogue of minimum aberration designs.

3.1. *Indicator function and orthogonality.* FPR study the orthogonality of designs using indicator functions. Although their concern is only about fractions with no replicates, most of their arguments are still valid for generalized indicator functions and designs with replicates. A key result in their paper links the coefficient of indicator functions to the orthogonality between contrasts.

Consider two contrasts X_I and X_J on the 2^s design $\mathcal{D}$. X_I and X_J are orthogonal on design $\mathcal{D}$ since $\sum_{\mathbf{x}\in\mathcal{D}} X_I(\mathbf{x})X_J(\mathbf{x}) = 0$ for $I \neq J$. Similarly, the two contrasts are orthogonal on design $\mathcal{A}$ if and only if

$$\sum_{\mathbf{x}\in\mathcal{A}} X_I X_J = 0. \tag{8}$$

Let $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$ be the indicator function of $\mathcal{A}$. Denote the difference between two sets $I \cup J - I \cap J$ by $I \triangle J$. Since $x_i^2 = 1$ on $\mathcal{D}$, we have $X_I X_J = X_{I \triangle J}$. The left-hand side of (8) can be simplified as follows:

$$\sum_{\mathbf{x} \in \mathcal{A}} X_I X_J = \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}} X_I X_J = \sum_{\mathbf{x} \in \mathcal{D}} \left(\sum_{K \in \mathcal{P}} b_K X_K \right) X_{I \triangle J}$$
$$= \sum_{K \in \mathcal{P}} b_K \sum_{\mathbf{x} \in \mathcal{D}} X_K X_{I \triangle J} = b_{I \triangle J} \sum_{\mathbf{x} \in \mathcal{D}} 1 = 2^s b_{I \triangle J}.$$

Therefore, X_I and X_J are orthogonal on $\mathcal{A}$ if and only if $b_{I \triangle J} = 0$ in $F_{\mathcal{A}}$. In particular, $b_I = 0$ if and only if X_I is orthogonal to the constant term $X_{\varnothing}$.

What should be mentioned here are the J-characteristics used by Deng and Tang (1999) as building blocks in defining their generalized aberration criterion. The J-characteristics of a design closely relate to coefficients of its indicator function and can be viewed as its orthogonality measure.

DEFINITION 3 [Deng and Tang (1999)]. Regard an $n \times s$ design as a set of s columns $\mathcal{A} = \{c_1, c_2, \ldots, c_s\}$. For any k-subset $d = \{d_1, d_2, \ldots, d_k\}$ of $\mathcal{A}$, define

$$J_k(d) = \left| \sum_{i=1}^{n} d_{i1} \cdots d_{ik} \right|,$$

where d_{ij} is the ith element of column d_j. The $J_k(d)$ values are called the J-characteristics of $\mathcal{A}$.

Let $I = \{i_1, i_2, \ldots, i_k\}$, and d be the k-subset $\{c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$. From the properties of indicator functions,

$$J_k(d) = \left| \sum_{\mathbf{x} \in \mathcal{A}} X_I \right| = \left| \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}} X_I \right| = \left| \sum_{\mathbf{x} \in \mathcal{D}} \sum_{J \in \mathcal{P}} b_J X_J X_I \right|$$
$$= \left| \sum_{J \in \mathcal{P}} b_J \sum_{\mathbf{x} \in \mathcal{D}} X_I X_J \right| = \left| b_I \sum_{\mathbf{x} \in \mathcal{D}} X_{\varnothing} \right| = 2^s |b_I|.$$

Therefore, the coefficients of indicator functions are the same as the signed J-characteristics up to a constant. Given the signed J-characteristics of a design, the design is uniquely determined. It should be noted here that, independently, Tang (2001) also observes the one-to-one relation between a factorial design and its signed J-characteristics.

3.2. *Aberration of regular fractional factorial designs.* Aberration describes the orthogonality of a regular fractional factorial design. Consider a two-level factorial design with s factors. For $I \in \mathcal{P}$, its corresponding contrast X_I is

$\prod_{i=1}^{s} x_i^{w_i}$, where $w_i = 1$ if $i \in I$ and 0 otherwise. Since $\mathcal{P}$ has a one-to-one map to the lattice $\{\mathbf{w} = (w_1, w_2, \ldots, w_s) \mid w_i = 0, 1\}$, the contrasts can be also denoted as $X^{\mathbf{w}}$. Indicator functions of regular fractions are discussed in detail by FPR. Without loss of generality, consider a regular 2^{s-k} fractional factorial design $\mathcal{A}$ generated by k generating relations, $\{X^{\mathbf{w}_{(1)}} = 1, \ldots, X^{\mathbf{w}_{(k)}} = 1\}$ such that $X^{\mathbf{w}_{(i)}} = 1$ for all $x \in \mathcal{A}$. The k generating relations generate a group of defining relations $\{X^{\mathbf{w}} = 1 \mid \mathbf{w} \in W_{\mathcal{A}}\}$, where

$$W_{\mathcal{A}} = \left\{ \mathbf{w} \,\middle|\, \mathbf{w} = \sum_{i=1}^{k} u_i \mathbf{w}_{(i)}; u_i = 0, 1; i = 1, \ldots, k \right\}. \tag{9}$$

The summation in (9) is as in Galois field GF(2). It can be easily verified that the polynomial form of the indicator function of $\mathcal{A}$ is

$$\begin{aligned} F_{\mathcal{A}} &= \frac{1}{2^k}(X^{\mathbf{w}_{(1)}} + 1)(X^{\mathbf{w}_{(2)}} + 1)\cdots(X^{\mathbf{w}_{(k)}} + 1) \\ &= \frac{1}{2^k} \sum_{\mathbf{w} \in W_{\mathcal{A}}} X^{\mathbf{w}} = \sum_{\mathbf{w} \in W_{\mathcal{A}}} \frac{2^{s-k}}{2^s} X^{\mathbf{w}}. \end{aligned} \tag{10}$$

Note in (10), the coefficient $b_{\mathbf{w}}$ of contrast $X^{\mathbf{w}}$ is $\frac{2^{s-k}}{2^s}$ if $\mathbf{w} \in W_{\mathcal{A}}$ and 0 otherwise.

A contrast $X^{\mathbf{w}}$ is also called a word with length $\|\mathbf{w}\| = \sum_i^s w_i$. The word-length pattern of a regular fractional factorial design $\mathcal{A}$ is defined as $\{\alpha_1(\mathcal{A}), \ldots, \alpha_s(\mathcal{A})\}$, where

$$\alpha_j(\mathcal{A}) = \#\{\mathbf{w} \in W_{\mathcal{A}} \mid \|\mathbf{w}\| = j\}. \tag{11}$$

A connection between the word-length pattern of a design and its indicator function can be observed from (10) and (11),

$$\alpha_j(\mathcal{A}) = \sum_{\|\mathbf{w}\| = j} \left(\frac{2^s}{n} b_{\mathbf{w}} \right)^2, \tag{12}$$

where n is the number of design points in $\mathcal{A}$ and $b_{\mathbf{w}}$ gives the coefficients of $X^{\mathbf{w}}$ in the indicator function. The aberration criterion of regular fractional factorial designs is defined as follows.

DEFINITION 4. Given two regular fractional factorial designs $\mathcal{A}_1$ and $\mathcal{A}_2$, design $\mathcal{A}_1$ is said to have less aberration than $\mathcal{A}_2$ if there exists r such that

$$\alpha_1(\mathcal{A}_1) = \alpha_1(\mathcal{A}_2), \ldots, \alpha_{r-1}(\mathcal{A}_1) = \alpha_{r-1}(\mathcal{A}_2), \alpha_r(\mathcal{A}_1) < \alpha_r(\mathcal{A}_2).$$

A design has minimum aberration if no other design has less aberration.

The aberration criterion extends the resolution criterion. The latter only compares the length of the shortest defining words while the former compares the numbers of defining words of each length. Therefore, designs with the same resolution might be different in aberration. For a detailed discussion, see Wu and Hamada (2000).

3.3. *Generalized aberration.* A nonregular two-level factorial design $\mathcal{A}$ does not have generating relations and defining relation groups. Therefore, (11) cannot be used to define the word-length pattern of a nonregular design. However, one can use (12) to define the word-length pattern of any two-level factorial design, regular or nonregular.

DEFINITION 5. Let $\mathcal{A}$ be a two-level factorial design and $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$ be its indicator function. Its word-length pattern is $\{\alpha_1(\mathcal{A}), \dots, \alpha_s(\mathcal{A})\}$, where

$$\alpha_j(\mathcal{A}) = \sum_{\|\mathbf{w}\|=j} \left(\frac{2^s}{n} b_{\mathbf{w}}\right)^2 = \sum_{\#I=j} \left(\frac{2^s}{n} b_I\right)^2.$$

Based on the generalized definition of word-length pattern, the definition of the aberration criterion for a regular fractional factorial can be applied to all two-level factorial designs, regular or nonregular, with or without replicates. The next theorem gives an important property of generalized word-length pattern which justifies this generalization. The following lemma was given by FPR for designs without replicates. Nonetheless, their proof applies directly to the general two-level factorial designs.

LEMMA 4 (FPR). *Let $\mathcal{A}$ be an $n \times s$ two-level factorial design. Let $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$. Then $b_{\varnothing} = n/2^s$.*

THEOREM 2. *Let $\mathcal{A}$ be a fraction of a 2^s design $\mathcal{D}$ with n runs. Its word-length pattern $\{\alpha_1(\mathcal{A}), \dots, \alpha_s(\mathcal{A})\}$ is as defined in Definition 5. We have*

$$\sum_{j=1}^{s} \alpha_j(\mathcal{A}) = 2^s (n_2/n^2) - 1, \tag{13}$$

where $n_2 = \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}}^2(\mathbf{x})$. For designs with no replicates,

$$\sum_{j=1}^{s} \alpha_j(\mathcal{A}) = \frac{2^s}{n} - 1. \tag{14}$$

PROOF. Let the indicator function of $\mathcal{A}$ be $F_{\mathcal{A}}(\mathbf{x}) = \sum_{I \in \mathcal{P}} b_I X_I$. Hence,

$$\sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}}^2(\mathbf{x}) = \sum_{\mathbf{x} \in \mathcal{D}} \left(\sum_{I \in \mathcal{P}} b_I X_I\right)^2 = \sum_{\mathbf{x} \in \mathcal{D}} \sum_{I, J \in \mathcal{P}} b_I b_J X_I(\mathbf{x}) X_J(\mathbf{x})$$
$$= \sum_{I, J \in \mathcal{P}} b_I b_J \sum_{\mathbf{x} \in \mathcal{D}} X_I(\mathbf{x}) X_J(\mathbf{x}) = 2^s \sum_{I \in \mathcal{P}} b_I^2.$$

From the definition, $\sum_{j=1}^{s}\alpha_j(\mathcal{A}) = \sum_{I\in\mathcal{P}-\{\varnothing\}} b_I^2/b_\varnothing^2$. From the lemma, $b_\varnothing = n/2^s$. Hence (13) is obtained. For designs with no replicates, $F_{\mathcal{A}}^2(\mathbf{x}) = F_{\mathcal{A}}(\mathbf{x})$ and $n_2 = n$; hence (14) is obtained. □

For a regular fractional factorial 2^{s-k} design $\mathcal{A}$, it is well known that fractions $\sum_{j=1}^{s}\alpha_j(\mathcal{A}) = 2^k - 1$. This identity can be easily obtained through the theory of defining contrast groups. It is, nonetheless, a special case of the above theorem. The sum of word-length patterns is constant for all designs with the same replication pattern. For designs with a high degree of replication, n_2 is large in (13). Therefore, they tend to have high aberration compared to designs with less replication.

Recently, Deng and Tang (1999) and Tang and Deng (1999) generalized resolution and aberration criterion to nonregular two-level designs based on the J-characteristics. Coincidentally, the definition of generalized aberration criterion in this paper is equivalent to the G_2-aberration criterion in Tang and Deng (1999). Without giving a particular reason, Tang and Deng (1999) seem to favor G_2-aberration criterion over other choice of G_p-aberration, which is equivalent to $\alpha_j(\mathcal{A}) = \sum_{\#I=j}(\frac{2^s}{n}b_I)^p$. Their intuition is well justified by Theorem 2. For other choice of p, the sum of word-length pattern is no longer constant for designs with the same size and same replication pattern.

One application of the above theorem is to search for minimum aberration designs using a uniformity criterion which will be discussed later in the paper.

4. Generalized aberration and uniformity. Uniformity (space filling property) is another popular design criterion. Fang and Mukerjee (2000) find a connection between aberration and a uniformity measure for regular two-level factorial designs. The uniformity measure that they used is defined in Hickernell (1998).

DEFINITION 6 (Hickernell). For design $\mathcal{A}$ as a collection of n design points in $[0,1]^s$, the centered L_2-discrepancy measure of uniformity is

$$\begin{aligned}\{CL_2(\mathcal{A})\}^2 = {} & \left(\frac{13}{12}\right)^s - \frac{2}{n}\sum_{\mathbf{x}\in\mathcal{A}}\prod_{j=1}^{s}\left(1+\frac{1}{2}|x_j-1/2|+\frac{1}{2}|x_j-1/2|^2\right)\\ & + \frac{1}{n^2}\sum_{\mathbf{x},\mathbf{x}'\in\mathcal{A}}\prod_{j=1}^{s}\left(1+\frac{1}{2}|x_j-1/2|+\frac{1}{2}|x_j'-1/2|-\frac{1}{2}|x_j-x_j'|\right).\end{aligned}\tag{15}$$

Rescaling a two-level factorial design to $[0,1]^s$ with levels at 1/4 and 3/4, a connection between aberration and uniformity is found by Fang and Mukerjee (2000).

THEOREM 3 (Fang and Mukerjee). *Let* $\mathcal{A}$ *be a regular design, involving* $n = 2^{s-k}$ *runs of a* 2^s *factorial. The centered* L_2*-discrepancy measure of uniformity in* $[0, 1]^s$

$$\{CL_2(\mathcal{A})\}^2 = \left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s + \left(\frac{9}{8}\right)\left\{1 + \sum_{r=1}^{s} \frac{\alpha_r(\mathcal{A})}{9^r}\right\}. \tag{16}$$

With the definition of generalized word-length pattern for nonregular designs, the above theorem extends to all two-level factorial designs, regular or nonregular, with or without replicates.

THEOREM 4. *Let* $\mathcal{A}$ *be an* $n \times s$ *run two-level factorial design. Then the* L_2*-discrepancy measure of uniformity*

$$\{CL_2(\mathcal{A})\}^2 = \left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s + \left(\frac{9}{8}\right)\left\{1 + \sum_{r=1}^{s} \frac{\alpha_r(\mathcal{A})}{9^r}\right\}. \tag{17}$$

PROOF. The proof follows the same argument in Fang and Mukerjee (2000). Note that "$\pi(x)'\mathbf{y}_d$" in their proof equals $2^s b_I$. □

It can be easily seen from Theorems 2 and 4 that minimum aberration designs should often agree with the most uniform design within the class of all two-level designs of the same run size, regular or nonregular, with replicates or without replicates. Following geometrical intuition, designs with a high degree of replication tend to be less uniform than those with less replicates. This is implied by Theorems 2 and 4. In general, designs with more replicates are less favored with respect to aberration and uniformity criteria. This provides another justification for the generalized aberration criterion.

Since the uniformity measure in (15) is much easier to compute than word-length pattern, it would be a very good searching criterion for minimum aberration design. Note that the first two terms of (15) need not be computed since they are constant for two-level designs, and the last term can be simplified to

$$\sum_{\mathbf{x},\mathbf{x}' \in \mathcal{A}} \prod_{j=1}^{s} \left(\tfrac{5}{4} - \tfrac{1}{2}|x_j - x_j'|\right).$$

With the uniformity measure as searching criterion, a columnwise–pairwise algorithm [Li and Wu (1997)] was used to search for minimum aberration designs within balanced 12×5 two-level factorial designs. The search lasted only a few seconds and found the well-known subdesign of 12-run Plackett–Burman design with no replicates. Its word-length pattern is $(0, 0, 10/9, 5/9, 0)$. The same method was applied for 16-run designs and found designs that match with those found by

Tang and Deng (1999). They construct the designs by only searching projections of the third Hadamard matrix of order 16. While searching within subdesigns of Hadamard matrices is very efficient, it is limited to designs with $4m$ runs. Nonetheless, their search can also be speeded up by using the uniformity criterion. Searching for minimum aberration designs within all two-level factorial designs of a run size takes a longer time, but it can be applied to supersaturated designs and designs with $4m + 2$ runs. With a columnwise–pairwise algorithm and an easy-to-compute searching criterion, it becomes plausible.

5. Conclusion.

Traditional factorial design theory focuses on regular fractions. But nonregular designs provide experimenters many more choices, especially more flexible run sizes. The group theory used to study regular fractions is not suitable for studying general factorial designs. This paper introduces indicator functions as a mathematical representation for general two-level factorial designs. It provides a single mathematical framework for regular and nonregular designs, and designs with or without replicates. In this paper, the effectiveness of indicator functions is explored by extending several existing results on regular designs to general factorial designs. First, word-length pattern and aberration criterion are generalized to general factorial designs. A justification of such generalization is also provided. Second, a relation between aberration and a uniformity measure is also extended to general factorial designs. The two criteria are very much consistent with each other. Designs of the same run size with a higher degree of replication tend to be less desirable with respect to both criteria.

This paper modifies the indicator function introduced by FPR to represent designs with replicates. One of the referees pointed out another approach to deal with replicates by Cohen, Di Bucchianico and Riccomagno (2001) which introduces the replication number as a new pseudo factor. Such an arrangement is useful in computing the Gröbner basis of the designs with replicates and is complementary to the indicator function approach proposed here. For details on other applications of computational commutative algebra in statistics, see Pistone, Riccomagno and Wynn (2001).

Acknowledgment. The author thanks both referees for their constructive suggestions that led to significant improvement of this paper.

REFERENCES

CHENG, C.-S. (1998). Some hidden projection properties of orthogonal arrays with strength three. *Biometrika* **85** 491–495.

COHEN, A. M., DI BUCCHIANICO, A. and RICCOMAGNO, E. (2001). Replications with Gröbner bases. In *mODa6*: *Advances in Model-Oriented Design and Analysis* (A. C. Atkinson, P. Hackl and W. Müller, eds.) 37–44. Physica, Heidelberg.

DENG, L.-Y. and TANG, B. (1999). Generalized resolution and minimum aberration criteria for Plackett–Burman and other nonregular factorial designs. *Statist. Sinica* **9** 1071–1082.

DEY, A. and MUKERJEE, R. (1999). *Fractional Factorial Plans*. Wiley, New York.

FANG, K.-T. and MUKERJEE, R. (2000). A connection between uniformity and aberration in regular fractions of two-level factorials. *Biometrika* **87** 193–198.

FONTANA, R., PISTONE, G. and ROGANTIN, M. P. (2000). Classification of two-level factorial fractions. *J. Statist. Plann. Inference* **87** 149–172.

FRIES, A. and HUNTER, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22** 601–608.

HICKERNELL, F. J. (1998). A generalized discrepancy and quadrature error bound. *Math. Comp.* **67** 299–322.

LI, W. and WU, C. F. J. (1997). Columnwise-pairwise algorithms with applications to the construction of supersaturated designs. *Technometrics* **39** 171–179.

LIN, D. K. J. and DRAPER, N. R. (1992). Projection properties of Plackett and Burman designs. *Technometrics* **34** 423–428.

PISTONE, G., RICCOMAGNO, E. and WYNN, H. P. (2001). *Algebraic Statistics*: *Computational Commutative Algebra in Statistics*. Chapman and Hall, Boca Raton, FL.

PISTONE, G. and WYNN, H. P. (1996). Generalized confounding with Gröbner bases. *Biometrika* **83** 653–666.

TANG, B. (2001). Theory of J-characteristics for fractional factorial designs and projection justification of minimum G_2-aberration. *Biometrika* **88** 401–407.

TANG, B. and DENG, L.-Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Ann. Statist.* **27** 1914–1926.

WANG, J. C. and WU, C. F. J. (1995). A hidden projection property of Plackett–Burman and related designs. *Statist. Sinica* **5** 235–250.

WU, C. F. J. and HAMADA, M. (2000). *Experiments*: *Planning*, *Analysis and Parameter Design Optimization*. Wiley, New York.

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