INDICATOR POLYNOMIAL FUNCTIONS AND THEIR APPLICATIONS IN TWO-LEVEL FACTORIAL DESIGNS

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Abstract

In this thesis, we discuss some properties of indicator polynomial functions. We extend some existing results from regular designs to non-regular designs. More general results which were not obtained even for regular designs are also provided.

First, we study indicator polynomial functions with one, two, or three words. Classification of indicator polynomial functions with three words are provided. Second, we consider the connections between resolutions of general two-level factorial designs. As special cases of our results, we generalize the results of Draper and Lin [14]. Next, we discuss the indicator polynomial functions of partial foldover design, especially, semifoldover designs. Using the indicator polynomial functions, we examine various possible semifoldover designs. We show that the semifoldover resolution $III.x$ design obtained by reversing the signs of all the factors can de-alias at least the same number of the main factors as the semifoldover design obtained by reversing the signs of one or more, but not all, the main factors. We also prove that the semifoldover resolution $IV.x$ designs can de-alias the same number of two-factor interactions as the corresponding full foldover designs. More general results are also provided. Finally, we present our conclusions and outline possible future work.

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Chapter 1

Introduction

1.1 General Introduction

In practice many processes or systems are affected by two or more factors. Thus scientists are often interested in the study of effects of several factors simultaneously. An experiment which involves several factors is called a *factorial experiment.*

Suppose a factorial design has m factors with each factor at two levels. A complete replicate of such a design would require 2^m observations and such a design is called a 2^m *factorial design* (see, for example, Figure 1). To distinguish it from *fractional* factorial designs (which are studied in this thesis) (see, for example, Figure 2), it is also called a *full factorial design.* This experimental design would enable the experimenter to investigate the individual effects of each factor and also to determine whether the factors interact or not. The experiment can be *replicated*, which means some runs of experiment are carried out two or more times.

	Factor		
run	A	B	C
1	-	-	-
2	+	-	-
3	-	+	-
4	+	+	-
5	-	-	+
6	+	-	+
7	-	+	+
8	+	+	+

Figure 1. 2^3 *Full Factorial Design*

	Factor		
run	A	B	C
2	+	-	-
3	-	+	-
5	-	-	+
8	+	+	+

Figure 2. 2^{3-1} *Fractional Factorial Design with Defining Relation* $ABC = 1$.

However, quite often in practice, full factorial design is infeasible both from time as well as resource points of view. For instance, if there are ten factors with each factor at two levels, then a full factorial design would need $2^{10} = 1024$ observations; if there are five factors with each factor at five levels, then it would need $3^5 = 3125$ runs. Thus, for large number of factors, full factorial designs may not be affordable in practice and *fractional factorial designs*, which consist of a subset or fraction of the runs, are more economic and are therefore commonly used in practice.

Fractional factorial designs have been studied for many years. In this area, many problems have a geometric, algebraic or combinatorial flavour. For example, if A and B are two factors of a full 2^2 factorial design, then the main effects A and B and the interaction effect AB with the identity element I form a group; if $x_1, x_2, \ldots, x_6$ are six factors such that $x_5 = x_1x_4$ and $x_6 = x_2x_3$, then the runs which satisfy these two conditions form a fraction of the full 2^6

factorial design; we call $x_5 = x_1x_4$ and $x_6 = x_2x_3$ as *generators* or *defining relation*, and $x_1x_4x_5 = x_2x_3x_6 = x_1x_2x_3x_4x_5x_6 = 1$, where $x_1x_2x_3x_4x_5x_6 = x_1x_4x_5 \cdot x_2x_3x_6$, is called a *complete defining relation*. Moreover, the set $G = \{1, x_1x_4x_5, x_2x_3x_6, x_1x_2x_3x_4x_5x_6\}$ forms a group and each element except 1 in G is called a *word*, an *interaction effect* or an *effect*, the fraction is called 2^{6-2} fractional factorial design, since this fraction has only 2^{6-2} runs, "6" represents 6 factors and "2" represents 2 generators; we also say, for example, x_5 is *aliased* with x_1x_4 since $x_5 = x_1x_4$. In general, a 2^{m-p} *design* is a fraction of a full m-factor design with p generators and, thus, contains 2^{m-p} runs. A fractional factorial design which has defining relations is called a *regular design*. This design has a group structure. Thus, the classical method for studying this area uses algebras such as linear algebra and finite groups. Early works in this direction have been summarized in Raktoe, Hedayat and Federer [25] and Dey [11]. For recent reviews, we refer to Dey and Mukerjee [12] and Wu and Hamada [30].

A fractional factorial design which has no generator or defining relation is called a *non-regular design*. Non-regular designs have not been well studied since these designs have no defining relation. However, sometimes non-regular designs are more useful than regular designs since they need fewer runs; see, for example, Addelman [2], Westlake [29] and Draper [13].

In 1996, Pistone and Wynn [23] introduced a method based on Gröbner bases (see; for example, Cox, Little and O'Shea [8] or Adams and Loustaunau [1]), an area in computational commutative algebra, to study the identifiability problem in experimental designs. Gröbner bases form a very useful tool to deal with problems in polynomial ring. The basic idea in their article is to represent the design as the solution of a set of polynomial equations. This application of Gröbner bases in experimental designs gives a completely new interface between computational commutative algebra and experimental designs, and it turns out to be a powerful tool in some areas of experimental designs (see Holliday, Pistone, Riccomagno and Wynn [17] and Bates, Giglio, Riccomagno and Wynn [4]).

In fact, it should not be surprising that there is such an interface between computational commutative algebra and statistics, since the mathematical structure of real random variables is a commutative ring, and other commutative rings and ideals appear naturally in distribution theory and modelling (see Pistone, Riccomagno and Wynn [24]). This interface attracts considerable interest from both the algebraic community (see Robbiano [26]) as well as from the statistical community since the publication of the paper by Pistone and Wynn [23]. As mentioned in the preface of the book *Algebraic Statistics* [24],

"Just as the introduction of vectors and matrices has greatly improved the mathematics of statistics, these new tools provide a further step forward by offering a constructive methodology for a basic mathematical tool in statistics and probability, that is to say a ring."

After the publication of Pistone and Wynn [23], Fontana, Pistone and Rogantin [15] introduced the indicator polynomial function (see Section 1.2) as a tool to study fractional factorial designs without replicates, which was subsequently extended to the case of replication by Ye [31]. Indicator polynomial functions unify regular designs and non-regular designs and provide an effective tool for studying non-regular designs.

In Section 1.2, we introduce indicator polynomial functions and review some of their properties. The definition of fractional resolution and the connections between regular fractional factorial designs of resolution III and V are presented in Section 1.3. In Section 1.4, we introduce foldover designs and semi-foldover designs. Some related work is also reviewed in this section. Finally, we present in Section 1.5 an outline and notation used in this thesis.

1.2 Indicator Polynomial Functions

Let D_{2^m} be the full two-level m-factor design, i.e.,

$$D_{2^m} = \{x = \{x_1, x_2, \ldots, x_m\} \mid x_i = 1 \text{ or } -1,\ i = 1, 2, \ldots, m\},$$

$$M = \{1, 2, \ldots, m\},$$

$$L_{2^m} = \{\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \mid \alpha_i = 1 \text{ or } 0\ \forall i \in M\},$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m},$$

and

$$\| \alpha \| = \sum_{i=1}^{m} \alpha_i.$$

Then, $\| \alpha \|$ is the number of letters of x^α.

Let $\mathcal{F}$ be any two-level m-factor design such that for any $x \in \mathcal{F}$, $x \in D_{2^m}$, but x might be repeated in $\mathcal{F}$. The *indicator polynomial function* of $\mathcal{F}$ is a function $f(x)$ defined on D_{2^m} such that

$$f(x) = \begin{cases} r_x & \text{if } x \in \mathcal{F} \\ 0 & \text{if } x \notin \mathcal{F}, \end{cases}$$

where r_x is the number of appearances of the point x in design $\mathcal{F}$. In particular, if $\mathcal{F} = \emptyset$ (i. e., there is no runs in $\mathcal{F}$) or $\mathcal{F}$ contains all the points in D_{2^m} (i.e., $\mathcal{F}$ is a full 2^m design), the indicator polynomial function of $\mathcal{F}$ is $f(x) = 0$ or $f(x) = r_x$ for any $x \in \mathcal{F}$, respectively. In this thesis, we assume that $\mathcal{F}$ contains some but not all the points in D_{2^m}.

Fontana, Pistone and Rogantin [15] and Ye [31] showed that the indicator polynomial function $f(x)$ of $\mathcal{F}$ can be uniquely represented by a polynomial function

$$f(x) = \sum_{\alpha \in L_{2^m}} b_\alpha x^\alpha, \tag{1.2.1}$$

where the coefficients $\{b_\alpha, \alpha \in L_{2^m}\}$ can be determined as

$$b_\alpha = \frac{1}{2^m} \sum_{x \in \mathcal{F}} x^\alpha. \tag{1.2.2}$$

In particular,

$$|\frac{b_\alpha}{b_0}| \leq 1 \tag{1.2.3}$$

and

$$b_0 = \frac{n}{2^m}, \tag{1.2.4}$$

where n is the total number of runs.

Thus, given a design, we can find the coefficients of its indicator polynomial function. For a regular design, its indicator polynomial function is easy to find. For example, if $x_5 = x_1x_4$ and $x_6 = x_2x_3x_4$ are generators of a two-level 6-factor design, then we can easily check that the corresponding indicator polynomial function is

$$f(x) = \frac{1}{2^2}(1 + x_1x_4x_5)(1 + x_2x_3x_4x_6) = \frac{1}{4}(1 + x_1x_4x_5 + x_2x_3x_4x_6 + x_1x_2x_3x_5x_6).$$

Conversely, given an indicator polynomial function of a two-level factorial design $\mathcal{F}$, we can check whether it represents a regular design or not. Proposition 1.2.1 below was proved by Fontana, Pistone and Rogantin [15] and Ye [32].

Proposition 1.2.1. *$\mathcal{F}$ is a regular design (with or without replicates) if and only if*

$$| b_\alpha / b_0 | = 1$$

for all nonzero b_α in the indicator polynomial function of $\mathcal{F}$.

Example 1.2.2. *[15] An indicator polynomial function of a two-level 5-factor design $\mathcal{F}$ without replicate is $f(x) = \frac{1}{2} - \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_2x_3x_4 + \frac{1}{4}x_2x_3x_5 + \frac{1}{4}x_1x_2x_3x_4x_5$. Since $|b_\alpha/b_0| \neq 1$ for all nonzero b_α, $\mathcal{F}$ is not a regular design. By (1.2.4), this design contains 16 runs. It is a half fraction of the full 2^5 factorial design.*

Any word in the indicator polynomial function indicates alias relations. For example, if $x^\alpha = x_1x_2x_4x_6$ is a word in an indicator polynomial function, then x_1x_2, x_1x_4, x_1x_6, x_2x_4, x_2x_6 and x_4x_6 are aliased with x_4x_6, x_2x_6, x_2x_4, x_1x_6, x_1x_4 and x_1x_2, respectively. If $|b_\alpha/b_0| = 1$, then they are fully aliased; if $|b_\alpha/b_0| < 1$, then they are partially aliased.

Let $\mathcal{F}$ be a fraction which does not allow replicates. Then, its *complementary fraction* contains the runs which are in D_{2^m} but not in $\mathcal{F}$. Proposition 1.2.3 is a part of Corollary 3.5 in Fontana, Pistone and Rogantin [15] which provides the relations of indicator polynomial functions of the two fractions.

Proposition 1.2.3. *If $\mathcal{F}$ and $\mathcal{F}'$ are complementary un-replicated fractions and b_α and b'_α are the coefficients of the respective indicator polynomial functions defined as in (1.2.1), then*

$$b_0 = 1 - b'_0 \quad \textit{and} \quad b_\alpha = -b'_\alpha, \quad \forall \alpha \neq 0.$$

When one or more factors are not important in a factorial design, one might be interested in the projection of the design. Some projection properties of fractional factorial designs have been studied (see, Wang and Wu [28] and Cheng [6]). Theorem 1.2.4 provide the indicator polynomial function of the projection given the indicator polynomial function of the original design and was provided by Fontana, Pistone and Rogantin [15] and Ye [31].

Theorem 1.2.4. *Let (1.2.1) be the indicator polynomial function of $\mathcal{F}$, $\mathcal{P}$ be its projection to $\{x_1, x_2, \ldots, x_l\}$, and*

$$S = \{\alpha \in L_{2^m} \mid \alpha_j = 0, \ \forall j = l+1, \ldots, m\},$$

Then the indicator polynomial function of $\mathcal{P}$ is

$$f_{\mathcal{P}} = 2^{m-l} \sum_{\alpha \in S} b_\alpha x^\alpha.$$

1.3 Resolutions and Their Connections

The traditional definition of resolution is defined through the complete defining relation, that is, the *resolution* of a regular design is the number of letters in the shortest word of the complete defining relation. Deng and Tang [10] and Tang and Deng [27] defined the generalized resolution, that is , fractional resolution, and generalized aberration criteria for non-regular designs. These criteria were redefined through indicator functions by Ye [31] and Li, Lin and Ye [19]. In this thesis, the definition of fractional resolutions by Li, Lin and Ye [19] is used.

First, Li, Lin and Ye [19] extended the traditional definition of the word to non-regular designs by calling each term (except the constant) in the indicator function of a design a *word.* If x^α is a word, its *length* is defined as

$$\| x^\alpha \| = \| \alpha \| + (1 - |b_\alpha / b_0|).$$

Thus for regular designs, since $|b_\alpha/b_0| = 1$, for each word x^α, its length is the number of letters of the word; for non-regular designs, the length of words may be fractional since $|b_\alpha/b_0|$ may be less than 1. In Example 1.2.2, the length of the word $x_1x_2x_3$ is 3.5.

Next, Li, Lin and Ye [19] defined the *extended word length pattern* of $\mathcal{F}$ as

$$(f_1, \ldots, f_{1+(n-1)/n}, f_2, \ldots, f_{2+(n-1)/n}, \ldots, f_m, \ldots, f_{m+(n-1)/n}),$$

where $f_{i+j/n}$ is the number of length $(i + j/n)$ words.

Finally, the *generalized resolution* is defined as the length of the shortest word. Thus the generalized resolution may be fractional. In this thesis, we will denote fractional resolutions by $N.x$, where N is an integer and x is a fraction. Thus, the resolution of the design in Example 1.2.2 is $III.5$.

Given the extended word length patterns of two designs, the *aberration* *criterion* is defined by sequentially comparing the two extended word length patterns from the shortest-length word to the longest-length word.

Resolution III^* regular designs are regular resolution III designs in which no two-factor interactions are confounded with one another. These designs are valuable in composite designs and were first examined by Hartley [16]. Draper and Lin [14] found the connection between resolutions III^* and V designs so that one can study resolution III^* designs through well-known resolution V designs.

The following Theorems and Corollaries are taken from Draper and Lin [14].

Theorem 1.3.1. *Any m-factor two-level fractional factorial design of resolution III^* forms a base that can be converted into a $(m-1)$-factor design of resolution V in the same number of runs.*

Corollary 1.3.2. *If m is the maximum number of factors that can be accommodated in a resolution III^* design, then the maximum number of factors that can be accommodated in a resolution V design with the same number of runs is at least $m-1$.*

Theorem 1.3.3. *Any $(m-1)$-factor two-level fractional factorial design of resolution V can be converted into a m-factor design of resolution III^* in the same number of runs.*

Corollary 1.3.4. *If $m-1$ is the maximum number of factors that can be accommodated in a resolution V design, then the maximum number of factors that can be accommodated in a resolution III^* design with the same number of runs is at least m.*

Theorem 1.3.5 is the extension of Theorem 1.3.1 by Draper and Lin [14].

Theorem 1.3.5. *Any m-factor two-level fractional factorial design of resolution $(2l-1)^*$ forms a base that can be converted into a $(m-1)$-factor design of resolution $(2l+1)$ in the same number of runs.*

1.4 Foldover Designs and Semifoldover Designs

When two effects are aliased, it is difficult to estimate one of them. Foldover is a classic technique to de-alias effects. Define a *foldover* of a factorial design as the procedure of adding a new fraction in which signs are reversed on one or more factors of the original design. The combined design has the double size of the original runs. A foldover design is also called a *full foldover design.*

For regular resolution III designs, some main effects are aliased with two-factor interactions. It is well-known (see, for example, [22]) that if we add to a resolution III fractional a second fraction in which the signs for all the factors are reversed, then the combined design has resolution IV.

For regular resolution IV designs, all the main effects are de-aliased with two-factor interactions. However, some two-factor interactions are aliased with each other. Box, Hunter, and Hunter [5] studied the foldover design obtained by reversing the sign of one factor. Montgomery and Runger [21] considered reversing the signs of one or two factors to de-alias as many two-factor interactions as possible. They stated that the complete defining relation of the combined design from a foldover consists of those effects in the complete defining relation of the original fraction that are not sign-reversed in the new fraction.

Li, Lin and Ye [19] studied foldover non-regular designs using indicator polynomial functions and extended above well-known results to non-regular designs. They provide the following three properties of indicator polynomial functions which are useful for studying foldover designs:

1. Let $f(x_1, x_2, \ldots, x_m)$ be the indicator polynomial function of a design. If the sign of factor x_1 is reversed, then the indicator polynomial function of the new design is $f(-x_1, x_2, \ldots, x_m)$.

2. Let $f_{\mathcal{F}_1}$ and $f_{\mathcal{F}_2}$ be the indicator polynomial functions of the designs $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then the indicator polynomial function of the combined design $\mathcal{F}_1 \cup \mathcal{F}_2$ is given by $f_{\mathcal{F}_1 \cup \mathcal{F}_2} = f_{\mathcal{F}_1} + f_{\mathcal{F}_2}$.

3. Let $f(x_1, x_2, \ldots, x_m)$ be the indicator polynomial function of a design. Without loss of generality, assume that the signs of factors $x_1, x_2, \ldots, x_r$ are reversed; then, the indicator polynomial function of the foldover design is $f(x_1, x_2, \ldots, x_m) + f(-x_1, -x_2, \ldots, -x_r, x_{r+1}, \ldots, x_m)$.

Although foldover designs can de-alias all the main effects for resolution $III.x$ designs and as many two-factor interactions as possible for resolution $IV.x$ designs, they involve twice the original runs. Therefore, it will be much more efficient to do a partial foldover. One of the partial foldover designs is the semi-foldover design.

Semifoldover designs are obtained by reversing signs of one or more factors in the original design and adds half of the new runs to the original designs. Thus, semifoldover designs save half of the original runs compared to the full foldover designs and are more valuable sometimes. A semifoldover design obtained by reversing signs of one factor in a resolution IV regular design with generators $x_1x_2x_3x_5 = 1$ and $x_2x_3x_4x_6 = 1$ was first studied by Daniel [9] and then investigated by Barnett *et al.* [3] through a case study.

Mee and Peralta [20] studied various possible semifoldover regular resolution III and IV designs.

Let

$$\mathcal{F}^{(e)} = \{x \in \mathcal{F} \mid z = e\}, \tag{1.4.1}$$

where $e = 1, -1$, and z is a main effect or an interaction. Denote by $\mathcal{F}_o$ the new fraction obtained by reversing the signs of $x_1, x_2, \ldots, x_r$. Then, we can add either the fraction $\mathcal{F}_o^{(1)}$ or the fraction $\mathcal{F}_o^{(-1)}$ to the original design to get the semifoldover design. In this case, according to Mee and Peralta's notation, we say that the semifoldover design is obtained by *foldover on* $x_1, x_2, \ldots, x_r$ and *subset on* z.

For resolution IV designs, Mee and Peralta [20] proved Theorem 1.4.1.

Theorem 1.4.1. *For any regular 2_{IV}^{m-p} design and any two factors x and y, the full foldover design obtained by folding over on x and the semifoldover design obtained by folding over on x and subsetting on y permit estimation of the same two-factor interactions, assuming that three-factor and higher-order interactions are negligible.*

Mee and Peralta [20] studied semifoldover resolution III design through an example. Although semifoldover resolution III designs usually can not de-alias as many two-factor interactions as the corresponding full foldover designs, Mee and Peralta [20] pointed out that the half new runs can be used as comfirmation runs which verify the validity of one's assessment of active versus inactive factors.

1.5 Outline and Notations

Define a *resolution $N.^*x$ design* is a resolution $N.x$ design such that its indicator polynomial function contains no $(N+1)$-letter word. The thesis is organized as follows:

In Chapter 2, we study some properties of indicator polynomial functions and $N.^*x$ design. We discuss indicator polynomial functions with one, two or three words. In particular, we show that the indicator polynomial functions with only one word must be a regular design or replicates of a regular design; there is no indicator polynomial function with only two words; we also classify the indicator polynomial functions which contain only three words.

In Chapter 3, we discuss the connections between designs of general two-level factorial designs. First, we prove that a resolution $(2l-1).^{*}\ x$ m-factor design can be converted into resolution $(2l+1).x$ $(m-1)$-factor design. The relations between designs of resolution $2l.x$ and $(2l-1).x$ are also provided. Next, we show that a resolution III^{*} m-factor design can be obtained from any design with resolution equal or bigger than V. Some illustrative examples are also provided.

In Chapter 4, we study indicator polynomial functions of partial foldover designs. We study indicator polynomial functions of semifoldover designs first. Then we extend them to partial foldover designs.

In Chapter 5, we discuss semifoldover resolution $III.x$ designs. We show that the semifoldover design obtained by folding over on all the factors can de-alias at least the same number of the main effects as the semifoldover design obtained by folding over on one or more, but not all, the main effects when subsetting on a same factor. We also study the semifoldover design obtained by subsetting on a two-factor interaction. Some illustrative examples are provided at the end of this chapter.

In Chapter 6, we consider semifoldover resolution $IV.x$ designs. After proving that the semifoldover non-regular design obtained by folding over on a factor and subsetting on a factor can de-alias the same number of the two-factor interactions as the corresponding full foldover design, we present a sufficient condition for a semifoldover design to de-alias the same number of the two-factor interactions as the corresponding full foldover design. Finally, we provide some illustrative examples.

In Chapter 7, We present some conclusions based on the results in this thesis. Then, several interesting problems for future work are outlined.

The notation used in this thesis are as follows:

In Chapter 2, we denote

$$\Omega = \{\alpha \in L_{2^m} \mid b_\alpha \neq 0 \text{ and } ||\alpha|| \neq 0\},$$

$$\Omega_1 = \{\alpha \in \Omega \mid ||\alpha|| \text{ is even } \} \cup \{\alpha \in L_{2^m} \mid ||\alpha|| = 0\},$$

and

$$\Omega_2 = \{\alpha \in \Omega \mid ||\alpha|| \text{ is odd } \}.$$

In Chapter 3, we denote

$$\Omega^{\mathcal{F}} = \{\alpha \in L_{2^m} \mid b_\alpha \neq 0 \text{ and } \parallel \alpha \parallel \neq 0\},$$

$$\Omega_o^{\mathcal{F}} = \{\alpha \in \Omega^{\mathcal{F}} \mid \ \parallel \alpha \parallel \text{ is odd}\},$$

and

$$\Omega_e^{\mathcal{F}} = \{\alpha \in \Omega^{\mathcal{F}} \mid \ \parallel \alpha \parallel \text{ is even}\}.$$

In Chapters 4 - 6, we denote

$$\mathcal{F}^{(e)} = \{x \in \mathcal{F} \mid z = e\}, \tag{1.5.1}$$

where $e = 1, -1$, and z is a main effect or an interaction.

Without loss of generality, we assume that the partial foldover design is obtained by reversing the signs of $x_1, x_2, \ldots, x_r$. Denote by Ω_e the set of all $\alpha \in \Omega$ such that there are 0 or even number of the first r entries which are 1 and Ω_o the set of all $\alpha \in \Omega$ such that there are odd number of the first r entries which are 1. Let

$$\mathcal{W}_e = \{x^\alpha \mid \alpha \in \Omega_e\},$$

$$\mathcal{W}_o = \{x^\alpha \mid \alpha \in \Omega_o\}.$$

Then, the indicator polynomial function (1.2.1) of $\mathcal{F}$ can be written as

$$f(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha. \tag{1.5.2}$$

Note that the constant term is in $\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha$.

Chapter 2

Properties of Indicator Polynomial Functions

2.1 Introduction

Indicator polynomial functions have been in the literature for several years, and yet only a few of their properties have been studied. In this chapter, we study some properties of indicator polynomial functions.

It is known that there is no regular design with only two words in their indicator polynomial functions, but there are regular designs with one or three words. Theoretically, one might be interested in knowing whether there exist non-regular designs with only one, two or three words in their indicator polynomial functions, and if they exist, what forms do those indicator polynomial functions have. In this chapter, we study some properties of indicator polynomial functions and, especially, indicator polynomial functions with only one, two or three words.

In Section 2.2, we study when a indicator polynomial function represents a half fraction and show that there is no $(2l + 1)$-factor design of resolution

$(2l-1).^{*}x$ when the run size of the design is not equal to 2^{2l}. Section 2.3 shows that the indicator polynomial functions with only one word must be a regular design or replicates of a regular design. Indicator polynomial functions with more than two words but only one odd or even word are also studied in this section. In Section 2.4, we establish that there is no indicator polynomial function with only two words. Indicator polynomial functions with more than two words but only two even words are also considered in this section. We prove that the indicator polynomial functions with only three words must have one or three even words and provide the forms of indicator polynomial functions for each case in Section 2.5.

In this chapter, we call a set of factors whose signs are reversed in the foldover design a *foldover plan* [19].

2.2 Indicator polynomial functions which represent half fractions and $N^{*}.x$ designs

Lemma 2.2.1. *Assume that $\mathcal{F}$ is a two-level m-factor design and (1.2.1) is its indicator polynomial function. Then the run size of $\mathcal{F}$ does not equal $2^m r$, where $r = f(1,1,\ldots,1)$, if and only if $\sum_{\alpha\in\Omega} b_\alpha \neq 0$.*

Proof. Since $f(1,1,\ldots,1) = b_0 + \sum_{\alpha\in\Omega} b_\alpha$, $\sum_{\alpha\in\Omega} b_\alpha = 0$ if and only if $b_0 = f(1,1,\ldots,1) = r$ and if and only if the run size of $\mathcal{F}$ equals $2^m r$. □

Proposition 2.2.2. *Assume that $\mathcal{F}$ is a two-level m-factor design. For any $x \in \mathcal{F}$, if all the words in its indicator polynomial function are odd words, then the sum of the number of replicates of the points $x = (x_1, x_2, \ldots, x_m)$ and $-x =$*

$(-x_1, -x_2, \ldots, -x_m)$ is $2b_0$. If all the words in its indicator polynomial function are even words, then the points x and $-x$ have the same number of replicates.

If there is no replicate and all the words in its indicator polynomial function are odd words, then either the point x or the point $-x$ is in $\mathcal{F}$ and $b_0 = 1/2$. In other words, if $b_0 \neq 1/2$, then there exists at least one $\alpha \in \Omega$ such that $\| \alpha \|$ is even.

Proof. Assume that (1.2.1) is the indicator polynomial function of $\mathcal{F}$. Then for any $x \in \mathcal{F}$, $f(x) = b_0 + \sum_{\alpha \in \Omega} b_\alpha x^\alpha$. If all the words are odd words, then $f(-x) = b_0 - \sum_{\alpha \in \Omega} b_\alpha x^\alpha$. Thus $f(x) + f(-x) = 2b_0$. If all the words are even words, the proof $f(x) = f(-x)$ follows similarly.

If there is no replicate and all the words in its indicator polynomial function are odd words, since $\#\mathcal{F} \neq 0, 2^m$, by Lemma 2.2.1, $\sum_{\alpha \in \Omega} b_\alpha \neq 0$. It then follows that $f(x) \neq f(-x)$, that is, either the point x or the point $-x$ is in $\mathcal{F}$. Thus $f(x) + f(-x)$ can only be 1. Therefore, $b_0 = 1/2$.

$\square$

Proposition 2.2.2 shows that a design with only odd words implies a half fraction. The result "If all the words in its indicator polynomial function are even words, then the points x and $-x$ have the same number of replicates" is also informally given by Cheng [7]. He showed that a design with only even words is a foldover of another design.

Example 1.2.2 shows that in the case of non-regular designs without replicate, if $b_0 = 1/2$ and $\#\Omega \geq 2$, then it is possible that all the words in the indicator polynomial function are odd words.

Proposition 2.2.3. *Assume that (1.2.1) is an indicator polynomial function of* *a two-level factorial design without replicates. If there exists a foldover plan such that the indicator polynomial function $g(x)$ of the foldover design contains no words, then this design is a half fraction.*

Proof. If the indicator polynomial function of the foldover design does not contain any word, then $g(x)$ must be 1. So $2b_0 = 1$, i.e., $b_0 = \frac{1}{2}$. □

Corollary 2.2.4. *Assume that (1.2.1) is an indicator polynomial function of a two-level factorial design without replicates. If there exists a main effect which is contained in all the words, then this design is a half fraction.*

Proof. Choosing the foldover plan as reversing the sign of the factor which is contained in all the words in $f(x)$, then the result. □

By Proposition 2.2.3, we also can get the result "if all the words in the indicator polynomial function are odd words, then the design is a half fraction" in Proposition 2.2.2 by reversing the signs of all the factors.

Note that a *resolution* $N.^{*}x$ *design* is a resolution $N.x$ design such that its indicator polynomial function contains no $(N+1)$-letter word.

Hartley [16] pointed out that there is no regular $2^{5-2}_{III^{*}}$ design. Proposition 2.2.5 shows that this is also true in general.

Proposition 2.2.5. *Assume that $\mathcal{F}$ is a (2l+1)-factor resolution $(2l-1).x$ design without replicate. Then, it is not a resolution $(2l-1).^{*}x$ design if $\#\mathcal{F} \neq 2^{2l}$.*

Proof. Since the design $\mathcal{F}$ has only $2l+1$ factors, it has no $(2l+2)$-letter word. If $\#\mathcal{F} \neq 2^{2l}$, then $b_0 \neq 1/2$. By Proposition 2.2.2, it must have a $(2l)$-letter word. □

Example 1.2.2 shows that when $\mathcal{F}$ is a 5-factor design and $\#\mathcal{F} = 2^4$, there exists a design of resolution $III.^{*}5$.

2.3 Indicator polynomial functions with one even or odd word

Proposition 2.3.1. *Assume that $\mathcal{F}$ is a two-level m-factor design and $f(x) = b_0 + b_\alpha x^\alpha$ is its indicator polynomial function. Then all the points in $\mathcal{F}$ have the same number of replicates $2b_0$ and $\mathcal{F}$ is a regular design. If there is no replicate, $|b_\alpha| = b_0 = \frac{1}{2}$.*

Proof. For any $x \in D_{2^m}$, x^α can only be 1 or -1. Thus, $f(x)$ equals either $b_0 + b_\alpha$ or $b_0 - b_\alpha$. Since for any $x \notin \mathcal{F}$, $f(x) = 0$, we have either $b_0 + b_\alpha = 0$ or $b_0 - b_\alpha = 0$. Consider $b_0 + b_\alpha = 0$. Let $b_0 - b_\alpha = a \neq 0$. Then, any point x such that $f(x) = b_0 - b_\alpha = a$ is in $\mathcal{F}$ and has the same number of replicates a. In this case, we have $a = 2b_0$ and $b_\alpha = -b_0$. When $b_0 - b_\alpha = 0$, the proof follows similarly. In this case, $a = 2b_0$ and $b_\alpha = b_0$. Thus, $|b_\alpha/b_0| = 1$. By Proposition 1.2.1, $\mathcal{F}$ is a regular design.

If there is no replicate, $2b_0 = 1$. Thus, $|b_\alpha| = b_0 = \frac{1}{2}$. □

Note that in this thesis, we assume that $\mathcal{F}$ does not contain all the points in D_{2^m}. If $\mathcal{F}$ contains all the points in D_{2^m}, then $|b_\alpha|$ may not equal b_0. For example, $f(x) = \frac{3}{2} + \frac{1}{2}x^\alpha$, for any $x \in \mathcal{F}$, $f(x) = 1$ or 2, this means $\mathcal{F}$ contains all the points in D_{2^m} and each point has one or two replications.

Lemma 2.3.2. *Assume that (1.2.1) is the indicator polynomial function of $\mathcal{F}$. Then the points $x = \{x_1, x_2, \ldots, x_m\}$ and $-x = \{-x_1, -x_2, \ldots, -x_m\}$ have different numbers of replicates if and only if $\sum_{\alpha \in \Omega_2} b_\alpha x^\alpha \neq 0$. Moreover,*

$$\sum_{\alpha \in \Omega_1} b_\alpha = \frac{1}{2}(f(1, 1, \ldots, 1) + f(-1, -1, \ldots, -1)) \tag{2.3.1}$$

and

$$\sum_{\alpha \in \Omega_2} b_\alpha = \frac{1}{2}(f(1,1,\ldots,1) - f(-1,-1,\ldots,-1)). \tag{2.3.2}$$

Proof. (1.2.1) can be written as

$$f(x) = \sum_{\alpha \in \Omega_1} b_\alpha x^\alpha + \sum_{\alpha \in \Omega_2} b_\alpha x^\alpha.$$

So

$$f(-x) = \sum_{\alpha \in \Omega_1} b_\alpha x^\alpha - \sum_{\alpha \in \Omega_2} b_\alpha x^\alpha.$$

Thus $\sum_{\alpha \in \Omega_2} b_\alpha x^\alpha \neq 0$ if and only if $f(x) \neq f(-x)$ and if and only if x and $-x$ have different numbers of replicates. Note that

$$f(1,1,\ldots,1) = \sum_{\alpha \in \Omega_1} b_\alpha + \sum_{\alpha \in \Omega_2} b_\alpha \tag{2.3.3}$$

and

$$f(-1,-1,\ldots,-1) = \sum_{\alpha \in \Omega_1} b_\alpha - \sum_{\alpha \in \Omega_2} b_\alpha, \tag{2.3.4}$$

we get (2.3.1) and (2.3.2). $\square$

Proposition 2.3.3. *Assume that (1.2.1) is the indicator polynomial function of a design which does not allow replicates; then,* $\sum_{\alpha \in \Omega_1} b_\alpha = \frac{1}{2}$ *and*

$$\sum_{\alpha \in \Omega_2} b_\alpha = \begin{cases} \frac{1}{2} & \text{if } (1,1,\ldots,1) \in \mathcal{F},\ (-1,-1,\ldots,-1) \notin \mathcal{F} \\ -\frac{1}{2} & \text{if } (1,1,\ldots,1) \notin \mathcal{F},\ (-1,-1,\ldots,-1) \in \mathcal{F} \end{cases}$$

if and only if $\sum_{\alpha \in \Omega_2} b_\alpha \neq 0$, *and*

$$\sum_{\alpha \in \Omega_1} b_\alpha = \begin{cases} 1 & \text{if } (1,1,\ldots,1),\ (-1,-1,\ldots,-1) \in \mathcal{F} \\ 0 & \text{if } (1,1,\ldots,1),\ (-1,-1,\ldots,-1) \notin \mathcal{F} \end{cases}$$

if and only if $\sum_{\alpha \in \Omega_2} b_\alpha = 0$.

Proof. By (2.3.3) and (2.3.4), $\sum_{\alpha\in\Omega_2} b_\alpha \neq 0$ if and only if $f(1,1,\ldots,1) \neq f(-1,-1,\ldots,-1)$. Note that $f(x)$ can only be 0 or 1, thus, $f(1,1,\ldots,1) \neq f(-1,-1,\ldots,-1)$ if and only if $f(1,1,\ldots,1) + f(-1,-1,\ldots,-1) = 1$ and

$$f(1,\ldots,1) - f(-1,\ldots,-1) = \begin{cases} 1 & \text{if } (1,\ldots,1) \in \mathcal{F},\ (-1,\ldots,-1) \notin \mathcal{F} \\ -1 & \text{if } (1,\ldots,1) \notin \mathcal{F},\ (-1,\ldots,-1) \in \mathcal{F}. \end{cases}$$

By Lemma 2.3.2, we get the first result.

On the other hand, by (2.3.3) and (2.3.4), $\sum_{\alpha\in\Omega_2} b_\alpha = 0$ if and only if $f(1,1,\ldots,1) = f(-1,-1,\ldots,-1)$ if and only if

$$f(1,\ldots,1) + f(-1,\ldots,-1) = \begin{cases} 2 & \text{if } (1,\ldots,1),\ (-1,\ldots,-1) \in \mathcal{F} \\ 0 & \text{if } (1,\ldots,1),\ (-1,\ldots,-1) \notin \mathcal{F}. \end{cases}$$

By Lemma 2.3.2, we get the second result. □

Corollary 2.3.4. *If there is only one odd word x^α in the indicator polynomial function of a design which does not allow replicates, then either $(1,1,\ldots,1)$ or $(-1,-1,\ldots,-1)$ is in $\mathcal{F}$ and*

$$b_\alpha = \begin{cases} \frac{1}{2} & \text{if } (1,1,\ldots,1) \in \mathcal{F},\ (-1,-1,\ldots,-1) \notin \mathcal{F} \\ -\frac{1}{2} & \text{if } (1,1,\ldots,1) \notin \mathcal{F},\ (-1,-1,\ldots,-1) \in \mathcal{F}. \end{cases} \tag{2.3.5}$$

Proposition 2.3.5. *If there is only one even word x^α in the indicator polynomial function of a design which does not allow replicates, then,*

$$b_0 = b_\alpha = \frac{1}{4} \text{ or } b_0 = -3b_\alpha = \frac{3}{4} \Longleftrightarrow \text{either } (1,1,\ldots,1) \text{ or } (-1,-1,\ldots,-1) \text{ is in } \mathcal{F},$$

$$b_0 = b_\alpha = \frac{1}{2}, b_0 = \frac{1}{3}b_\alpha = \frac{1}{4} \text{ or } b_0 = 3b_\alpha = \frac{3}{4} \Longleftrightarrow (1,1,\ldots,1),\ (-1,-1,\ldots,-1) \in \mathcal{F},$$

and

$$b_0 = -b_\alpha = \frac{1}{4}, \frac{1}{2} \text{ or } \frac{3}{4} \Longleftrightarrow (1,1,\ldots,1),\ (-1,-1,\ldots,-1) \notin \mathcal{F}.$$

Proof. The indicator polynomial function of the foldover design obtained by folding over on all the factors is $g(x) = 2b_0 + 2b_\alpha x^\alpha$. There exist y and z such that $g(y) = 2b_0 + 2b_\alpha$ and $g(z) = 2b_0 - 2b_\alpha$. Thus $b_0 = \frac{1}{4}(g(y) + g(z))$. Since $g(x)$ can be $0, 1$ or 2, $b_0 = \frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$.

Note that $b_\alpha \neq 0$, we get

$$b_0 = b_\alpha = \frac{1}{4} \text{ or } b_0 = -3b_\alpha = \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = \frac{1}{2},$$

$$b_0 = b_\alpha = \frac{1}{2}, b_0 = \frac{1}{3}b_\alpha = \frac{1}{4} \text{ or } b_0 = 3b_\alpha = \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = 1,$$

and

$$b_0 = -b_\alpha = \frac{1}{4}, \frac{1}{2} \text{ or } \frac{3}{4} \Longleftrightarrow b_0 + b_\alpha = 0,$$

by Proposition 2.3.3, we get the results. □

2.4 Indicator polynomial functions which contain two special words

In this section, we prove that when all the runs in a design have the same number of replicates, the indicator polynomial function of this design can not contain only two words. To prove this, we need the Remark 2.4.1 below.

Remark 2.4.1. Let x^α and x^β be two different words. Then, we can choose a point $y \in D_{2^m}$ such that $y^\alpha = y^\beta = \pm 1$ or $y^\alpha = -y^\beta = \pm 1$. This point can be chosen as follows:

1. When all the factors in x^α are also in x^β. Assume that x_i is in x^α and x_j is in x^β but not in x^α. We can choose a point y such that its ith entry is

± 1 and other entries are 1 so that $y^\alpha = y^\beta = \pm 1$ or its ith entry is ± 1, jth entry is -1 and other entries are 1 so that $y^\alpha = -y^\beta = \pm 1$.

2. When there exists a factor x_i which is in x^α but not in x^β and a factor x_j which is in x^β but not in x^α. We can choose a point y such that its ith and jth entries are ± 1 and other entries are 1 so that $y^\alpha = y^\beta = \pm 1$ or its ith entry is ± 1, jth entry is ∓ 1 and other entries are 1 so that $y^\alpha = -y^\beta = \pm 1$.

Now, we are ready to prove Theorem 2.4.1.

Theorem 2.4.1. *There is no two-level factorial design such that all the points in it have the same number of replicates and its indicator polynomial function has only two words.*

Proof. Assume that $f(x)$ is the indicator polynomial function of a design $\mathcal{F}_1$ which does not allow replicates. Let $\mathcal{F}_2$ be the design which contains the same points as $\mathcal{F}_1$ and each point has n replicates. Then, using the formula (1.2.2), it is easy to check that the indicator polynomial function of $\mathcal{F}_2$ is $nf(x)$. Thus, if $f(x)$ can not contain only two words, the indicator polynomial function of $\mathcal{F}_2$ also can not contain only two words.

Now we establish that $f(x)$ can not be only two words.

Assume that there exists a design such that its indicator polynomial function is $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta$. By Remark 2.4.1, we can choose a point y such that $y^\alpha = y^\beta = 1$ and a point z such that $z^\alpha = z^\beta = -1$. Then, $f(y) = b_0 + b_\alpha + b_\beta$ and $f(z) = b_0 - b_\alpha - b_\beta$, and thus, $b_0 = \frac{1}{2}(f(y) + f(z))$, which can only be $0, \frac{1}{2}$, and 1. But b_0 can not be 0 and 1, therefore $b_0 = \frac{1}{2}$. We can also choose another point h such that $h^\alpha = -h^\beta = 1$, then, we get $b_\beta = \frac{1}{2}(f(y) - f(h))$, which can be

$\pm\frac{1}{2}$. Similarly, $b_\alpha = \pm\frac{1}{2}$. Since x^α and x^β can only be 1 and -1, $f(x)$ can never be an integer.

We can also prove the result through the following three cases using Proposition 2.3.3 and Proposition 2.3.5:

1. If x^α is an odd word and x^β is an even word, then, by Proposition 2.3.3 and Proposition 2.3.5, $f(x) = \frac{1}{4} \pm \frac{1}{2}x^\alpha + \frac{1}{4}x^\beta$ or $\frac{3}{4} \pm \frac{1}{2}x^\alpha - \frac{1}{4}x^\beta$, which are impossible since when $x^\beta = -1$, $f(x)$ can not be an integer.

2. If both x^α and x^β are odd words, then, $\sum_{\alpha\in\Omega_1} b_\alpha = b_0 = \frac{1}{2}$. By Proposition 2.3.3, $b_\alpha + b_\beta$ can only be $\pm\frac{1}{2}$. Thus $f(x) = \frac{1}{2} + b_\alpha x^\alpha + (\pm\frac{1}{2} - b_\alpha)x^\beta$. Choosing a point y such that $y^\alpha = -1$ and $y^\beta = 1$, then $f(y) = (\frac{1}{2} \pm \frac{1}{2}) - 2b_\alpha$. Since $f(y)$ can only be 1 or 0, $b_\alpha = 0$ or $\frac{1}{2}$ if $f(y) = 1 - 2b_\alpha$ and $-\frac{1}{2}$ or 0 if $f(y) = -2b_\alpha$. Since b_α and b_β can not be 0, all the solutions are impossible.

3. If both x^α and x^β are even words, then, since $\sum_{\alpha\in\Omega_2} b_\alpha = 0$, by Proposition 2.3.3, $b_0 + b_\alpha + b_\beta = 1$ or 0.

   (a) If $b_0 + b_\alpha + b_\beta = 1$, then $f(x) = b_0 + b_\alpha x^\alpha + (1 - b_0 - b_\alpha)x^\beta$. When $x^\alpha = x^\beta = -1$, we get $b_0 = \frac{1}{2}$. Then $f(x) = \frac{1}{2} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta$, which is impossible by the proof of (2).

   (b) If $b_0 + b_\alpha + b_\beta = 0$, then $f(x) = b_0 + b_\alpha x^\alpha + (-b_0 - b_\alpha)x^\beta$. When $x^\alpha = x^\beta = -1$, we get $b_0 = \frac{1}{2}$. which is again impossible by the proof of (2).

$\square$

Remark 2.4.2. Proposition 2.3.1 implies that when all the runs in a design have the same number of replicates and $b_0 \neq \frac{1}{2}$, then there are at least two words in the indicator polynomial function. By Theorem 2.4.1, if $b_0 \neq \frac{1}{2}$, then there are at least three words in the indicator polynomial function.

Lemma 2.4.2. *If there exists a foldover plan of an un-replicated two-level factorial design such that the indicator polynomial function of the foldover design has only two words, then this design is a half fraction. Moreover, if the two words are x^α and x^β, then $b_\alpha = \pm\frac{1}{4}$ and $b_\beta = \pm\frac{1}{4}$.*

Proof. Assume that (1.2.1) is the indicator polynomial function of the original design and the indicator polynomial function of the foldover design is $g(x) = 2b_0 + 2b_\alpha x^\alpha + 2b_\beta x^\beta$. Then, we can choose $y, z \in D_{2^m}$ such that $g(y) = 2b_0 + 2b_\alpha + 2b_\beta$ and $g(z) = 2b_0 - 2b_\alpha - 2b_\beta$. So, $b_0 = \frac{1}{4}(g(y) + g(z))$. Since $g(x)$ can only be 0, 1, or 2, b_0 can only be $\frac{1}{4}, \frac{1}{2}$, or $\frac{3}{4}$. We can also choose $h \in D_{2^m}$ such that $g(h) = 2b_0 - 2b_\alpha + 2b_\beta$. So $b_\alpha = \frac{1}{4}(g(y) - g(h))$, which can be $\pm\frac{1}{4}$ and $\pm\frac{1}{2}$. Similarly, $b_\beta = \pm\frac{1}{4}, \pm\frac{1}{2}$.

1. When $b_0 = \frac{1}{4}$. If $|b_\alpha| = |b_\beta| = \frac{1}{2}$, then $g(x) = \frac{1}{2} \pm x^\alpha \pm x^\beta$, which can not be an integer. If $|b_\alpha| = |b_\beta| = \frac{1}{4}$, then $g(x) = \frac{1}{2} \pm \frac{1}{2}x^\alpha \pm \frac{1}{2}x^\beta$, which can also not be an integer. If one of b_α and b_β, say b_α, such that $|b_\alpha| = \frac{1}{2}$ and another one $|b_\beta| = \frac{1}{4}$, then $g(x) = \frac{1}{2} \pm x^\alpha \pm \frac{1}{2}x^\beta$, which may be negative for some points in D_{2^m}. Thus $b_0 \neq \frac{1}{4}$.

2. When $b_0 = \frac{1}{2}$. If $|b_\alpha| = |b_\beta| = \frac{1}{2}$, then $g(x) = 1 \pm x^\alpha \pm x^\beta$, which may be negative. If one of b_α and b_β, say b_α, such that $|b_\alpha| = \frac{1}{2}$ and another one $|b_\beta| = \frac{1}{4}$, then $g(x) = 1 \pm x^\alpha \pm \frac{1}{2}x^\beta$, which can never be an integer.

If $|b_\alpha| = |b_\beta| = \frac{1}{4}$, then $g(x) = 1 \pm \frac{1}{2}x^\alpha \pm \frac{1}{2}x^\beta$, which is always an integer between 0 and 2. Thus, when $b_0 = \frac{1}{2}$, $|b_\alpha| = |b_\beta| = \frac{1}{4}$.

3. When $b_0 = \frac{3}{4}$. If $|b_\alpha| = |b_\beta| = \frac{1}{2}$ or $\frac{1}{4}$, then we can similarly get $g(x)$ can not be an integer. If one of b_α and b_β, say b_α, such that $|b_\alpha| = \frac{1}{2}$ and another one $|b_\beta| = \frac{1}{4}$, then $g(x) = \frac{3}{2} \pm x^\alpha \pm \frac{1}{2}x^\beta$, which may equal to 3 for some points in D_{2^m}. Thus, $b_0 \neq \frac{3}{4}$.

□

Theorem 2.4.3 provides the coefficients of the two even words in more detail if there are two even words in the indicator polynomial function.

Theorem 2.4.3. *If the indicator polynomial function of a two-level un-replicated factorial design $\mathcal{F}$ has more than two words but only two of them are even words, say x^α and x^β, then this design must be a half fraction and*

$$\begin{cases} b_\alpha = -b_\beta = \pm\frac{1}{4} & \text{if } (1,1,\dots,1) \text{ or } (-1,-1,\dots,-1) \in \mathcal{F} \\ b_\alpha = b_\beta = \frac{1}{4} & \text{if } (1,1,\dots,1), (-1,-1,\dots,-1) \in \mathcal{F} \\ b_\alpha = b_\beta = -\frac{1}{4} & \text{if } (1,1,\dots,1), (-1,-1,\dots,-1) \notin \mathcal{F} \end{cases} \tag{2.4.1}$$

Proof. Assume that the indicator polynomial function of a design is $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + \sum_{\gamma \in \Omega_2} b_\gamma x^\gamma$, where x^α and x^β are even words. Then, the indicator polynomial function of the foldover design obtained by reversing the signs of all the factors is $g(x) = 2b_0 + 2b_\alpha x^\alpha + 2b_\beta x^\beta$. By Lemma 2.4.2, $b_0 = \frac{1}{2}$.

1. If $\sum_{\alpha \in \Omega_2} b_\alpha \neq 0$, by Proposition 2.3.3, $b_0 + b_\alpha + b_\beta = \frac{1}{2}$. Since $b_0 = \frac{1}{2}$, $b_\alpha + b_\beta = 0$. By Lemma 2.4.2, $b_\alpha = -b_\beta = \pm\frac{1}{4}$.

2. If $\sum_{\alpha \in \Omega_2} b_\alpha = 0$, by Proposition 2.3.3, $b_0 + b_\alpha + b_\beta = 1$ or 0.

   If $b_0 + b_\alpha + b_\beta = 1$, then $b_\alpha + b_\beta = \frac{1}{2}$. By Lemma 2.4.2, $b_\alpha = b_\beta = \frac{1}{4}$.

   If $b_0 + b_\alpha + b_\beta = 0$, then $b_\alpha + b_\beta = -\frac{1}{2}$. By Lemma 2.4.2, $b_\alpha = b_\beta = -\frac{1}{4}$.

□

When there are two odd words, say x^α and x^β, in the indicator polynomial function, it is hard to say their coefficients when either $(1, 1, \ldots, 1)$ or $(-1, -1, \ldots, -1)$ is in $\mathcal{F}$, but when both $(1, 1, \ldots, 1)$ and $(-1, -1, \ldots, -1)$ are either in $\mathcal{F}$ or not in $\mathcal{F}$ in the case of replicates, the sum of the two coefficients is equal to 0 by Proposition 2.3.3. Thus, $b_\alpha = -b_\beta$.

2.5 Indicator polynomial functions with only three words

In this section, we discuss indicator polynomial functions with only three words and give the classification of the indicator polynomial functions.

Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma$ is the indicator polynomial function of an un-replicated design $\mathcal{F}$. By Remark 2.4.1, given $\alpha, \beta \in \Omega$, there exists a point $x \in D_{2^m}$ such that x^α and x^β have either the same sign or different signs. Given α, β and x, i.e., given x^α and x^β, x^γ is either 1 or -1. The following claims will be used later in this section.

Claim 1 Given the indicator polynomial function $f(x)$, if there exist $y, z \in D_{2^m}$ such that $y^\alpha = z^\alpha$ and $y^\beta = z^\beta$, but $y^\gamma \neq z^\gamma$, then $b_\gamma = \pm\frac{1}{2}$.

By assumptions, $f(y) - f(z) = b_\gamma(y^\gamma - z^\gamma)$. Since $f(y) - f(z)$ can only be $0, 1$ and -1 and $y^\gamma - z^\gamma$ can only be ± 2, $b_\gamma = \pm\frac{1}{2}$.

Claim 2 There is no indicator polynomial function of three words satisfies $|b_\alpha| = |b_\beta| = |b_\gamma| = b_0 = \frac{1}{2}$.

In this case, $|b_\alpha/b_0| = |b_\beta/b_0| = |b_\gamma/b_0| = 1$, and thus by Proposition 1.2.1, the design is a regular design. But, then $|b_\alpha| = |b_\beta| = |b_\gamma| = b_0$ has to equal $\frac{1}{4}$.

Claim 3 There is no indicator polynomial function which has the form

$$f(x) = \frac{1}{2} + bx^\alpha - bx^\beta \pm \frac{1}{2}x^\gamma. \tag{2.5.1}$$

When x^α and x^β have the same sign, $f(x) = 0$ or 1. When x^α and x^β have different signs, $f(x) = 1 \pm 2b$ or $\pm 2b$. For $f(x)$ to be 0 or 1, $|b|$ has to equal $\frac{1}{2}$. This is impossible by *Claim 2*.

Now, we are ready to prove Theorem 2.5.1.

Theorem 2.5.1. *Assume that $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + b_\gamma x^\gamma$ is the indicator polynomial function of an un-replicated design $\mathcal{F}$. Then, either one or all of the three words are even words and $\mathcal{F}$ is either a $\frac{1}{4}$ fraction or a $\frac{3}{4}$ fraction. More specifically,*

1. *When there is only one even word, say x^γ,*

   (a) if $\mathcal{F}$ is a $\frac{1}{4}$ fraction, then, $f(x)$ has the forms:

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1, \ldots, 1) \in \mathcal{F}, (-1, \ldots, -1) \notin \mathcal{F} \\ \frac{1}{4} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma & \text{if } (1, \ldots, 1) \notin \mathcal{F}, (-1, \ldots, -1) \in \mathcal{F} \\ \frac{1}{4} \pm \frac{1}{4}x^\alpha \mp \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma & \text{if } (1, \ldots, 1), (-1, \ldots, -1) \notin \mathcal{F}. \end{cases} \tag{2.5.2}$$

(b) *if* $\mathcal{F}$ *is a* $\frac{3}{4}$ *fraction, then,* $f(x)$ *has the forms:*

$$f(x)=\begin{cases}\frac{3}{4}+\frac{1}{4}x^{\alpha}+\frac{1}{4}x^{\beta}-\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1)\in\mathcal{F},(-1,\ldots,-1)\notin\mathcal{F}\\ \frac{3}{4}-\frac{1}{4}x^{\alpha}-\frac{1}{4}x^{\beta}-\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1)\notin\mathcal{F},(-1,\ldots,-1)\in\mathcal{F}\\ \frac{3}{4}\mp\frac{1}{4}x^{\alpha}\pm\frac{1}{4}x^{\beta}+\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1),(-1,\ldots,-1)\in\mathcal{F}.\end{cases} \quad (2.5.3)$$

2. *When all the words are even words,*

(a) *if* $\mathcal{F}$ *is a* $\frac{1}{4}$ *fraction, then,* $f(x)$ *has the forms:*

$$f(x)=\begin{cases}\frac{1}{4}+\frac{1}{4}x^{\alpha}+\frac{1}{4}x^{\beta}+\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1),(-1,\ldots,-1)\in\mathcal{F}\\ \frac{1}{4}+\frac{1}{4}x^{\alpha}-\frac{1}{4}x^{\beta}-\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1),(-1,\ldots,-1)\notin\mathcal{F}.\end{cases} \quad (2.5.4)$$

(b) *if* $\mathcal{F}$ *is a* $\frac{3}{4}$ *fraction, then,* $f(x)$ *has the forms:*

$$f(x)=\begin{cases}\frac{3}{4}-\frac{1}{4}x^{\alpha}+\frac{1}{4}x^{\beta}+\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1),(-1,\ldots,-1)\in\mathcal{F}\\ \frac{3}{4}-\frac{1}{4}x^{\alpha}-\frac{1}{4}x^{\beta}-\frac{1}{4}x^{\gamma} & \textit{if } (1,\ldots,1),(-1,\ldots,-1)\notin\mathcal{F}.\end{cases} \quad (2.5.5)$$

Proof. 1. If all the three words are odd words, then, by Proposition 2.2.2, $b_0=\frac{1}{2}$. By Proposition 2.3.3, either $(1,1,\ldots,1)$ or $(-1,-1,\ldots,-1)$ is in $\mathcal{F}$ and

$$f(x)=\begin{cases}\frac{1}{2}+b_\alpha x^{\alpha}+b_\beta x^{\beta}+(\frac{1}{2}-b_\alpha-b_\beta)x^{\gamma} & \text{if } (1,1,\ldots,1)\in\mathcal{F}\\ \frac{1}{2}+b_\alpha x^{\alpha}+b_\beta x^{\beta}+(-\frac{1}{2}-b_\alpha-b_\beta)x^{\gamma} & \text{if } (-1,-1,\ldots,-1)\in\mathcal{F}.\end{cases}$$

(a) When $f(x)=\frac{1}{2}+b_\alpha x^{\alpha}+b_\beta x^{\beta}+(\frac{1}{2}-b_\alpha-b_\beta)x^{\gamma}$. Since there exists a point x such that $x^{\alpha}=-x^{\beta}=1$, if $x^{\gamma}=1$, then, $f(x)=1-2b_\beta$, which yields $b_\beta=\frac{1}{2}$. So $f(x)=\frac{1}{2}+b_\alpha x^{\alpha}+\frac{1}{2}x^{\beta}-b_\alpha x^{\gamma}$, which is the form (2.5.1), by *Claim 3*, this is impossible. If $x^{\gamma}=-1$, then, $f(x)=2b_\alpha$, which yields $b_\alpha=\frac{1}{2}$. So $f(x)=\frac{1}{2}+\frac{1}{2}x^{\alpha}+b_\beta x^{\beta}-b_\beta x^{\gamma}$. This is again the form (2.5.1).

(b) When $f(x) = \frac{1}{2} + b_\alpha x^\alpha + b_\beta x^\beta + (-\frac{1}{2} - b_\alpha - b_\beta)x^\gamma$. There exists a point x such that $x^\alpha = -x^\beta = 1$. If $x^\gamma = 1$, then, $f(x) = -2b_\beta$, which yields $b_\beta = -\frac{1}{2}$. So $f(x) = \frac{1}{2} + b_\alpha x^\alpha - \frac{1}{2}x^\beta - b_\alpha x^\gamma$, which is the form (2.5.1), by *Claim 3*, this is impossible. If $x^\gamma = -1$, then, $f(x) = 1 + 2b_\alpha$, which yields $b_\alpha = -\frac{1}{2}$. So $f(x) = \frac{1}{2} - \frac{1}{2}x^\alpha + b_\beta x^\beta - b_\beta x^\gamma$. This is also impossible by *Claim 3*.

2. If there are two even words, say x^α and x^β, in the three words, then, by Corollary 2.3.4 and Theorem 2.4.3, we get

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + \frac{1}{2}x^\gamma & \text{if } (1,\ldots,1) \in \mathcal{F}, (-1,\ldots,-1) \notin \mathcal{F} \\ \frac{1}{2} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{2}x^\gamma & \text{if } (1,\ldots,1) \notin \mathcal{F}, (-1,\ldots,-1) \in \mathcal{F}, \end{cases}$$

which has the form (2.5.1). By *Claim 3*, this is impossible.

3. If there is one even word, say x^γ, in the three words, then, by Proposition 2.3.3 and Proposition 2.3.5, the indicator polynomial function has the following possible forms.

(a) $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$ or $\frac{3}{4} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta - \frac{1}{4}x^\gamma$, if $(1, 1, \ldots, 1) \in \mathcal{F}$, $(-1, -1, \ldots, -1) \notin \mathcal{F}$.

i. When $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$, considering various cases, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$\frac{1}{2}$	impossible
2	-1	-1	1	0	
			-1	$-\frac{1}{2}$	impossible
3	1	-1	1	$2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$2b_\alpha - \frac{1}{2}$	$b_\alpha = \frac{1}{4}, \frac{3}{4}$
4	-1	1	1	$1 - 2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{1}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, -\frac{1}{4}$

Since $b_\gamma \neq \pm\frac{1}{2}$, by *Claim 1*, for each case, $x^\gamma \equiv 1$ or -1. From *Case 4*, $b_\alpha = \frac{1}{2}, \frac{1}{4}$ and $-\frac{1}{4}$. If $b_\alpha = \frac{1}{2}$, then, $b_\beta = \frac{1}{2} - b_\alpha = 0$, a contradiction. If $b_\alpha = -\frac{1}{4}$, then, in *Case 3*, $f(x)$ can not be an integer. If $b_\alpha = \frac{1}{4}$, then, $f(x)$ in other 3 cases can be 0 or 1. Thus $b_\alpha = \frac{1}{4}$. This gives the first form of (2.5.2).

ii. When $f(x) = \frac{3}{4} + b_\alpha x^\alpha + (\frac{1}{2} - b_\alpha)x^\beta - \frac{1}{4}x^\gamma$, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$\frac{3}{2}$	impossible
2	-1	-1	1	0	
			-1	$\frac{1}{2}$	impossible
3	1	-1	1	$2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$2b_\alpha + \frac{1}{2}$	$b_\alpha = \frac{1}{4}, -\frac{1}{4}$
4	-1	1	1	$1 - 2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{3}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, \frac{3}{4}$

Similarly, we get $b_\alpha = \frac{1}{4}$. This gives the first form of (2.5.3).

(b) $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (-\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$ or $\frac{3}{4} + b_\alpha x^\alpha + (-\frac{1}{2} - b_\alpha)x^\beta - \frac{1}{4}x^\gamma$, if $(1, 1, \ldots, 1) \notin \mathcal{F}$, $(-1, -1, \ldots, -1) \in \mathcal{F}$.

i. When $f(x) = \frac{1}{4} + b_\alpha x^\alpha + (-\frac{1}{2} - b_\alpha)x^\beta + \frac{1}{4}x^\gamma$. Considering various cases, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$-\frac{1}{2}$	impossible
2	-1	-1	1	1	
			-1	$\frac{1}{2}$	impossible
3	1	-1	1	$1 + 2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$\frac{1}{2} + 2b_\alpha$	$b_\alpha = -\frac{1}{4}, \frac{1}{4}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$-\frac{1}{2} - 2b_\alpha$	$b_\alpha = -\frac{1}{4}, -\frac{3}{4}$

Similar as the discussion in (3a), we get $b_\alpha = -\frac{1}{4}$. This gives the second form of (2.5.2).

ii. When $f(x) = \frac{3}{4} + b_\alpha x^\alpha + (-\frac{1}{2} - b_\alpha)x^\beta - \frac{1}{4}x^\gamma$, we have the table below.

$Case\ \#$	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$\frac{1}{2}$	impossible
2	-1	-1	1	1	
			-1	$\frac{3}{2}$	impossible
3	1	-1	1	$1+2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$\frac{3}{2}+2b_\alpha$	$b_\alpha = -\frac{1}{4}, -\frac{3}{4}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{1}{2}-2b_\alpha$	$b_\alpha = -\frac{1}{4}, \frac{1}{4}$

Similarly, we get $b_\alpha = -\frac{1}{4}$. This gives the second form of (2.5.3).

(c) $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{3}{4}x^\gamma$, $\frac{1}{2} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{1}{2}x^\gamma$ or $\frac{3}{4} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{1}{4}x^\gamma$, if $(1,1,\ldots,1), (-1,-1,\ldots,-1) \in \mathcal{F}$.

i. When $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{3}{4}x^\gamma$, we have the table below.

$Case\ \#$	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$-\frac{1}{2}$	impossible
2	-1	-1	1	1	
			-1	$-\frac{1}{2}$	impossible
3	1	-1	1	$1+2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$-\frac{1}{2}+2b_\alpha$	$b_\alpha = \frac{1}{4}, \frac{3}{4}$
4	-1	1	1	$1-2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$-\frac{1}{2}-2b_\alpha$	$b_\alpha = -\frac{1}{4}, -\frac{3}{4}$

From *Case 3* and *4*, $f(x)$ can not always be an integer for any b_α. Thus, $f(x)$ can not be this form.

ii. When $f(x) = \frac{1}{2} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{1}{2}x^\gamma$. This is the form (2.5.1) and, so, is impossible by *Claim 3*.

iii. When $f(x) = \frac{3}{4} + b_\alpha x^\alpha - b_\alpha x^\beta + \frac{1}{4}x^\gamma$, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$\frac{1}{2}$	impossible
2	-1	-1	1	1	
			-1	$\frac{1}{2}$	impossible
3	1	-1	1	$1 + 2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$\frac{1}{2} + 2b_\alpha$	$b_\alpha = -\frac{1}{4}, \frac{1}{4}$
4	-1	1	1	$1 - 2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{1}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, -\frac{1}{4}$

Since $b_\gamma = \frac{1}{4}$, similar as the discussion in (3a), we get $b_\alpha = \pm\frac{1}{4}$. This gives the third form of (2.5.3).

(d) $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{1}{4}x^\gamma$, $\frac{1}{2} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{1}{2}x^\gamma$ or $\frac{3}{4} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{3}{4}x^\gamma$, if $(1, 1, \ldots, 1), (-1, -1, \ldots, -1) \notin \mathcal{F}$.

i. When $f(x) = \frac{1}{4} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{1}{4}x^\gamma$, we have the table below.

$Case$ #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$\frac{1}{2}$	impossible
2	-1	-1	1	0	
			-1	$\frac{1}{2}$	impossible
3	1	-1	1	$2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{1}{2} + 2b_\alpha$	$b_\alpha = -\frac{1}{4}, \frac{1}{4}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$\frac{1}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, -\frac{1}{4}$

Since $b_\gamma = -\frac{1}{4}$, similar as the discussion in (3a), we get $b_\alpha = \pm\frac{1}{4}$. This gives the third form of (2.5.2).

ii. When $\frac{1}{2} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{1}{2}x^\gamma$. This is the form (2.5.1). By *Claim 3*, this is impossible.

iii. When $f(x) = \frac{3}{4} + b_\alpha x^\alpha - b_\alpha x^\beta - \frac{3}{4}x^\gamma$, we have the table below.

$Case$ #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$\frac{3}{2}$	impossible
2	-1	-1	1	0	
			-1	$\frac{3}{2}$	impossible
3	1	-1	1	$2b_\alpha$	$b_\alpha = \frac{1}{2}$
			-1	$\frac{3}{2} + 2b_\alpha$	$b_\alpha = -\frac{3}{4}, -\frac{1}{4}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$\frac{3}{2} - 2b_\alpha$	$b_\alpha = \frac{1}{4}, \frac{3}{4}$

From the *Case 3* and *4*, we can see that $f(x)$ can not be an integer for any b_α. Therefore, $f(x)$ can not have this form.

4. If all the three words are even words, then $\sum_{\alpha\in\Omega_2} b_\alpha = 0$. By Proposition 2.3.3, $f(x)$ has two possible forms.

   (a) $f(x) = b_0+b_\alpha x^\alpha+b_\beta x^\beta+(1-b_0-b_\alpha-b_\beta)x^\gamma$, if $(1,1,\ldots,1), (-1,-1,\ldots,-1) \in \mathcal{F}$.

   Considering various cases, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	1	
			-1	$2(b_0+b_\alpha+b_\beta)-1$	$b_0+b_\alpha+b_\beta=\frac{1}{2},1$
2	-1	-1	1	$1-2(b_\alpha+b_\beta)$	$b_\alpha+b_\beta=0,\frac{1}{2}$
			-1	$-1+2b_0$	$b_0=\frac{1}{2}$
3	1	-1	1	$1-2b_\beta$	$b_\beta=\frac{1}{2}$
			-1	$2(b_0+b_\alpha)-1$	$b_0+b_\alpha=\frac{1}{2},1$
4	-1	1	1	$1-2b_\alpha$	$b_\alpha=\frac{1}{2}$
			-1	$2(b_0+b_\beta)-1$	$b_0+b_\beta=\frac{1}{2},1$

   i. Show that $b_0 \neq \frac{1}{2}$. Assume that $b_0 = \frac{1}{2}$. Then, for *Case 4*, if $b_\alpha = \frac{1}{2}$, then $f(x)$ has the form (2.5.1), which is impossible; if $b_0 + b_\beta = \frac{1}{2}$, then, $b_\beta = 0$, a contradiction; if $b_0 + b_\beta = 1$, then, $b_\beta = \frac{1}{2}$ and $f(x)$ also has the form (2.5.1), which is impossible. Thus, in *Case 2*, that is, when $x^\alpha = x^\beta = -1$, x^γ must be 1, which needs

$$b_\alpha + b_\beta = 0 \qquad (2.5.6)$$

or

$$b_\alpha + b_\beta = \frac{1}{2}. \tag{2.5.7}$$

ii. Show that $b_\beta \neq \frac{1}{2}$. Assume that $b_\beta = \frac{1}{2}$. Then, since $b_0 \neq 0$, $b_\alpha + b_\beta \neq \frac{1}{2}$. So $b_\alpha + b_\beta = 0$ by (2.5.6) and (2.5.7) and, thus, $b_\alpha = -\frac{1}{2}$. Therefore, for Case 4, we have $b_0 + b_\beta = \frac{1}{2}$ or 1, which yields $b_0 = 0$ or $\frac{1}{2}$, respectively, a contradiction. Thus, in *Case 3*, that is, when $x^\alpha = -x^\beta = 1$, x^γ must be -1, which needs

$$b_0 + b_\alpha = \frac{1}{2} \tag{2.5.8}$$

or

$$b_0 + b_\alpha = 1. \tag{2.5.9}$$

iii. Show that $b_\alpha \neq \frac{1}{2}$. Assume that $b_\alpha = \frac{1}{2}$. Then, by (2.5.8) and (2.5.9), $b_0 = 0$ or $\frac{1}{2}$, which is impossible. Thus, in *Case 4*, that is, when $x^\alpha = -x^\beta = -1$, x^γ must be -1, which needs

$$b_0 + b_\beta = \frac{1}{2} \tag{2.5.10}$$

or

$$b_0 + b_\beta = 1. \tag{2.5.11}$$

iv. By (i), (2.5.6) and (2.5.7), we know that $b_0 + b_\alpha + b_\beta$ can not be $\frac{1}{2}$ or 1. Thus, in *Case 1*, that is, when $x^\alpha = x^\beta = 1$, x^γ must be 1 and $f(x) = 1$.

Now, by (2.5.6), (2.5.8) and (2.5.11), we get $b_0 = \frac{3}{4}$, $b_\alpha = -b_\beta = -b_\gamma = -\frac{1}{4}$, which gives the first form of (2.5.5). By (2.5.6), (2.5.9) and (2.5.10), we get $b_0 = \frac{3}{4}$, $b_\alpha = -b_\beta = b_\gamma = \frac{1}{4}$, which also gives the first form of (2.5.5). By (2.5.7), (2.5.8) and (2.5.10), we get $b_0 = b_\alpha = b_\beta = b_\gamma = \frac{1}{4}$, which gives the first form of (2.5.4). By (2.5.7), (2.5.9) and (2.5.11), we get $b_0 = \frac{3}{4}$, $b_\alpha = b_\beta = -b_\gamma = \frac{1}{4}$, which again gives the first form of (2.5.5). All the other combinations of the equations (2.5.6) or (2.5.7), (2.5.8) or (2.5.9), and (2.5.10) or (2.5.11) lead to the solutions with b_0 equals $\frac{1}{2}$, contradictions.

(b) $f(x) = b_0 + b_\alpha x^\alpha + b_\beta x^\beta + (-b_0 - b_\alpha - b_\beta)x^\gamma$, if $(1, 1, \ldots, 1), (-1, -1, \ldots, -1) \notin \mathcal{F}$.

Considering various cases, we have the table below.

Case #	x^α	x^β	x^γ	$f(x)$	$b_\alpha, b_\beta, b_\gamma$
1	1	1	1	0	
			-1	$2(b_0 + b_\alpha + b_\beta)$	$b_0 + b_\alpha + b_\beta = 0, \frac{1}{2}$
2	-1	-1	1	$-2(b_\alpha + b_\beta)$	$b_\alpha + b_\beta = 0, -\frac{1}{2}$
			-1	$2b_0$	$b_0 = \frac{1}{2}$
3	1	-1	1	$-2b_\beta$	$b_\beta = -\frac{1}{2}$
			-1	$2(b_0 + b_\alpha)$	$b_0 + b_\alpha = 0, \frac{1}{2}$
4	-1	1	1	$-2b_\alpha$	$b_\alpha = -\frac{1}{2}$
			-1	$2(b_0 + b_\beta)$	$b_0 + b_\beta = 0, \frac{1}{2}$

i. Show that $b_0 \neq \frac{1}{2}$. Assume that $b_0 = \frac{1}{2}$. Then, for *Case 4*, if $b_\alpha = -\frac{1}{2}$, then $f(x)$ has the form (2.5.1), which is impossible; if $b_0 + b_\beta = 0$, then, $b_\beta = -\frac{1}{2}$ and $f(x)$ also has the form (2.5.1), which is again impossible; $b_0 + b_\beta$ can not be $\frac{1}{2}$, since, then, $b_\beta = 0$.

Thus, when $x^{\alpha} = x^{\beta} = -1$, x^{γ} must be 1, which needs

$$b_{\alpha} + b_{\beta} = 0 \tag{2.5.12}$$

or

$$b_{\alpha} + b_{\beta} = -\frac{1}{2}. \tag{2.5.13}$$

ii. Show that $b_{\beta} \neq -\frac{1}{2}$. Assume that $b_{\beta} = -\frac{1}{2}$. Then $b_{\alpha}+b_{\beta} \neq -\frac{1}{2}$. So $b_{\alpha} + b_{\beta} = 0$ by (2.5.12) and (2.5.13) and, thus, $b_{\alpha} = \frac{1}{2}$. Therefore, for *Case 4*, we have $b_0 + b_{\beta} = 0$ or $\frac{1}{2}$, which yields $b_0 = \frac{1}{2}$ or 1, respectively, a contradiction. Thus, when $x^{\alpha} = -x^{\beta} = 1$, x^{γ} must be -1, which needs

$$b_0 + b_{\alpha} = 0 \tag{2.5.14}$$

or

$$b_0 + b_{\alpha} = \frac{1}{2}. \tag{2.5.15}$$

iii. Show that $b_{\alpha} \neq -\frac{1}{2}$. Assume that $b_{\alpha} = -\frac{1}{2}$. Then, by (2.5.14) and (2.5.15), $b_0 = \frac{1}{2}$ or 1, which is impossible. Thus, when $x^{\alpha} = -x^{\beta} = -1$, x^{γ} must be -1, which needs

$$b_0 + b_{\beta} = 0 \tag{2.5.16}$$

or

$$b_0 + b_{\beta} = \frac{1}{2}. \tag{2.5.17}$$

iv. By (i), (2.5.12) and (2.5.13), we know that $b_0 + b_{\alpha} + b_{\beta}$ can not be 0 or $\frac{1}{2}$. Thus, when $x^{\alpha} = x^{\beta} = 1$, x^{γ} must be 1 and $f(x) = 0$.

Now, by (2.5.12), (2.5.14) and (2.5.17), we get $b_0 = -b_{\alpha} = b_{\beta} = -b_{\gamma} = \frac{1}{4}$, which gives the second form of (2.5.4). By (2.5.12), (2.5.15) and

(2.5.16), we get $b_0 = b_\alpha = -b_\beta = -b_\gamma = \frac{1}{4}$, which also gives the second form of (2.5.4). By (2.5.13), (2.5.14) and (2.5.16), we get $b_0 = -b_\alpha = -b_\beta = b_\gamma = \frac{1}{4}$, which again gives the second form of (2.5.4). By (2.5.13), (2.5.15) and (2.5.17), we get $b_0 = \frac{3}{4}$, $b_\alpha = b_\beta = b_\gamma = -\frac{1}{4}$, which gives the second form of (2.5.5). All the other combinations of the equations (2.5.12) or (2.5.13), (2.5.14) or (2.5.15), and (2.5.16) or (2.5.17) lead to the solutions with b_0 equals $\frac{1}{2}$ or 0, contradictions.

□

By Proposition 1.2.3, the $\frac{3}{4}$ fractions with the forms of indicator polynomial functions in (2.5.3) and (2.5.5) are corresponding complementary fractions of the $\frac{1}{4}$ fractions with the forms of indicator polynomial functions in (2.5.2) and (2.5.4). For example, when there is one even word, the fraction $f(x) = \frac{3}{4} + \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma$ is the complementary fraction of the fraction $f(x) = \frac{1}{4} - \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma$; when all the three words are even words, the fraction $f(x) = \frac{3}{4} - \frac{1}{4}x^\alpha + \frac{1}{4}x^\beta + \frac{1}{4}x^\gamma$ is the complementary fraction of the fraction $f(x) = \frac{1}{4} + \frac{1}{4}x^\alpha - \frac{1}{4}x^\beta - \frac{1}{4}x^\gamma$.

Chapter 3

Connections Between the Resolutions of General Two-Level Factorial Designs

3.1 Introduction

Regular resolution III^* designs are regular resolution III designs in which no two-factor interactions are confounded with one another. Draper and Lin [14] showed that resolution III^* m-factor regular designs can be converted into resolution V $(m-1)$-factor regular designs and, conversely, resolution V m-factor regular designs can be converted into resolution III^* $(m+1)$-factor regular designs. In this chapter, using indicator polynomial functions, we not only extend these results to general two-level factorial designs, but also obtain even more general results.

Remember that a *resolution $N.^*x$ design* is a resolution $N.x$ design such that its indicator polynomial function contains no $(N+1)$-letter word.

In Section 3.2, we provide a way to convert resolution $(2l-1).^{*}x$ designs to resolution $(2l+1).x$ designs. A link between resolution $(2l-1).x$ designs and resolution $2l.x$ designs is also presented in this section. In Section 3.3, we show that resolution $III.^{*}x$ designs can be obtained from designs whose resolutions are equal or bigger than V.

3.2 Changing Resolutions by Converting a m-Factor Design into a $(m-1)$-Factor Design

In this section, we extend Theorems 1 and 2 of Draper and Lin [14] to general two-level factorial designs. A more general theorem is also proved in this section. As another special case of this theorem, a relation between designs of resolution $(2l-1).x$ and resolution $2l.x$ is also presented. For this purpose, we use the same transformations as Draper and Lin [14] used in their work.

Assume that $x_1, x_2, \ldots, x_m$ are the m factors that form a two-level factorial design $\mathcal{F}$ with the indicator polynomial function (1.2.1). Let x_k be any of the m factors and let

$$y_j = x_k x_j,\ j = 1, 2, \ldots, m\ (j \neq k). \tag{3.2.1}$$

Then $y_1, y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m$ form a $(m-1)$-factor two-level factorial design $\hat{\mathcal{F}}$. Define

$$\alpha'_i = \alpha_i,\ i = 1, 2, \ldots, m\ (i \neq k).$$

Then

$$x^{\alpha} = x_k^{\alpha_k}[\prod_{i \neq k}(x_k y_i)^{\alpha_i}] = x_k^{\|\alpha\|} y^{\alpha'}, \tag{3.2.2}$$

and

$$\| \alpha' \| = \begin{cases} \| \alpha \| & \text{if } \alpha_k = 0 \\ \| \alpha' \| - 1 & \text{if } \alpha_k = 1. \end{cases}$$

Lemma 3.2.1 gives the indicator polynomial function of $\hat{\mathcal{F}}$. Lemma 3.2.2 provides the resolution of $\hat{\mathcal{F}}$ and follows from Lemma 3.2.1 and the above discussion directly.

Lemma 3.2.1. *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the corresponding $(m-1)$-factor design. If (1.2.1) is the indicator polynomial function of $\mathcal{F}$, then*

$$g(y) = 2b_0 + 2 \sum_{\alpha \in \Omega_e^{\mathcal{F}}} b_\alpha y^{\alpha'}$$

is the indicator polynomial function of $\hat{\mathcal{F}}$.

Proof. By (3.2.2), the indicator polynomial function of $\mathcal{F}$ can be written as

$$f(x) = b_0 + x_k \sum_{\alpha \in \Omega_o^{\mathcal{F}}} b_\alpha y^{\alpha'} + \sum_{\alpha \in \Omega_e^{\mathcal{F}}} b_\alpha y^{\alpha'}.$$

This can also be seen as the indicator polynomial function of the design with the factors $y_1, y_2, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_m$. When it is projected onto $y_1, y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m$, the resulting projected design is $\hat{\mathcal{F}}$. By Theorem 1.2.4, the indicator polynomial function of the projected design is

$$g(y) = 2b_0 + 2 \sum_{\alpha \in \Omega_e^{\mathcal{F}}} b_\alpha y^{\alpha'}.$$

$\square$

Although $g(y)$ is only related to the even words in $f(x)$, the words in $g(y)$ can be odd words. When there is only one even word in $f(x)$, $g(y)$ has only one word. By Proposition 2.3.1, the design is a regular design. When all the words in $f(x)$ are odd words, $\hat{\mathcal{F}}$ is a full two-level $(m-1)$-factor design with $2b_0$ replicates for each point in $\hat{\mathcal{F}}$.

Lemma 3.2.2. *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the corresponding $(m-1)$-factor design. Assume that $2r$ is the number of letters in the shortest even word of $\Omega_e^{\mathcal{F}}$. Let*

$$A = \{\alpha \in \Omega_e^{\mathcal{F}} \mid \parallel \alpha \parallel = 2r\}.$$

Then, the resolution of $\hat{\mathcal{F}}$ is

$$R_{\hat{\mathcal{F}}} = \begin{cases} (2r-1).x & \text{if there exists an } \alpha \in A \text{ s.t. } \alpha_k = 1 \\ 2r.x & \text{otherwise.} \end{cases}$$

Theorem 3.2.3 shows the relation between the resolutions of the original design $\mathcal{F}$ and the resolutions of the transformed design $\hat{\mathcal{F}}$.

Theorem 3.2.3. *Let $\mathcal{F}$ be a m-factor two-level fractional factorial design with the indicator polynomial function (1.2.1). Assume that $2r$ is the number of letters in the shortest even word of $\Omega_e^{\mathcal{F}}$. Then regardless of what resolution $\mathcal{F}$ is, $\mathcal{F}$ can always be converted into a $(m-1)$-factor design $\hat{\mathcal{F}}$ of resolution $(2r-1).x$ in the same number of runs. If there exists a $k \in \{1, 2, \dots, m\}$ such that for any $\alpha \in A, \alpha_k \neq 1$, then $\mathcal{F}$ can be converted into a $(m-1)$-factor design $\hat{\mathcal{F}}$ of resolution $2r.x$.*

Proof. Let $\alpha \in A$. Take any k such that $\alpha_k = 1$. By Lemma 3.2.2, we get the first result. The second result follows from Lemma 3.2.2 directly. □

The following corollaries are obtained readily from Theorem 3.2.3. Corollary 3.2.4 is a generalization of Theorem 1.3.1 and Theorem 1.3.5 which were obtained by Draper and Lin [14], while Corollary 3.2.5 is an extension of Corollary 1.3.2 obtained by Draper and Lin [14].

Corollary 3.2.4. *Let $\mathcal{F}$ be a m-factor two-level fractional factorial design of resolution $(2l-1).^*x$. If there is a $(2l+2)$-letter word in the indicator polynomial function of $\mathcal{F}$, then $\mathcal{F}$ can be converted into a $(m-1)$-factor design $\hat{\mathcal{F}}$ of resolution $(2l+1).x$ in the same number of runs.*

Corollary 3.2.5. *Assume that a design of resolution $III.^*x$ can be converted into a design of resolution $V.x$. Then, if m is the maximum number of factors that can be accommodated in the design of resolution $III.^*x$, then the maximum number of factors that can be accommodated in the design of resolution $V.x$ with the same number of runs is at least $m-1$.*

Example 3.2.6. *An indicator polynomial function of a 6-factor design is*

$$\begin{aligned} f(x) &= \frac{1}{4}+\frac{1}{8}x_1x_4x_5+\frac{1}{8}x_2x_3x_6-\frac{1}{8}x_1x_5x_6-\frac{1}{8}x_2x_3x_4-\frac{1}{8}x_2x_5x_6 \\ &\quad -\frac{1}{8}x_1x_3x_6-\frac{1}{8}x_2x_4x_5-\frac{1}{8}x_1x_3x_4+\frac{1}{4}x_1x_2x_3x_4x_5x_6. \end{aligned}$$

This is a resolution $III.^*5$ design. Take, for example, $k=6$ (one can take any i, $i=1,2,\ldots,6$), that is, $y_i=x_6x_i$, $i=1,2,\ldots,5$. Since $f(x)$ only contains one even-letter word $x_1x_2x_3x_4x_5x_6$, by Corollary 3.2.4, $\mathcal{F}$ can be converted into a resolution V design and

$$g(y)=\frac{1}{2}+\frac{1}{2}y_1y_2y_3y_4y_5.$$

For the regular resolution III^* design, when there are at least two 3-letter words in the indicator polynomial function, there is always a 6-letter word in the indicator polynomial function. But when there is only one 3-letter word, it is possible that there is no 6-letter word in the indicator polynomial function. Draper and Lin pointed out in their Example 3 that when there is only one 3-letter word in the defining relation, one may get a resolution V design by deleting one variable in the 3-letter word.

Example 3.2.7 shows that when a resolution III^* design has only one 3-letter word in its indicator polynomial function, it may be converted into two designs with different resolutions. Example 3.2.8 shows that one can not possibly convert it to a resolution V design by deleting a variable in the 3-letter word, but possibly convert it to a design with resolution higher that V.

Example 3.2.7. *An indicator polynomial function of a 8-factor design with generators $x_7 = x_1x_4$ and $x_8 = x_2x_3x_5x_6$ is*

$$f(x) = \frac{1}{4} + \frac{1}{4}x_1x_4x_7 + \frac{1}{4}x_2x_3x_5x_6x_8 + \frac{1}{4}x_1x_2x_3x_4x_5x_6x_7x_8.$$

This is a resolution III^* design and $2r = 8$. If we take $k = 1$, then after the transformation (3.2.1), $\mathcal{F}$ is converted into the design of resolution VII and its indicator polynomial function is

$$g(y) = \frac{1}{2} + \frac{1}{2}y_2y_3y_4y_5y_6y_7y_8.$$

If we delete, for example, the variable x_1 as in Example 3 of Draper and Lin [14], then this design can also be converted into the design of resolution V and its indicator polynomial function is

$$h(y) = \frac{1}{2} + \frac{1}{2}y_2y_3y_5y_6y_8.$$

Example 3.2.8. *An indicator polynomial function of a 10-factor design with generators $x_9 = x_1x_4$ and $x_{10} = x_2x_3x_5x_6x_7x_8$ is*

$$f(x) = \frac{1}{4} + \frac{1}{4}x_1x_4x_9 + \frac{1}{4}x_2x_3x_5x_6x_7x_8x_{10} + \frac{1}{4}x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}.$$

This design can not be converted into a design of resolution V by removing one variable in the 3-letter word as Example 3 of Draper and Lin [14]. If we *delete, for example, the variable x_1, then the indicator polynomial function of the resulting design is*

$$g(y) = \frac{1}{2} + \frac{1}{2} y_2 y_3 y_5 y_6 y_7 y_8 y_{10},$$

which is a resolution VII design.

Example 3.2.9 shows that the condition, that a $(2l+2)$-letter word is needed in the indicator polynomial function of $\mathcal{F}$ in Corollary 3.2.4, is necessary even in the regular case.

Example 3.2.9. *An indicator polynomial function of a 10-factor design with generators $x_9 = x_1x_4x_5x_7$ and $x_{10} = x_2x_3x_6x_8$ is*

$$f(x) = \frac{1}{4} + \frac{1}{4}x_1x_4x_5x_7x_9 + \frac{1}{4}x_2x_3x_6x_8x_{10} + \frac{1}{4}x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}.$$

This is a resolution V^* design and $2r = 10$. If we take $k = 2$, then after the transformation (3.2.1), $\mathcal{F}$ is converted into a resolution IX (not VII) design and its indicator polynomial function is

$$g(y) = \frac{1}{2} + \frac{1}{2} y_1 y_3 y_4 y_5 y_6 y_7 y_8 y_9 y_{10}.$$

Corollaries 3.2.10 and 3.2.11 provide connections between two-level designs of resolution $(2l-1).x$ and resolution $2l.x$. In particular, when $l = 2$, they show connections between two-level designs of resolution $III.x$ and resolution $IV.x$.

Corollary 3.2.10. *Let $\mathcal{F}$ be a two-level m-factor design of resolution $(2l-1).x$. If there is a $2l$-letter word in the indicator polynomial function of $\mathcal{F}$ and there exists a $k \in \{1, 2, \ldots, m\}$ such that for any $\alpha \in A, \alpha_k \neq 1$, then $\mathcal{F}$ can be converted into a $(m-1)$-factor design $\hat{\mathcal{F}}$ of resolution $2l.x$ in the same number of runs.*

Corollary 3.2.11. *Let $\mathcal{F}$ be a two-level m-factor design of resolution $2l.x$. Then, $\mathcal{F}$ can be converted into a $(m-1)$-factor design $\hat{\mathcal{F}}$ of resolution $2l-1.x$ in the same number of runs.*

Example 3.2.12. *An indicator polynomial function of a 7-factor regular design with generators $x_5 = x_1x_2x_4$, $x_6 = x_1x_3$, and $x_7 = x_2x_3$ is*

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{1}{8}x_1x_3x_6 + \frac{1}{8}x_2x_3x_7 + \frac{1}{8}x_1x_2x_4x_5 + \frac{1}{8}x_4x_5x_6x_7 \\ &\quad + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_2x_3x_4x_5x_6 + \frac{1}{8}x_1x_3x_4x_5x_7. \end{aligned}$$

This is a resolution III design. Since there exists a k $(= 3)$ such that x_3 is not in all the 4-letter words, $\mathcal{F}$ can be converted into a 6-factor design of resolution IV and its indicator polynomial function is

$$g(y) = \frac{1}{4} + \frac{1}{4}y_1y_2y_4y_5 + \frac{1}{4}y_4y_5y_6y_7 + \frac{1}{4}y_1y_2y_6y_7.$$

Example 3.2.13. *An indicator polynomial function of a 7-factor design is*

$$f(x) = \frac{3}{4} + \frac{1}{4}x_1x_3x_4x_7 + \frac{1}{4}x_1x_2x_4x_5 + \frac{1}{4}x_2x_3x_5x_7 + \frac{1}{2}x_2x_3x_4x_6x_7.$$

This is a resolution $4\frac{2}{3}$ design. If we take $k = 1$, then $\mathcal{F}$ is converted into a 6-factor design of resolution III and its indicator polynomial function is

$$g(y) = \frac{3}{2} + \frac{1}{2}y_3y_4y_7 + \frac{1}{2}y_2y_4y_5 + \frac{1}{2}y_2y_3y_5y_7.$$

3.3 Obtaining a Resolution III^* Design by Converting a m-Factor Design into a $(m+1)$-Factor Design

Assume that

$$y_i = (x_k x_l)x_i, \ i = 1, 2, \ldots, m, \ y_{m+1} = x_k x_l.$$

Then, $y_1, y_2, \ldots, y_{m+1}$ form a $(m+1)$-factor two-level factorial design and

$$x_l = y_k, \ x_k = y_l, \text{ and } x_i = y_k y_l y_i, \ \forall i \neq k, l.$$

Since

$$x^\alpha = [\prod_{\substack{i=1 \\ i\neq k,l}}^{m} (y_k y_l y_i)^{\alpha_i}] y_k^{\alpha_l} y_l^{\alpha_k} = [\prod_{\substack{i=1 \\ i\neq k,l}}^{m} (y_i)^{\alpha_i}] y_k^{\|\alpha\|-\alpha_k} y_l^{\|\alpha\|-\alpha_l},$$

we define

$$\alpha'_i = \begin{cases} \alpha_i, & \text{if } 1 \leq i \leq m \text{ and } i \neq k, l, \\ \| \alpha \| - \alpha_k (\text{mod} 2), & \text{if } i = k, \\ \| \alpha \| - \alpha_l (\text{mod} 2), & \text{if } i = l, \\ 0, & \text{if } i = m+1. \end{cases} \tag{3.3.1}$$

Then $x^\alpha = y^{\alpha'}$.

Lemma 3.3.1 gives the indicator polynomial function of the transformed design.

Lemma 3.3.1. *Let $\mathcal{F}$ be a two-level m-factor design and $\hat{\mathcal{F}}$ be the transformed $(m+1)$-factor design. If (1.2.1) is the indicator polynomial function of $\mathcal{F}$, then the indicator polynomial function of $\hat{\mathcal{F}}$ is*

$$g(y) = \frac{1}{2}(\sum_{\alpha \in L_{2^m}} b_\alpha y^{\alpha'} + \sum_{\alpha \in L_{2^m}} b_\alpha y^{\alpha' + \varphi}),$$

where φ is a $1 \times (m+1)$ vector such that k, l and $(m+1)$-th entries are 1 and all others are 0.

Proof. The indicator polynomial function of the $2^{(m+1)-1}$ design with defining relation $y_{m+1} = y_k y_l$ is

$$f_0(y) = \frac{1}{2}(1 + y_k y_l y_{m+1}).$$

From above discussion, the right hand side of (1.2.1) can be written as $\sum_{\alpha \in L_{2m}} b_\alpha y^{\alpha'}$. Thus, the indicator polynomial function of the design formed by the factors $y_1, y_2, \ldots, y_m$ is

$$f_1(y) = \sum_{\alpha \in L_{2m}} b_\alpha y^{\alpha'}.$$

Therefore, the indicator polynomial function of $\hat{\mathcal{F}}$ is

$$\begin{aligned} g(y) &= f_0(y) f_1(y) \\ &= \frac{1}{2}(1 + y_k y_l y_{m+1}) \sum_{\alpha \in L_{2m}} b_\alpha y^{\alpha'} \\ &= \frac{1}{2}\Big(\sum_{\alpha \in L_{2m}} b_\alpha y^{\alpha'} + \sum_{\alpha \in L_{2m}} b_\alpha y^{\alpha' + \varphi}\Big). \end{aligned}$$

□

The following theorem is a generalization of Theorem 1.3.3 which was obtained by Draper and Lin [14].

Theorem 3.3.2. *Any m-factor two-level fractional factorial design $\mathcal{F}$ of resolution V or bigger can be converted into a $(m+1)$-factor design $\hat{\mathcal{F}}$ of resolution III^*.*

Proof. By (3.3.1),

$$\| \alpha' \| = \begin{cases} \| \alpha \| - 2, & \text{if } \| \alpha \| \text{ is odd, and } \alpha_k = 1, \alpha_l = 1, \\ \| \alpha \| + 2, & \text{if } \| \alpha \| \text{ is odd, and } \alpha_k = 0, \alpha_l = 0, \\ \| \alpha \|, & \text{otherwise,} \end{cases} \tag{3.3.2}$$

$$(\alpha' + \varphi)_i = \begin{cases} \alpha_i, & \text{if } 1 \leq i \leq m \text{ and } i \neq k, l, \\ \| \alpha \| - \alpha_k + 1 (\text{mod } 2), & \text{if } i = k, \\ \| \alpha \| - \alpha_l + 1 (\text{mod } 2), & \text{if } i = l, \\ 1, & \text{if } i = m + 1, \end{cases}$$

and therefore

$$\| \alpha' + \varphi \| = \begin{cases} \| \alpha \| - 1, & \text{if } \| \alpha \| \text{ is even, and } \alpha_k = 1, \alpha_l = 1, \\ \| \alpha \| + 3, & \text{if } \| \alpha \| \text{ is even, and } \alpha_k = 0, \alpha_l = 0 \\ \| \alpha \| + 1, & \text{otherwise.} \end{cases} \tag{3.3.3}$$

By (3.3.2) and (3.3.3), for any $\alpha \in \Omega^{\mathcal{F}}$ such that $\| \alpha \| \geq 5$, $\| \alpha \|$ and $\| \alpha' + \varphi \|$ are all at least 3 but not equal to 4. From Lemma 3.3.1, the indicator polynomial function of $\hat{\mathcal{F}}$ has a word $y^{\varphi} = y_k y_l y_{m+1}$ and its length is 3. Thus, the resolution of $\hat{\mathcal{F}}$ is III^*. □

The following corollary is an extension of Corollary 1.3.4 obtained by Draper and Lin [14].

Corollary 3.3.3. *If $(m - 1)$ is the maximum number of factors that can be accommodated in a design $\mathcal{F}$ of resolution $V.x$, then the maximum number of factors that can be accommodated in a resolution III^* design with the same number of runs is at least m.*

Example 3.3.4. *An indicator polynomial function of a two-level 9-factor design is*

$$f(x) = \frac{3}{4} + \frac{1}{4}x_1x_3x_4x_6x_7 + \frac{1}{4}x_2x_3x_5x_6x_9 + \frac{1}{4}x_1x_2x_4x_5x_7x_9 + \frac{1}{2}x_1x_2x_5x_6x_8x_9.$$

If we take $k = 1$ and $l = 6$, then $x_1x_3x_4x_6x_7 = y_3y_4y_7$, $x_2x_3x_5x_6x_9 = y_1y_2y_3y_5y_9$, $x_1x_2x_4x_5x_7x_9 = y_1y_2y_4y_5y_7y_9$, and $x_1x_2x_5x_6x_8x_9 = y_1y_2y_5y_6y_8y_9$. Thus, by Lemma 3.3.1, the indicator polynomial function of the transformed design is

$$\begin{aligned} g(y) &= \frac{3}{8} + \frac{3}{8}y_1y_6y_{10} + \frac{1}{8}y_3y_4y_7 + \frac{1}{8}y_1y_2y_3y_5y_9 + \frac{1}{8}y_1y_2y_4y_5y_7y_9 + \frac{1}{4}y_1y_2y_5y_6y_8y_9 \\ &\quad + \frac{1}{8}y_1y_3y_4y_6y_7y_{10} + \frac{1}{8}y_2y_3y_5y_6y_9y_{10} + \frac{1}{4}y_2y_5y_8y_9y_{10} + \frac{1}{8}y_2y_4y_5y_6y_7y_9y_{10}. \end{aligned}$$

The word lengths of the 3-letter words $y_1y_6y_{10}$ and $y_3y_4y_7$ are 3 and $3\frac{2}{3}$, respectively. Thus, the resolution of the transformed design is III. Since there is no 4-letter word in its indicator polynomial function, the transformed design is therefore of resolution III^*.

If we take $k = 4$ and $l = 8$ (note that x_8 is not in any 5-letter word), then $x_1x_3x_4x_6x_7 = y_1y_3y_6y_7y_8$, $x_2x_3x_5x_6x_9 = y_2y_3y_4y_5y_6y_8y_9$, $x_1x_2x_4x_5x_7x_9 = y_1y_2y_4y_5y_7y_9$, and $x_1x_2x_5x_6\ x_8x_9 = y_1y_2y_5y_6y_8y_9$. Thus, the indicator polynomial function of the transformed design is

$$\begin{aligned} g(y) &= \frac{3}{8} + \frac{3}{8}y_4y_8y_{10} + \frac{1}{8}y_1y_3y_6y_7y_8 + \frac{1}{4}y_1y_2y_5y_6y_8y_9 + \frac{1}{8}y_1y_2y_4y_5y_7y_9 \\ &\quad + \frac{1}{8}y_1y_3y_4y_6y_7y_{10} + \frac{1}{8}y_2y_3y_5y_6y_9y_{10} + \frac{1}{4}y_1y_2y_4y_5y_6y_9y_{10} \\ &\quad + \frac{1}{8}y_2y_3y_4y_5y_6y_8y_9 + \frac{1}{8}y_1y_2y_5y_7y_8y_9y_{10}. \end{aligned}$$

There is only one 3-letter word $y_4y_8y_{10}$ and its word length is 3. Note that there is no 4-letter word in the indicator polynomial function and hence this is also a *resolution III^* design. However, if we compare the two transformed designs by the minimum aberration criteria, the second design is better since it has only one 3-letter word.*

Hence, when we choose k or l, it is better to choose the one whose factor is not contained in any 5-letter word.

Chapter 4

Indicator Polynomial Functions of Partial Foldover Designs

4.1 Introduction

Partial foldover designs save half or more of the original runs comparing to corresponding full foldover designs. The new runs which are added to the original design is used to de-alias some effects. Mee and Peralta [20] studied various possible semifoldover regular designs. John [18] considered to add a fraction of $\frac{1}{4}$ original runs to the original regular design. We study partial foldover general two-level factorial designs, regular or non-regular.

The powerful tool we use for this purpose is indicator polynomial functions. As we mentioned earlier in Section 1.2, indicator polynomial functions provide alias structure of the designs, that is, any word in the indicator polynomial functions implies an alias relation. Conversely, if a word is not in the indicator polynomial function, then the alias relations caused by this word are de-aliased. Note that for a regular design, if two effects are aliased with the same effect, then this two effects are also aliased; but for a non-regular design, this is not true in general.

For example ([15]), an indicator polynomial function of a 5-factor non-regular design is:

$$f(x) = \frac{1}{2} + \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2x_4 + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_1x_2x_4x_5.$$

Although x_3 and x_4 are partially aliased with x_1x_2, x_3 and x_4 are not (partially) aliased. Also, x_3x_5 and x_4x_5 are partially aliased with x_1x_2, but x_3x_5 and x_4x_5 are not (partially) aliased since x_3x_4 is not in the indicator polynomial function.

In Section 4.2, we study the indicator polynomial functions of semifoldover designs. In Section 4.3, we extend some results in Section 4.2 to a more general case and, especially, the indicator polynomial functions of partial foldover designs obtained by adding a fraction of $\frac{1}{4}$ original runs are obtained.

4.2 Indicator Polynomial Functions of Semifoldover Designs

In this section, we study indicator polynomial functions of semifoldover designs.

Lemma 4.2.1 below is obtained directly from the properties in Section 1.4, and this result will be utilized later. One can see that the words that are left are those in the original design which are not sign-reversed in the new fraction. This shows that the "Foldover Rule 1" in Montgomery [21] is also true for non-regular designs. Thus, foldover of a non-regular design can also de-alias all (or as many as possible) the two-factor interactions which contain factors of interest as foldover of a regular design.

Lemma 4.2.1. *Assume that (1.5.2) is the indicator polynomial function of $\mathcal{F}$. If we fold over on $x_1, x_2, \ldots, x_r$, then the indicator polynomial function of the full foldover design is*

$$f_c(x) = 2 \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha. \tag{4.2.1}$$

Lemma 4.2.2 below is useful for getting indicator polynomial functions of semifoldover designs, and is easy to verify.

Lemma 4.2.2. *Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be fractions of a two-level factorial design and $f_a(x)$ and $f_b(x)$ be the corresponding indicator polynomial functions, respectively. Then, the indicator polynomial function of $\mathcal{F}_a \cap \mathcal{F}_b$ is $f_a(x) f_b(x)$.*

Lemma 4.2.3 provides the indicator polynomial function of $\mathcal{F}^{(e)}$.

Lemma 4.2.3. *Let (1.5.2) is the indicator polynomial function of $\mathcal{F}$, then*

$$f^{(e)}(x) = \frac{1}{2}(1 + ez) f(x) = \frac{1}{2}(1 + ez)\Big(\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha\Big) \tag{4.2.2}$$

is the indicator polynomial function of $\mathcal{F}^{(e)}$.

Proof. The indicator polynomial function of the half fraction which satisfies $ez = 1$ (or $z = e$) is $f_h(x) = 1/2 + ez/2$. By Lemma 4.2.2, the indicator polynomial function of $\mathcal{F}^{(e)}$ is

$$f^{(e)}(x) = f(x) f_h(x) = \frac{1}{2}(1 + ez) f(x).$$

We can also prove this in the following way by using the definition of indicator polynomial functions:

Since $\mathcal{F} = \mathcal{F}^{(e)} \cup \mathcal{F}^{(-e)}$, $D_{2^m} = \mathcal{F}^{(e)} \cup \mathcal{F}^{(-e)} \cup (D_{2^m} \setminus \mathcal{F})$. To prove (4.2.2) is the indicator polynomial function of $\mathcal{F}^{(e)}$, we consider the value of $f^{(e)}(x)$ in the three subsets separately.

1. If $x \in \mathcal{F}^{(e)}$, then $x \in \mathcal{F}$ and $z = e$. It follows that $f(x) = 1$ and $ez = 1$. Thus $f^{(e)}(x) = f(x) = 1$.

2. If $x \in \mathcal{F}^{(-e)}$, then $z = -e$. It yields $ez = -1$ and, therefore, $f^{(e)}(x) = 0$.

3. If $x \in D_{2^m} \setminus \mathcal{F}$, then $f(x) = 0$. It follows that $f^{(e)}(x) = 0$.

By (1),(2) and (3), we obtain

$$f^{(e)}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{F}^{(e)} \\ 0 & \text{if } x \in D_{2^m} \setminus \mathcal{F}^{(e)}. \end{cases}$$

By the uniqueness of the indicator polynomial function, we know that $f^{(e)}(x)$ is the indicator polynomial function of $\mathcal{F}^{(e)}$.

□

Remember that $\mathcal{F}_o$ is the new fraction obtained by reversing the signs of $x_1, x_2, \ldots, x_r$. Note that for regular designs, number of runs in the fraction $\mathcal{F}_o^{(e)}$ is exactly half runs of the original design. But for non-regular designs, number of runs in the fraction $\mathcal{F}_o^{(e)}$ may be less or more than half runs of the original design.

Proposition 4.2.4 below provides the indicator polynomial function of the semifoldover design. It shows that the words in the two semifoldover designs obtained by adding the fraction $\mathcal{F}_o^{(1)}$ or the fraction $\mathcal{F}_o^{(-1)}$ to the original design are the same. Thus, the two semifoldover designs have the same alias sets.

Proposition 4.2.4. *Assume that (1.5.2) is the indicator polynomial function of $\mathcal{F}$. If we subset on a main effect or an interaction effect ez, $e = 1, -1$, then the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is*

$$f_1(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + ez \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha.$$

The indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ *is*

$$f_2(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - ez \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha.$$

Proof. By Property (1) in Section 1.4, the indicator polynomial function of $\mathcal{F}_o$ is

$$f_o(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha.$$

If we subset on ez, then, by Lemma 4.2.3, the indicator polynomial function of $\mathcal{F}_o^{(e)}$ is given by

$$f_o^{(e)}(x) = \frac{1}{2}\{\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha + ez \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - ez \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha\}.$$

By (4.2.2) and the Property (2) in Section 1.4, the results follow easily. □

Note that any word in $f_c(x)$ is also in $f_1(x)$ and $f_2(x)$ and, consequently, all the alias relations in the full foldover design are still in the semifoldover design. Thus, the semifoldover design can not de-alias any additional two-factor interactions than the full foldover design.

It is well known that for a regular design $\mathcal{F}$, its run size must have the form 2^{m-p} if there are m factors and p generators, thus, the combined fraction $\mathcal{F} \cup \mathcal{F}_o^{(e)}$ is a non-regular design since its run size is $2^{m-p} + \frac{1}{2} \cdot 2^{m-p} = \frac{3}{2} \cdot 2^{m-p}$. But for a non-regular design $\mathcal{F}$, it is hard to see if the run size of the combined fraction $\mathcal{F} \cup \mathcal{F}_o^{(e)}$ has the form 2^{m-p}, so we can not tell whether it can be a regular design. Proposition 4.2.5 provides the indicator polynomial function of $\mathcal{F} \cup \mathcal{F}_o^{(e)}$ which allow us to answer this question easily.

Proposition 4.2.5. *Assume that (1.5.2) is the indicator polynomial function of* $\mathcal{F}$*. If we subset on* ez, $e = 1, -1$, *then the indicator polynomial function* $f_s(x) = \sum_{\alpha \in L_{2m}} c_\alpha x^\alpha$ *of the combined semifoldover design* $\mathcal{F} \cup \mathcal{F}_o^{(e)}$ *is*

$$f_s(x) = \frac{3}{2} \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + \frac{1}{2} \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha + \frac{1}{2} ez(- \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha + \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha). \qquad (4.2.3)$$

Proof. By the proof of Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}_o^{(e)}$ is

$$f_o^{(e)}(x) = \frac{1}{2}\{\sum_{\alpha\in\Omega_e} b_\alpha x^\alpha - \sum_{\alpha\in\Omega_o} b_\alpha x^\alpha + ez\sum_{\alpha\in\Omega_e} b_\alpha x^\alpha - ez\sum_{\alpha\in\Omega_o} b_\alpha x^\alpha\}.$$

Thus, the indicator polynomial function of the combined semifoldover design $\mathcal{F}\cup\mathcal{F}_o^{(e)}$ is $f_s(x) = f(x) + f_o^{(e)}(x)$, which is (4.2.3). □

One can see that the constant of the indicator polynomial function (4.2.3) is $c_0 = \frac{3}{2}b_0$, but other coefficients are $c_\alpha = \frac{1}{2}b_\alpha$ or $-\frac{1}{2}b_\alpha$. Thus, by Proposition 1.2.1, (4.2.3) represents a regular design if and only if

$$|c_\alpha/c_0| = 1$$

and if and only if

$$|b_\alpha/b_0| = 3,$$

which is impossible by (1.2.3). Therefore, this combined fraction can never be a regular design no matter the original design is a regular design or a non-regular design.

4.3 Extensions

In Section 4.2, we obtained the indicator polynomial functions of semifoldover design. In this section, we consider the addition of a smaller fraction to the original fraction and extend some results in Section 4.2 to a more general case.

It is easy to check that Lemma 4.2.2 can be extended to a more general case as follows.

Lemma 4.3.1. *Let $\mathcal{F}_1$, $\mathcal{F}_2, \cdots, \mathcal{F}_p$ be fractions of a two-level factorial design and $f_1(x), f_2(x), \cdots, f_p(x)$ be the corresponding indicator polynomial functions. Then, the indicator polynomial function of $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \cdots \cap \mathcal{F}_p$ is $f_1(x)f_2(x)\cdots f_p(x)$.*

Let

$$\mathcal{F}^{(e_1,e_2,\ldots,e_p)} = \{x \in \mathcal{F} \mid z_1 = e_1, z_2 = e_2, \ldots, z_r = e_p\}, \tag{4.3.1}$$

where $e_i = 1$ or -1, $i = 1, 2, \ldots, p$, and z_i be a main effect or an interaction. We can add a smaller fraction $\mathcal{F}^{(e_1,e_2,\ldots,e_p)}$ to the original design to get a partial foldover design.

Lemma 4.3.2 provides the indicator polynomial function of $\mathcal{F}^{(e_1,e_2,\ldots,e_p)}$.

Lemma 4.3.2. *Assume that (1.5.2) is the indicator polynomial function of $\mathcal{F}$, then*

$$f^{(e_1,e_2,\ldots,e_p)}(x) = \frac{1}{2^p} f(x) \prod_{i=1}^{p} (1 + e_i z_i) \tag{4.3.2}$$

is the indicator polynomial function of $\mathcal{F}^{(e_1,e_2,\ldots,e_p)}$.

Proof. The indicator polynomial function of the half fraction which satisfies $e_i z_i = 1$ (or $z_i = e_i$) is $f_i(x) = \frac{1}{2}(1 + e_i z_i)$. By Lemma 4.3.1, we get the results. $\square$

From the proof of Proposition 4.2.4, we know that the indicator polynomial function of $\mathcal{F}_o$ is

$$f_o(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha.$$

Thus, by Lemma 4.3.2, the indicator polynomial function of $\mathcal{F}_o^{(e_1,e_2,\ldots,e_p)}$ is given by

$$f_o^{(e_1,e_2,\ldots,e_p)}(x) = \frac{1}{2^p} f_o(x) \prod_{i=1}^{p} (1 + e_i z_i). \tag{4.3.3}$$

By (4.3.2) and (4.3.3), we can get indicator polynomial functions of partial foldover designs.

Proposition 4.3.3 below provides the indicator polynomial functions of the double semifoldover designs which is obtained by adding $\frac{1}{4}$ fraction to the original design.

Proposition 4.3.3. *Assume that (1.5.2) is the indicator polynomial function of $\mathcal{F}$. If we subset on $e_j z_j$, $j = 1, 2$, then the indicator polynomial function of $\mathcal{F}^{(e_1,e_2)} \cup \mathcal{F}_o^{(e_1,e_2)}$ is*

$$f_1(x) = \frac{1}{2}(1 + e_1 z_1)(1 + e_2 z_2) \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha.$$

The indicator polynomial function of $\mathcal{F}^{(e_1,-e_2)} \cup \mathcal{F}_o^{(e_1,e_2)}$ is

$$f_2(x) = \frac{1}{2}(1 + e_1 z_1)(\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e_2 z_2 \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha).$$

The indicator polynomial function of $\mathcal{F}^{(-e_1,e_2)} \cup \mathcal{F}_o^{(e_1,e_2)}$ is

$$f_3(x) = \frac{1}{2}(1 + e_2 z_2)(\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e_1 z_1 \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha).$$

The indicator polynomial function of $\mathcal{F}^{(-e_1,-e_2)} \cup \mathcal{F}_o^{(e_1,e_2)}$ is

$$f_4(x) = \frac{1}{2}[(1 + e_1 e_2 z_1 z_2) \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - (e_1 z_1 + e_2 z_2) \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha].$$

Proof. By (4.3.2) and (4.3.3), we get

$$f_o^{(e_1,e_2)}(x) = \frac{1}{2^2}(1 + e_1 z_1)(1 + e_2 z_2) f_o(x),$$

$$f^{(e_1,e_2)}(x) = \frac{1}{2^2}(1 + e_1 z_1)(1 + e_2 z_2) f(x),$$

$$f^{(e_1,-e_2)}(x) = \frac{1}{2^2}(1 + e_1 z_1)(1 - e_2 z_2) f(x),$$

$$f^{(-e_1,e_2)}(x) = \frac{1}{2^2}(1 - e_1 z_1)(1 + e_2 z_2) f(x),$$
$$f^{(-e_1,-e_2)}(x) = \frac{1}{2^2}(1 - e_1 z_1)(1 - e_2 z_2) f(x).$$

Thus the indicator polynomial function of $\mathcal{F}^{(e_1,e_2)} \cup \mathcal{F}_o^{(e_1,e_2)}$ is

$$\begin{aligned} f_1(x) &= f^{(e_1,e_2)}(x) + f_o^{(e_1,e_2)}(x) \\ &= \frac{1}{2^2}(1 + e_1 z_1)(1 + e_2 z_2) f(x) + \frac{1}{2^2}(1 + e_1 z_1)(1 + e_2 z_2) f_o(x) \\ &= \frac{1}{2}(1 + e_1 z_1)(1 + e_2 z_2) \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha. \end{aligned}$$

Similarly, we can get other indicator polynomial functions. $\square$

Chapter 5

Semifolding Resolution $III.x$ Designs

5.1 Introduction

It is well known that folding over a resolution III design can de-alias all the main effects. Li, Lin and Ye [19] studied foldover non-regular designs using indicator polynomial functions. They showed that the foldover non-regular designs obtained by folding over on all the main effects can also de-alias all the main effects. Mee and Peralta [20] considered semifoldover resolution III designs through a example. Although semifoldover resolution III designs can not de-alias all the main effects as the corresponding foldover designs, but Mee and Peralta [20] pointed out that the half new runs can be used as confirmation runs which verify the validity of one's assessment of active versus inactive factors.

In this chapter, we assume that $\mathcal{F}$ is a resolution $III.x$ design. From a practical point of view, we assume that $\mathcal{F}$ is a design without replicates. When we say a main effect can be de-aliased, we mean it can be de-aliased with its aliased two-factor interactions and ignore its aliased three-factor and higher-order interactions.

We study a semifoldover design obtained from a two-level resolution $III.x$ factorial design, regular and non-regular. We examine when a semifoldover design can de-alias one or more main effects. In Section 5.2, we consider semifoldover designs obtained by subsetting on a main effect and provide necessary and sufficient conditions for the semifoldover designs de-alias a main effect. We show that the semifoldover design, obtained by foldover on all the factors, can de-alias at least the same number of factors as the semifoldover designs obtained by folding over on one or more, but not all, main effects. In Section 5.3, we consider a semifoldover design obtained by subsetting on a two-factor interaction and provide necessary and sufficient conditions for the semifoldover designs de-alias a main effect. Finally, we present a number of illustrative examples in Section 5.4 which compare various semifoldover designs in more detail.

5.2 Subsetting on a main effect

When subsetting on a main effect, by Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + e \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_j}.$$

The indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j}.$$

Since $f_1(x)$ contains a one-letter word x_j, any main effect $x_h(\neq x_j)$ is (partially) aliased with at least the two-factor interaction $x_j x_h$ in the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$, and therefore can not be de-aliased in this fraction.

Proposition 5.2.1 provides a sufficient and necessary condition for a partial foldover design to de-alias a main effect from its aliased two-factor interactions.

Proposition 5.2.1. *Assume that the semifoldover design is obtained by folding over on all the main effects and subsetting on a main effect ex_j, $e = 1, -1$, $j = 1, 2, \ldots, m$. Then, x_j can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ and a main effect $x_h (\neq x_j)$ can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if it is not in any three-letter word which contains x_j.*

Proof. Let x_h be any main effect. Since $\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha$ contains four and higher-letter words, any word in $x_h(\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha)$ is at least three-letter word.

If $x_h = x_j$, then $x_h(\sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j}) = \sum_{\alpha \in \Omega_o} b_\alpha x^\alpha$, which contains only three and higher-letter words. It follows that $x_h = x_j$ can be de-aliased in the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$.

If $x_h \neq x_j$, then, for any five or higher-letter word x^α, $x_h x^{\alpha+\phi_j}$ is at least a three-letter word. If a three-letter word x^α does not contain x_j, then $x_h x^{\alpha+\phi_j}$ is either a three-letter word, if x_h is in x^α, or a five-letter word, if x_h is not in x^α. If a three-letter word x^α contains x_j, then $x_h x^{\alpha+\phi_j}$ is a three-letter word if and only if x_h is not in x^α. Thus, all the words in $x_h(\sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j})$ are three or higher-letter words if and only if x_h is not in any three-letter word which contains x_j. It follows that x_h can be de-aliased from its aliased main effects and two-factor interactions in the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if x_h is not in any three-letter word which contains x_j. □

From Proposition 5.2.1, we should choose the x_j which is in as few three-letter words as possible. When there exists a factor which is not in any three-letter word of the indicator polynomial function of the original design (in this case, the factor is not aliased with any two factor interaction), then subset on this factor permits the semifoldover design de-alias all the main effects from their aliased two-factor interactions.

Remark 5.2.1. Mee and Peralta [20] found in their example that the semifoldover design obtained by folding over on all the main effects (say S_1) can de-alias most main effects. This is true in general when subsetting on the same main effect x_j. Similar to Proposition 5.2.1, it is not difficult to prove that a main effect x_h can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ obtained by folding over on one or more, but not all, main effects (say S_2) if and only if x_h is not in any three-letter word in $\mathcal{W}_e$ and, if $x_h \neq x_j$, any three and four-letter word which contains x_j in $\mathcal{W}_o$. It means that if x_h can be de-aliased in S_2, then it is not in any three-letter word which contains x_j. Thus, the semifoldover design obtained by folding over on one or more main effects can not de-alias more main effects than the semifoldover design obtained by folding over on all the main effects if subsetting on the same main effect. Thus, we only need to consider the semifoldover design obtain by folding over on all the main effects and subsetting on a main effect.

5.3 Subsetting on a two-factor interaction

When subsetting on a two-factor interaction $x_i x_j$, then, by Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + e \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha + \phi_i + \phi_j}$$

and the indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha + \phi_i + \phi_j}.$$

Note that $f_1(x)$ contains a two-letter word x_ix_j, x_i and x_j are aliased in the fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$.

Proposition 5.3.1. *Assume that the semifoldover design is obtained by folding over on one or more main effects and subsetting on ex_ix_j, $e = 1, -1$, $i, j = 1, 2, \ldots, m$. Then, $x_i(x_j)$ can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if $x_i(x_j)$ is not in any three-letter word in $\mathcal{W}_e$ and $x_j(x_i)$ is not in any three-letter word in $\mathcal{W}_o$. $x_h(\neq x_i, x_j)$ can be de-aliased in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ if and only if x_ix_j and x_h is not in any three-letter word and x_h is not in any four and five-letter word which contains x_ix_j in $\mathcal{W}_e$; and can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if x_ix_j is not in any three-letter word in $\mathcal{W}_o$, x_h is not in any three-letter word in $\mathcal{W}_e$, not in any three-letter word which contains x_i or x_j and not in any four and five-letter word which contains x_ix_j in $\mathcal{W}_o$.*

Proof. Let x_h be any main effect. The results can be obtained from the following facts:

a. All the words in $x_h(\sum_{\alpha \in \Omega_e} b_\alpha x^\alpha)$ are at least three-letter words if and only if x_h is not in any three-letter word in $\mathcal{W}_e$.

b. If $x_h = x_i(x_j)$, then $x_h(\sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_i+\phi_j}) = \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j(\phi_i)}$. Thus, any word in it is a three or higher-letter word if and only if $x_j(x_i)$ is not in any three-letter word in $\mathcal{W}_o$. It follows that $x_h = x_j(x_i)$ can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if $x_i(x_j)$ is not in any three-letter word in $\mathcal{W}_e$ and $x_j(x_i)$ is not in any three-letter word in $\mathcal{W}_o$.

c. If $x_h \neq x_i(x_j)$, then, for any six or higher-letter word x^α, $x_h x^{\alpha+\phi_i+\phi_j}$ is at least a three-letter word. For a three-letter word x^α, $x_h x^{\alpha+\phi_i+\phi_j}$ is at least a three-letter word if and only if x^α does not contain x_ix_j and x_h is not in x^α

if it contains x_i or x_j. For a four or five-letter word x^α, $x_h x^{\alpha+\phi_i+\phi_j}$ is at least a three-letter word if and only if x_h is not in x^α if it contains $x_i x_j$. □

Corollary 5.3.2. *If the semifoldover design is obtained by folding over on all the main effects and subsetting on ex_ix_j, $e = 1, -1$, $i, j = 1, 2, \ldots, m$, then, $x_i(x_j)$ can be de-aliased in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if $x_j(x_i)$ is not in any three-letter word. Any main effect $x_h(\neq x_i, x_j)$ can be de-aliased in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ if and only if it is not in any four-letter word which contains x_ix_j, and in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ if and only if it is not in any three-letter word which contains x_i or x_j, in any five-letter word which contains x_ix_j and any three-letter word does not contain x_ix_j .*

From Proposition 5.2.1, Proposition 5.3.1 and Corollary 5.3.2, one can see that it is hard to say which semifoldover design can de-alias more main effects. Examples presented in the next section will explain this in detail.

5.4 Illustrative examples

In this section, we give some examples which explain how Proposition 5.2.1, Proposition 5.3.1 and Corollary 5.3.2 can be applied to get the semifoldover designs and also compare which semifoldover design can de-alias more main effects.

Example 5.4.1 shows that for this example, if subsetting on a proper main effect, the semifoldover design obtained by folding over on all the main effects can de-alias more main effects than the semifoldover designs obtained by folding over on all the main effects and subsetting on any two-factor interaction.

Example 5.4.1. *An indicator polynomial function of a 7-factor regular design*

$\mathcal{F}$ with generators $x_5 = x_1x_2$, $x_6 = x_1x_3$ and $x_7 = x_2x_3x_4$ is

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{1}{8}x_1x_2x_5 + \frac{1}{8}x_1x_3x_6 + \frac{1}{8}x_2x_3x_5x_6 + \frac{1}{8}x_2x_3x_4x_7 \\ &+\frac{1}{8}x_4x_5x_6x_7 + \frac{1}{8}x_1x_3x_4x_5x_7 + \frac{1}{8}x_1x_2x_4x_6x_7. \end{aligned}$$

Since there are two three-letter words, the alias sets are: $\{x_1, x_2x_5, x_3x_6\}$, $\{x_2, x_1x_5\}$, $\{x_3, x_1x_6\}$, $\{x_5, x_1x_2\}$ and $\{x_6, x_1x_3\}$.

When folding over on all the main effects, let $x_j = x_2$, then the indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{1}{8}x_2x_3x_5x_6 + \frac{1}{8}x_2x_3x_4x_7 + \frac{1}{8}x_4x_5x_6x_7 + \frac{1}{8}x_1x_5 \\ &+\frac{1}{8}x_1x_2x_3x_6 + \frac{1}{8}x_1x_2x_3x_4x_5x_7 + \frac{1}{8}x_1x_4x_6x_7. \end{aligned}$$

Since $x_3(f(x) - \frac{1}{8})$ and $x_6(f(x) - \frac{1}{8})$ contains only three or higher-letter words, x_3 and x_6 are de-aliased with their aliased two-factor interactions in the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. This can also be done by subsetting on $x_j = x_5$. Similarly, subsetting on x_3 or x_6 permits the semifoldover design de-alias x_2 and x_5 from their aliased two-factor interactions.

Note that x_4 and x_7 are not in any three-letter word, let $x_j = x_4$, for example, then, the indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{1}{8}x_2x_3x_5x_6 + \frac{1}{8}x_2x_3x_4x_7 + \frac{1}{8}x_4x_5x_6x_7 + \frac{1}{8}x_1x_2x_4x_5 \\ &+\frac{1}{8}x_1x_3x_4x_6 + \frac{1}{8}x_1x_3x_5x_7 + \frac{1}{8}x_1x_2x_6x_7, \end{aligned}$$

which contains only four-letter words. Thus, any main effect in this combined fraction is not aliased with two-factor interactions.

Since x_1 is in both three-letter words, subsetting on this factor does not permit the semifoldover design de-alias any other main effect from their aliased two-factor interactions.

Since x_1 is not in any four-letter word, subsetting on x_1x_i, $i = 2, \ldots, 7$ can de-alias all the main effects except x_1 and x_i in the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$. Since x_4 and x_7 are not in the three-letter words, subsetting on x_kx_h, $k = 1, \ldots, 7$, $h = 4, 7$, $k \neq h$, can de-alias x_k in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. However, subsetting on x_4x_7 can de-alias both x_4 and x_7 in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. Subsetting on any other two-factor interaction can not de-alias any main effect in both fractions.

Examples 5.4.2 and 5.4.3 show that when there are several three-letter words but no four-letter word in the indicator polynomial function, subsetting on a two-factor interaction usually can de-alias more main effects than subsetting on a main effect.

Example 5.4.2. *[15] An indicator polynomial function of a two-level 5-factor non-regular design $\mathcal{F}$ is*

$$f(x) = \frac{1}{2} - \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_2x_3x_4 + \frac{1}{4}x_2x_3x_5 + \frac{1}{4}x_1x_2x_3x_4x_5.$$

There are three three-letter words in the indicator polynomial function. The partial alias sets are: $\{x_1, x_2x_3\}$, $\{x_2, x_1x_3\}$, $\{x_2, x_3x_4\}$, $\{x_2, x_3x_5\}$, $\{x_3, x_1x_2\}$, $\{x_3, x_2x_4\}$, $\{x_3, x_2x_5\}$, $\{x_4, x_2x_3\}$, $\{x_5, x_2x_3\}$.

When folding over on all the main effects, note that x_1, x_4 and x_5 are in different three-letter words, subsetting on any of them permits the semifoldover design separate the first and the last two alias sets. But subsetting on x_2 or x_3 can not separate any set since all the three-letter words contain x_2 and x_3. Since there is no four-letter word in the indicator polynomial function, subsetting on any two-factor interaction x_ix_j, $i, j = 1, \ldots, 5$, $i \neq j$, separates all the partial alias sets, but x_i is aliased with x_j, in the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$. However, since *any main effect is in some three-letter word and the five-letter word contains all the main effects and possible two-factor interactions, subsetting on any two-factor interaction can not de-alias any main effect in* $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$.

Example 5.4.3. *An indicator polynomial function of a 6-factor non-regular design* $\mathcal{F}$ *is*

$$\begin{aligned} f(x) &= \frac{1}{4} + \frac{1}{8}x_1x_4x_5 + \frac{1}{8}x_2x_3x_6 - \frac{1}{8}x_1x_5x_6 - \frac{1}{8}x_2x_3x_4 - \frac{1}{8}x_2x_5x_6 \\ &\quad -\frac{1}{8}x_1x_3x_6 - \frac{1}{8}x_2x_4x_5 - \frac{1}{8}x_1x_3x_4 + \frac{1}{4}x_1x_2x_3x_4x_5x_6. \end{aligned}$$

The partial alias sets are: $\{x_1, x_3x_4\}$, $\{x_1, x_3x_6\}$, $\{x_1, x_4x_5\}$, $\{x_1, x_5x_6\}$, $\{x_2, x_3x_4\}$, $\{x_2, x_3x_6\}$, $\{x_2, x_4x_5\}$, $\{x_2, x_5x_6\}$, $\{x_3, x_1x_4\}$, $\{x_3, x_1x_6\}$, $\{x_3, x_2x_4\}$, $\{x_3, x_2x_6\}$, $\{x_4, x_1x_3\}$, $\{x_4, x_1x_5\}$, $\{x_4, x_2x_3\}$, $\{x_4, x_2x_5\}$, $\{x_5, x_1x_4\}$, $\{x_5, x_1x_6\}$, $\{x_5, x_2x_4\}$, $\{x_5, x_2x_6\}$, $\{x_6, x_1x_3\}$, $\{x_6, x_1x_5\}$, $\{x_6, x_2x_3\}$, $\{x_6, x_2x_5\}$.

When folding over on all the main effects, subsetting on any main effect x_i, $i = 1, 2, \ldots, 6$, *only permits the semifoldover design part the alias set which contains* x_i, *since the three-letter words which contain* x_i *include all the main effects. Since there is no four-letter word in the indicator polynomial function, subsetting on any two-factor interaction* x_ix_j, $i, j = 1, 2, \ldots, 6$, *can de-alias the other four main effects in* $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$, *but can not de-alias any main effect in* $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$, *since all the main effects are contained in some three-letter words which contain* x_i, x_j, *or* x_ix_j.

Examples 5.4.4 and 5.4.5 show that when we properly choose the main effect and the two-factor interaction, the semifoldover designs obtained by folding over on all the factors and subsetting on a main effect and a two-factor interaction, respectively, can de-alias the same number of main effects. Example 5.4.5

also shows that the semifoldover design obtained by folding over on a proper main effect and subsetting on a proper two-factor interaction can also de-alias all the main effects. Moreover, Example 5.4.5 shows that the semifoldover design obtained by folding over on one factor can de-alias more main effects than the semifoldover design obtained by folding over on all the main effects when subsetting on the same two-factor interaction.

Example 5.4.4. *(Montgomery [22], p.690) A 9-factor regular design $\mathcal{F}$ with generators $x_5 = x_1x_2x_3$, $x_6 = x_2x_3x_4$, $x_7 = x_1x_3x_4$, $x_8 = x_1x_2x_4$ and $x_9 = x_1x_2x_3x_4$. There are four three-letter words, $x_1x_6x_9$, $x_2x_7x_9$, $x_3x_8x_9$ and $x_4x_5x_9$, in its indicator polynomial function. The aliases are: $\{x_1, x_6x_9\}$, $\{x_2, x_7x_9\}$, $\{x_3, x_8x_9\}$, $\{x_4, x_5x_9\}$, $\{x_5, x_4x_9\}$, $\{x_6, x_1x_9\}$, $\{x_7, x_2x_9\}$, $\{x_8, x_3x_9\}$ and $\{x_9, x_1x_6, x_2x_7, x_3x_8, x_4x_5\}$.*

When folding over on all the main effects, subsetting on x_i, $i = 1, 2, \ldots, 8$, permits the semifoldover design de-alias 7-factors. For instance, if $x_j = x_1$, then $x_1, x_2, x_3, x_4, x_5, x_7$ and x_8 can be de-aliased from their aliased two-factor interactions in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. However, if $x_j = x_9$, then the semifoldover design can only de-alias x_9 from its aliased two-factor interactions. There are fourteen four-letter words in the indicator polynomial function, but none of them contains x_9. Thus, subsetting on x_ix_9, $i = 1, 2, \ldots, 8$, de-alias all the main effects in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ except x_i and x_9. Since x_1x_6, x_2x_7, x_3x_8 or x_4x_5 is in only one of the three-letter words and not in any five-letter words, subsetting on any of them can de-alias all the main effects which are not in the two-factor interaction except x_9 in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. But, subsetting on any other two-factor interaction can only de-alias x_9 in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$.

Example 5.4.5. *[15] An indicator polynomial function of a two-level 5-factor* *non-regular design $\mathcal{F}$ is*

$$f(x) = \frac{1}{2} + \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2x_3x_4 + \frac{1}{4}x_1x_2x_3x_5 - \frac{1}{4}x_1x_2x_3x_4x_5.$$

There is one three-letter word in the indicator polynomial function. The partial alias sets are: $\{x_1, x_2x_3\}$, $\{x_2, x_1x_3\}$ and $\{x_3, x_1x_2\}$.

When folding over on all the main effects, since x_4 and x_5 are not in the three-letter word, subsetting on x_4 or x_5 permits the semifoldover design separate all the partial alias sets. But subsetting on x_k, $k = 1, 2, 3$, only part the partial alias set which contains x_k. Since x_4x_5 is not in any four-letter word, subsetting on x_4x_5 permit the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ de-alias x_1, x_2 and x_3 from their aliased two-factor interactions. But subsetting on x_ix_j, $k = 1, 2, 3$, $l = 4, 5$, can only de-alias x_k in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. Subsetting on any other two-factor interactions can not de-alias any main effect in both fractions.

When folding over on any x_k, $k = 1, 2, 3$, all the words belong to $\mathcal{W}_o$. Thus, subsetting on x_4x_5 can separate all the partial alias sets in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$; subsetting on x_kx_l can separate the partial alias sets which do not contain x_k in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ and separate the partial alias set which contains x_k in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$, that is, subsetting on x_kx_l can separate all the partial alias sets; subsetting on any other two-factor interaction can only de-alias one main effect in the partial alias sets which do not contain x_k in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$.

When folding over on x_l, the three-letter word belongs to $\mathcal{W}_e$ and, therefore, can not de-alias any main effect.

Note that all the words in the indicator polynomial functions of Examples 5.4.2 and 5.4.5 contain one or two same factors (This is only possible for non-regular designs). For the designs which have this property, when folding over on the factor which is contained in all the words, $\mathcal{W}_e = \{1\}$. By Corollary 2.2.4, the constant in the indicator polynomial function of the original design is $\frac{1}{2}$. So, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is $f_1(x) = \frac{1}{2} + \frac{1}{2}x_k x_l$. Therefore, all the main effects except x_k and x_l can be de-aliased in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$.

Example 5.4.6 shows that for some designs, the semifoldover design almost can not de-alias any main effect.

Example 5.4.6. *(Montgomery [22], p.685) A 7-factor regular design $\mathcal{F}$ with generators $x_4 = x_1x_2$, $x_5 = x_1x_3$, $x_6 = x_2x_3$, $x_7 = x_1x_2x_3$. There are seven three-letter words, $x_1x_2x_4$, $x_1x_3x_5$, $x_2x_3x_6$, $x_4x_5x_6$, $x_3x_4x_7$, $x_2x_5x_7$, and $x_1x_6x_7$, and seven four-letter words: $x_2x_3x_4x_5$, $x_1x_3x_4x_6$, $x_1x_2x_5x_6$, $x_1x_2x_3x_7$, $x_1x_4x_5x_7$, $x_2x_4x_6x_7$, and $x_3x_5x_6x_7$. The alias sets caused by the three-letter words are: $\{x_1, x_2x_4, x_3x_5, x_6x_7\}$, $\{x_2, x_1x_4, x_3x_6, x_5x_7\}$, $\{x_3, x_1x_5, x_2x_6, x_4x_7\}$, $\{x_4, x_1x_2, x_3x_7, x_5x_6\}$, $\{x_5, x_1x_3, x_4x_6,$
$x_2x_7\}$, $\{x_6, x_2x_3, x_4x_5, x_1x_7\}$, $\{x_7, x_3x_4, x_2x_5, x_1x_6\}$.*

When folding over on all the main effects, subsetting on any main effect x_i, $i = 1, 2, \ldots, 7$, only permits the semifoldover design part the alias set which contains x_i, since the three-letter words which contain x_i include all the main effects. There are seven four-letter words in its indicator polynomial function and any two-factor interaction is in two four-letter words. Thus, subsetting on any two-factor interaction can only de-alias one main effect in $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$. For a two-factor interaction x_ix_j, any main effect which is not x_i and x_j is in some three-letter word which contains either x_i or x_j, therefore, subsetting on any two-factor interaction can not de-alias any main effect in $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$.

When folding over on one or more factors, subsetting on any two factor interaction can not de-alias any main effect.

Chapter 6

Semifolding Resolution $IV.x$ Designs

6.1 Introduction

Montgomery and Runger [21] showed that a foldover on one or more factors for regular resolution IV designs can de-alias all (or as many as possible) the two-factor interactions that we are interested in. Mee and Peralta [20] studied various cases when semifolding a regular resolution IV design. They showed that the semifoldover resolution IV design obtained by folding over on one factor and subsetting on a main effect can estimate all the the two-factor interactions as the corresponding full foldover design. The full foldover design obtained from a non-regular design has been studied by Li, Lin and Ye [19] using indicator polynomial functions.

In this chapter, we assume that $\mathcal{F}$ is a resolution $IV.x$ design which does not allow replicates. When we say a main effect can be de-aliased, we mean it can be de-aliased with its aliased two-factor interactions and ignore its aliased three-factor and higher-order interactions.

In this chapter, we study a semifoldover design obtained from a general two-level resolution $IV.x$ factorial design, regular and non-regular. We examine when a semifoldover design can de-alias all (or as many as possible) the two-factor interactions that we are interested in as the full foldover design. In Section 6.2, we show that a semifoldover design, obtained by foldover on a factor of interest for a non-regular resolution $IV.\ x$ design, can de-alias all the two-factor interactions which contain that particular factor. We also discuss the same problem for the semifoldover design, obtained by foldover on a factor of interest and subset on a two or three-factor interaction for a general factorial design. In Section 6.3, we consider a semifoldover design obtained by reversing the signs of two or more factors for a general factorial design and provide a sufficient condition for de-aliasing as many two-factor interactions as the full foldover design. Finally, we present in Section 6.4 number of illustrative examples.

6.2 Folding Over on a Main Factor

Mee and Peralta [20] showed that a semifoldover design, obtained by folding over on a main effect and subsetting on a main effect for a regular resolution IV design, can estimate as many two-factor interactions as the full foldover design. In particular, they showed that the semifoldover design can de-alias all the two-factor interactions which contain the factor of interest. In this section, we study this problem for a non-regular design. We also investigate the cases when the semifoldover design is obtained by subsetting on a two and three-factor interactions.

Let ϕ_i be the $1 \times m$ vector with the ith entry being 1 and all other entries being 0.

Theorem 6.2.1. *Let $\mathcal{F}$ be a two-level m-factor design of resolution IV.x with the indicator polynomial function (1.2.1). Assume that we fold over on the main effect x_h and subset on a main effect ex_j, $e = 1, -1$, $j = 1, 2, \ldots, m$. Then, the semifoldover design can de-alias all the two-factor interactions which contain x_h from other two-factor interactions as the corresponding full foldover design.*

Proof. By Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha \in \Omega_0} b_\alpha x^\alpha + e \sum_{\alpha \in \Omega_0} b_\alpha x^{\alpha + \phi_j}.$$

The indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha \in \Omega_0} b_\alpha x^\alpha - e \sum_{\alpha \in \Omega_1} b_\alpha x^{\alpha + \phi_j}.$$

If $x_h = x_j$, then any word in $\sum_{\alpha \in \Omega_1} b_\alpha x^{\alpha + \phi_j}$ is at least a three-letter word and does not contain x_h. Since x_h does not appear in any word in $\sum_{\alpha \in \Omega_0} b_\alpha x^\alpha$, all the words in $x_h x_{k_1}(f_2(x) - b_0)$ are at least three-letter words for any two-factor interaction $x_h x_{k_1}$, $k_1 \neq h$. Thus, $x_h x_{k_1}$ can be de-aliased with other two-factor interactions in the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$.

If $x_h \neq x_j$, then any four-letter word in $\sum_{\alpha \in \Omega_1} b_\alpha x^{\alpha + \phi_j}$ is either a three-letter word which does not contain x_j or a five-letter word which contains x_j. It then follows that all the words in $x_h x_j(f_2(x) - b_0)$ are at least three-letter words. Thus, $x_h x_j$ can be de-aliased with other two-factor interactions in the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. Since any word in $f_1(x) - ex_j$ is at least a three-letter word and x_h does not appear in any word in $f_1(x)$, all the words in $x_h x_{k_2}(f_1(x) - b_0)$ are at least three-letter words for any two-factor interaction $x_h x_{k_2}$, $k_2 \neq 1, j$. Thus, $x_h x_{k_2}$ can be de-aliased with other two-factor interactions in the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$.

From the above discussion, we observe that the semifoldover design separates the two-factor interactions which contain x_h from their aliased chains. Thus, they can be estimated in the semifoldover design, and the left alias chains can be estimated in the original design.

□

Propositions 6.2.2 and 6.2.4 below provide necessary and sufficient conditions for a semifoldover design, when subset on a two or three-factor interaction, to de-alias all the two-factor interactions which contain x_h with other two-factor interactions.

Proposition 6.2.2. *Let $\mathcal{F}$ be a two-level m-factor design of resolution IV.x. Assume that we fold over on a main effect x_h and subset on a two-factor interaction ex_hx_j, $e = 1, -1$, $j = 1, \ldots, h-1, h+1, \ldots, m$. Then, the semifoldover design can de-alias all the two-factor interactions which contain x_h with other two-factor interactions if and only if x_j is not in any four-letter word of $\mathcal{F}$ which contains x_h.*

Proof. By Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha\in\Omega_e} b_\alpha x^\alpha + e \sum_{\alpha\in\Omega_e} b_\alpha x^{\alpha+\phi_h+\phi_j}$$

and the indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha\in\Omega_e} b_\alpha x^\alpha - e \sum_{\alpha\in\Omega_o} b_\alpha x^{\alpha+\phi_h+\phi_j}.$$

If x_j is not in any four-letter word which contains x_h, then, for any $\alpha \in \Omega_o$, x^α has the form $x_h \prod_{i=1}^{l} x_{k_i}$ or $x_hx_j \prod_{i=1}^{l} x_{k_i}$, $3 \leq l \leq m-2$, $k_i \neq h, j$. Thus, $x^{\alpha+\phi_h+\phi_j}$ has the form $x_j \prod_{i=1}^{l} x_{k_i}$ or $\prod_{i=1}^{l} x_{k_i}$. Note that any two-factor interaction which can be de-aliased in the full foldover design is not in the four-letter words which do not contain x_h; one can then check that all the two-factor interactions which contain x_h can be de-aliased in the fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. Thus, the semifoldover design can de-alias all the two-factor interactions which contain x_h with other two-factor interactions.

If x_j is in some four-letter word which contains x_h, let that the four-letter word be of the form $x^\alpha = x_h x_j x_{k_1} x_{k_2}$. Then, $\alpha \in \Omega_o$. Thus, $f_2(x)$ contains a two-letter word $x_{k_1} x_{k_2}$. It follows that $x_h x_{k_1}(f_2(x) - b_0)$ contains a two-factor interaction $x_h x_{k_2}$ and $x_h x_{k_2}(f_2(x) - b_0)$ contains a two-factor interaction $x_h x_{k_1}$. Therefore, $x_h x_{k_1}$ and $x_h x_{k_2}$ are aliased in the fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$. Similarly, since $f_1(x)$ has a two-letter word $x_h x_j$, $x_h x_{k_1}$ and $x_h x_{k_2}$ are aliased with $x_j x_{k_1}$ and $x_j x_{k_2}$ (these two-factor interactions may be different from those which $x_h x_{k_1}$ and $x_h x_{k_2}$ are aliased with in the original design) in the fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$, respectively. Therefore, the semifoldover design can not de-alias all the two-factor interactions which contain x_h with other two-factor interactions. $\square$

If the original design is a regular design and all the two-factor interactions which contain x_h can be de-aliased in the semifoldover design, then the semifoldover design can estimate as many two-factor interactions as the full foldover design. Mee and Peralta [20] explained, in terms of degree of freedom, that subset on a two-factor interaction usually does not permit a semifoldover design to estimate as many two-factor interactions as the full foldover design. Since such an x_j in Proposition 6.2.2 does not exist for many resolution IV designs, Proposition 6.2.2 also explains, from a different point of view, the reason why subset on a two-factor interaction does not allow a semifoldover design to estimate as many two-factor interactions as the full foldover design (Note that for a two-factor interaction which does not contain x_h to be de-aliased with the two-factor interaction which contains x_h, it also needs a strong condition).

Corollary 6.2.3. *Let $\mathcal{F}$ be a two-level m-factor regular resolution IV design. Assume that we fold over on a main effect x_h and subset on a two-factor interaction ex_hx_j, $e = 1, -1$, $j = 1, \ldots, h-1, h+1, \ldots, m$. Then, the semifoldover design can estimate as many two-factor interactions as the full foldover design if x_j is not in any four-letter word of $\mathcal{F}$ which contains x_h.*

Proposition 6.2.4. *Let $\mathcal{F}$ be a two-level m-factor design of resolution IV.x. Assume that we foldover on the main effect x_h and subset on a three-factor interaction $ex_hx_jx_k$, $e = 1, -1$, $j, k \neq h$. Then, the semifoldover design can de-alias all the two-factor interactions which contain x_h with other two-factor interactions if and only if x_jx_k is not in any four and five-letter words in either $\mathcal{W}_e$ or $\mathcal{W}_o$.*

Proof. By Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + e \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_h+\phi_j+\phi_k}.$$

The indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_h+\phi_j+\phi_k}.$$

A two-factor interaction which contains x_h has one of the two forms: $x_hx_{k_t}$ and x_hx_p, $k_t \neq h, j, k$, $p = j, k$. One can similarly check that all the words in $x_hx_p(f_2(x) - b_0)$, $p = j, k$, are at least three-letter words. And if x_jx_k is not in any four and five-letter words in $\mathcal{W}_e$, then all the words in $x_hx_{k_t}(f_1(x) - b_0)$ are at least three-letter words; if $x_j x_k$ is not in any four and five-letter words in $\mathcal{W}_o$, then all the words in $x_h x_{k_t}(f_2(x) - b_0)$ are at least three-letter words.

Therefore, any two-factor interaction which contains x_h can be de-aliased in either the combined fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ or the combined fraction $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$.

If $x_j x_k$ is in some four-letter word, say $x_j x_k x_{k_1} x_{k_2}$, in $\mathcal{W}_e$, then, $x_h x_{k_1}(f_1(x) - b_0)$ contains a one-letter word x_{k_2} and $x_h x_{k_2}(f_1(x) - b_0)$ contains a one-letter word x_{k_1}. Thus, $x_h x_{k_1}$ and $x_h x_{k_2}$ are aliased with x_{k_1} and x_{k_2} in the fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$, respectively. If $x_j x_k$ is in some five-letter word, say $x_j x_k x_{k_1} x_{k_2} x_{k_3}$, in $\mathcal{W}_e$, then $x_h x_{k_1}(f_1(x) - b_0)$, $x_h x_{k_2}(f_1(x) - b_0)$ and $x_h x_{k_3}(f_1(x) - b_0)$ contain two-factor interactions $x_{k_2} x_{k_3}$, $x_{k_1} x_{k_3}$ and $x_{k_1} x_{k_2}$, respectively. It then follows that $x_h x_{k_1}$, $x_h x_{k_2}$ and $x_h x_{k_3}$ are aliased with $x_{k_2} x_{k_3}$, $x_{k_1} x_{k_3}$ and $x_{k_1} x_{k_2}$ in the fraction $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$, respectively. Similarly, if $x_j x_k$ is in some four or five-letter word in $\mathcal{W}_o$, then some two-factor interactions which contain x_h are aliased with some main or two-factor interactions, respectively. Thus, the semifoldover design can not de-alias all the two-factor interactions which contain x_h. $\square$

Corollary 6.2.5. *Let $\mathcal{F}$ be a two-level m-factor regular resolution IV design. Assume that we foldover on the main effect x_h and subset on a three-factor interaction $ex_h x_j x_k$, $e = 1, -1$, $j, k \neq h$. Then, the semifoldover design can estimate as many two-factor interactions as the corresponding full foldover design if $x_j x_k$ is not in any four and five-letter words in either $\mathcal{W}_e$ or $\mathcal{W}_o$.*

6.3 Folding Over on R Factors

Mee and Peralta [20] pointed out that it is not always true that folding over on two factors permits the semifoldover design to estimate as many two-factor interactions as the full foldover design. In this section, we consider the case when foldover on two or more factors for general two-level factorial designs. In particular, when the original design is a regular design, we provide a sufficient condition for the semifoldover design, obtained by reversing the signs of r factors and subset on a main effect, to estimate as many two-factor interactions as the full foldover design.

Note that if the full foldover design separates a alias set in the original design, then the alias set is divided to two alias sets, say set A and set B, in the full foldover design. All the two-factor interactions which have the forms $x_p x_q$, $p = 1, 2, \ldots, r$ and $q = r+1, r+2, \ldots, m$, belong to one alias set, say set A. Since the alias relations in set A are also in the semifoldover design, if one two-factor interaction in set A can be de-aliased with the same two-factor interactions as the full foldover design in one combined fraction, then, the set A can be separated from other two-factor interactions as the full foldover design, although in the case of non-regular designs, some two-factor interactions in set A may be (partially) aliased with some other two-factor interactions.

Theorem 6.3.1. *Let $\mathcal{F}$ be a two-level m-factor design of resolution IV.x. Assume that we fold over on the main effects $x_1, x_2, \ldots, x_r$, $r \geq 2$, and subset on the factor ex_j, $e = 1, -1$. Then, for any alias set which can be separated to set A and set B in the full foldover design, the semifoldover design can also separate set A from other two-factor interactions if there exists one two-factor interaction in set A which either contains x_j or not in any four and five-letter word in either $\mathcal{W}_e$ or $\mathcal{W}_o$.*

Proof. By Proposition 4.2.1, any word in the full foldover design is also in the semifoldover design. Thus if two two-factor interactions are aliased in the full foldover design, they will also be aliased in the semifoldover design.

By Proposition 4.2.4, the indicator polynomial function of $\mathcal{F}^{(e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_1(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha + e \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_j}$$

and the indicator polynomial function of $\mathcal{F}^{(-e)} \cup \mathcal{F}_o^{(e)}$ is

$$f_2(x) = \sum_{\alpha \in \Omega_e} b_\alpha x^\alpha - e \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j}. \tag{6.3.1}$$

From the above discussion, we only need to show that the two-factor interactions which satisfy the condition in this theorem can be de-aliased with the same two-factor interactions as the full foldover design. To prove this, we need to show that any word in either $x_p x_q \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_j}$ or $x_p x_q \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j}$ is at least a three-letter word.

For any word $x^\alpha \in \mathcal{W}$, x^α has the form $x^\alpha = \prod_{i=1}^{h_1} x_{k_i}$, $4 \leq h_1 \leq m$ or $x^\alpha = x_j \prod_{i=1}^{h_2} x_{k_i}$, $3 \leq h_2 \leq m-1$, where $k_i = 1, \ldots, j-1, j+1, \ldots, m$. Thus,

$$x^{\alpha+\phi_j} = x_j \prod_{i=1}^{h_1} x_{k_i} \text{ or } x^{\alpha+\phi_j} = \prod_{i=1}^{h_2} x_{k_i}. \tag{6.3.2}$$

(1) If x_j appears in the two-factor interaction $x_p x_q$, then, by (6.3.2), one can check that for any word $x^\alpha \in \mathcal{W}_o$, $x_p x_q x^{\alpha+\phi_j}$ is at least a three-letter word. Thus, all the words in $x_p x_q \sum_{\alpha \in \Omega_o} b_\alpha x^{\alpha+\phi_j}$ are at least three-letter words.

(2) If x_j does not appear in the two-factor interaction $x_p x_q$ and $x_p x_q$ is not in any four and five-letter words which contain x_j in $\mathcal{W}_e$, then, by (6.3.2), one can check that any word in $x_p x_q \sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_j}$ is at least a three-letter word (Note that the only one-letter word x_j in $\sum_{\alpha \in \Omega_e} b_\alpha x^{\alpha+\phi_j}$ also becomes the three-letter word $x_p x_q x_j$). Similarly, if $x_p x_q$ is not in any four and five-letter words which

contain x_j in $\mathcal{W}_o$, then any word in $x_p x_q \sum_{\alpha\in\Omega_o} b_\alpha x^{\alpha+\phi_j}$ is at least a three-letter word.

□

Corollary 6.3.2. *Let $\mathcal{F}$ be a two-level m-factor regular resolution IV design. Then, foldover on the main effects $x_1, x_2, \ldots, x_r$, $r \geq 2$, and subset on the factor ex_j, $e = 1, -1$, permits the semifoldover design to estimate as many two-factor interactions as the full foldover design if for any alias set which can be separated in the full foldover design, there exists one two-factor interaction which has the form $x_p x_q$, $p = 1, 2, \ldots, r$, and $q = 1, 2, \ldots, m$, and either contains x_j or not in any four and five-letter word in either $\mathcal{W}_e$ or $\mathcal{W}_o$.*

6.4 Illustrative Examples

In this section, we study semifoldover designs obtained by folding over on two or more factors through examples.

Example 6.4.1 below was first considered by Daniel [9] and then by Mee and Peralta [20] with foldover on one factor. It was also discussed through a case study by Barnett *et al.* [3]. Here, we discuss the case of foldover on two factors with the use of indicator polynomial functions.

Example 6.4.1. *A six-factor design with generators $x_1x_2x_3x_5 = 1$ and $x_2x_3x_4x_6 = 1$. Its indicator polynomial function is*

$$f(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_2x_3x_4x_6 + \frac{1}{4}x_1x_4x_5x_6.$$

If we foldover on x_1 and x_2, then by Proposition 4.2.1, the indicator polynomial *function of the full foldover design is*

$$f_c(x) = \frac{1}{2} + \frac{1}{2}x_1x_2x_3x_5.$$

Thus, the full foldover design can de-alias the following two-factor interactions: x_1x_4, x_1x_6, x_2x_4, x_2x_6, x_3x_4, x_3x_6, x_4x_5, x_4x_6 *and* x_5x_6. *One can check that subset on any main effect permits the semifoldover design to estimate as many two-factor interactions as the full foldover design.*

For instance, if we subset on x_1, *then by Proposition 4.2.4, the indicator polynomial function of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$f_1(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_1 + \frac{1}{4}x_2x_3x_5 \tag{6.4.1}$$

and the indicator polynomial function of $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$f_2(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 - \frac{1}{4}x_4x_5x_6 + \frac{1}{4}x_1x_2x_3x_4x_6. \tag{6.4.2}$$

Since x_1x_4 *and* x_1x_6 *both contain* x_1, *by the proof of Theorem 6.3.1, any word in* $x_1x_4(f_2(x) - 1/4)$ *and* $x_1x_6(f_2(x) - 1/4)$ *is at least a three-letter word. So,* x_1x_4 *and* x_1x_6 *can be de-aliased with other two-factor interactions in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$. *Since* x_2x_4 *and* x_2x_6 *are not in any word which contains* x_1, *they can be de-aliased with other two-factor interactions in both fractions* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *and* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$.

If we subset on x_3, *then the indicator polynomial function of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$f_1(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_3 + \frac{1}{4}x_1x_2x_5 \tag{6.4.3}$$

and the indicator polynomial function of $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$f_2(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 - \frac{1}{4}x_1x_3x_4x_5x_6 - \frac{1}{4}x_2x_4x_6. \tag{6.4.4}$$

Since x_1x_4 and x_1x_6 are not in any word which contains x_3, they can be de-aliased with other two-factor interactions in both the fractions. Since x_2x_4 and x_2x_6 are not in any word in $\mathcal{W}_e$, they can be de-aliased with other interactions in the fraction $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$. One can check these using the indicator polynomial functions.

Example 6.4.2. *Mee and Peralta [20] considered the 2_{IV}^{7-3} design with generators $x_5 = x_1x_2x_3$, $x_6 = x_2x_3x_4$ and $x_7 = x_1x_3x_4$. The indicator polynomial function of this design is*

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{1}{8}x_1x_2x_3x_5 + \frac{1}{8}x_2x_3x_4x_6 + \frac{1}{8}x_1x_4x_5x_6 + \frac{1}{8}x_1x_3x_4x_7 \\ &\quad + \frac{1}{8}x_2x_4x_5x_7 + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_3x_5x_6x_7. \end{aligned}$$

If we foldover on x_1 and x_2, then the indicator polynomial function of the full foldover design is

$$f_c(x) = \frac{1}{4} + \frac{1}{4}x_1x_2x_3x_5 + \frac{1}{4}x_1x_2x_6x_7 + \frac{1}{4}x_3x_5x_6x_7.$$

The full foldover design separates the two-factor alias sets as follows:

(1) $\{x_1x_3, x_2x_5, x_4x_7\} \rightarrow \{x_1x_3, x_2x_5\}$ *and* $\{x_4x_7\}$

(2) $\{x_1x_5, x_2x_3, x_4x_6\} \rightarrow \{x_1x_5, x_2x_3\}$ *and* $\{x_4x_6\}$

(3) $\{x_5x_6, x_3x_7, x_1x_4\} \rightarrow \{x_5x_6, x_3x_7\}$ *and* $\{x_1x_4\}$

(4) $\{x_1x_6, x_2x_7, x_4x_5\} \rightarrow \{x_1x_6, x_2x_7\}$ *and* $\{x_4x_5\}$

(5) $\{x_1x_7, x_2x_6, x_3x_4\} \rightarrow \{x_1x_7, x_2x_6\}$ *and* $\{x_3x_4\}$

(6) $\{x_3x_6, x_5x_7, x_2x_4\} \rightarrow \{x_3x_6, x_5x_7\}$ *and* $\{x_2x_4\}$

(7) $\{x_1x_2, x_3x_5, x_6x_7\} \rightarrow$ *(no change)*

If we subset on x_1*, then the indicator polynomial function of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$\begin{aligned} f_1(x) &= \frac{1}{8} + \frac{1}{8}x_1x_2x_3x_5 + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_3x_5x_6x_7 + \frac{1}{8}x_1 \\ &\quad + \frac{1}{8}x_2x_3x_5 + \frac{1}{8}x_2x_6x_7 + \frac{1}{8}x_1x_3x_5x_6x_7 \end{aligned}$$

and the indicator polynomial function of $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$\begin{aligned} f_2(x) &= \frac{1}{8} + \frac{1}{8}x_1x_2x_3x_5 + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_3x_5x_6x_7 - \frac{1}{8}x_1x_2x_3x_4x_6 \\ &\quad - \frac{1}{8}x_4x_5x_6 - \frac{1}{8}x_3x_4x_7 - \frac{1}{8}x_1x_2x_4x_5x_7. \end{aligned}$$

The four-letter words which contain x_1 *in* $\mathcal{W}_e$ *are* $x_1x_2x_3x_5$ *and* $x_1x_2x_6x_7$*, and in* $\mathcal{W}_o$ *are* $x_2x_3x_4x_6$*,* $x_1x_4x_5x_6$*,* $x_1x_3x_4x_7$ *and* $x_2x_4x_5x_7$*. One can check that the semi-foldover design can estimate as many two-factor interactions as the full foldover design, since the condition of Theorem 6.3.1 is satisfied in this case.*

For instance, for the first alias set, since x_1x_3 *contains* $x_j = x_1$*, by the proof of Theorem 6.3.1, it can be de-aliased with* x_4x_7 *in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$*. So,* x_2x_5 *can also be de-aliased with* x_4x_7 *in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$*, this can also be explained by the reason that* x_2x_5 *is not in any four-letter word which contains* x_1 *in* $\mathcal{W}_o$*. For the third alias set, since* x_1x_4 *contains* x_1*, it can be de-aliased with* x_5x_6 *and* x_3x_7 *in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$*. And for the sixth alias set, since* x_2x_4 *is not in any four-letter word which contains* x_1 *in* $\mathcal{W}$*, it can be de-aliased with* x_3x_6 *and* x_5x_7 *in both fractions* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *and* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$*. We can check these in detail using the indicator polynomial functions of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *and* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$*. Since*

$$\begin{aligned} x_1x_3\left(f_2(x) - \frac{1}{8}\right) &= \frac{1}{8}x_2x_5 + \frac{1}{8}x_2x_3x_6x_7 + \frac{1}{8}x_1x_5x_6x_7 - \frac{1}{8}x_2x_4x_6 \\ &\quad - \frac{1}{8}x_1x_3x_4x_5x_6 - \frac{1}{8}x_1x_4x_7 - \frac{1}{8}x_2x_3x_4x_5x_7 \end{aligned}$$

and

$$x_2x_5\left(f_2(x)-\frac{1}{8}\right) = \frac{1}{8}x_1x_3+\frac{1}{8}x_1x_5x_6x_7+\frac{1}{8}x_2x_3x_6x_7-\frac{1}{8}x_1x_3x_4x_5x_6$$
$$-\frac{1}{8}x_2x_4x_6-\frac{1}{8}x_2x_3x_4x_5x_7-\frac{1}{8}x_1x_4x_7,$$

x_1x_3 *and* x_2x_5 *are still aliased with each other, but de-aliased with* x_4x_7. *Since*

$$x_1x_4\left(f_2(x)-\frac{1}{8}\right) = \frac{1}{8}x_2x_3x_4x_5+\frac{1}{8}x_2x_4x_6x_7+\frac{1}{8}x_1x_3x_4x_5x_6x_7-\frac{1}{8}x_2x_3x_6$$
$$-\frac{1}{8}x_1x_5x_6-\frac{1}{8}x_1x_3x_7-\frac{1}{8}x_2x_5x_7,$$

x_1x_4 *is de-aliased with other two-factor interactions. Note that*

$$x_2x_4\left(f_1(x)-\frac{1}{8}\right) = \frac{1}{8}x_1x_3x_4x_5+\frac{1}{8}x_1x_4x_6x_7+\frac{1}{8}x_2x_3x_4x_5x_6x_7+\frac{1}{8}x_1x_2x_4$$
$$+\frac{1}{8}x_3x_4x_5+\frac{1}{8}x_4x_6x_7+\frac{1}{8}x_1x_2x_3x_4x_5x_6x_7$$

and

$$x_2x_4\left(f_2(x)-\frac{1}{8}\right) = \frac{1}{8}x_1x_3x_4x_5+\frac{1}{8}x_1x_4x_6x_7+\frac{1}{8}x_2x_4x_3x_5x_6x_7-\frac{1}{8}x_1x_3x_6$$
$$-\frac{1}{8}x_2x_5x_6-\frac{1}{8}x_2x_3x_7-\frac{1}{8}x_1x_5x_7,$$

and consequently x_2x_4 *is de-aliased with all other two-factor interactions in both fractions.*

Similarly, one can check that subset on any main effect permits the semi-foldover design to de-alias as many two-factor interactions as the full foldover design.

Example 6.4.3. *Montgomery ([22], p.691). The* 2_{IV}^{9-4} *design is with generators* $x_6 = x_2x_3x_4x_5$, $x_7 = x_1x_3x_4x_5$, $x_8 = x_1x_2x_4x_5$ *and* $x_9 = x_1x_2x_3x_5$. *The indicator* *polynomial function of this design is*

$$\begin{aligned} f(x) &= \frac{1}{16} + \frac{1}{16}x_2x_3x_4x_5x_6 + \frac{1}{16}x_1x_3x_4x_5x_7 + \frac{1}{16}x_1x_2x_6x_7 + \frac{1}{16}x_1x_2x_4x_5x_8 \\ &+ \frac{1}{16}x_1x_3x_6x_8 + \frac{1}{16}x_2x_3x_7x_8 + \frac{1}{16}x_4x_5x_6x_7x_8 + \frac{1}{16}x_1x_2x_3x_5x_9 \\ &+ \frac{1}{16}x_1x_4x_6x_9 + \frac{1}{16}x_2x_4x_7x_9 + \frac{1}{16}x_3x_5x_6x_7x_9 + \frac{1}{16}x_3x_4x_8x_9 \\ &+ \frac{1}{16}x_2x_5x_6x_8x_9 + \frac{1}{16}x_1x_5x_7x_8x_9 + \frac{1}{16}x_1x_2x_3x_4x_6x_7x_8x_9. \end{aligned}$$

If we foldover on x_1 and x_2, then the indicator polynomial function of the full foldover design is

$$\begin{aligned} f_c(x) &= \frac{1}{8} + \frac{1}{8}x_1x_2x_6x_7 + \frac{1}{8}x_1x_2x_4x_5x_8 + \frac{1}{8}x_4x_5x_6x_7x_8 + \frac{1}{8}x_1x_2x_3x_5x_9 \\ &+ \frac{1}{8}x_3x_5x_6x_7x_9 + \frac{1}{8}x_3x_4x_8x_9 + \frac{1}{8}x_1x_2x_3x_4x_6x_7x_8x_9. \end{aligned}$$

Thus, the full foldover design separates the two-factor interaction alias sets as *follows:*

(1) $\{x_1x_3, x_6x_8\} \rightarrow \{x_1x_3\}$ *and* $\{x_6x_8\}$

(2) $\{x_1x_4, x_6x_9\} \rightarrow \{x_1x_4\}$ *and* $\{x_6x_9\}$

(3) $\{x_1x_8, x_3x_6\} \rightarrow \{x_1x_8\}$ *and* $\{x_3x_6\}$

(4) $\{x_1x_9, x_4x_6\} \rightarrow \{x_1x_9\}$ *and* $\{x_4x_6\}$

(5) $\{x_2x_3, x_7x_8\} \rightarrow \{x_2x_3\}$ *and* $\{x_7x_8\}$

(6) $\{x_2x_4, x_7x_9\} \rightarrow \{x_2x_4\}$ *and* $\{x_7x_9\}$

(7) $\{x_2x_8, x_3x_7\} \rightarrow \{x_2x_8\}$ *and* $\{x_3x_7\}$

(8) $\{x_2x_9, x_4x_7\} \rightarrow \{x_2x_9\}$ *and* $\{x_4x_7\}$

(9) $\{x_1x_6, x_2x_7, x_3x_8, x_4x_9\} \rightarrow \{x_1x_6, x_2x_7\}$ *and* $\{x_3x_8, x_4x_9\}$

(10) $\{x_1x_2, x_6x_7\} \rightarrow$ *(no change)*

(11) $\{x_1x_7, x_2x_6\} \rightarrow$ *(no change)*

(12) $\{x_3x_4, x_8x_9\} \rightarrow$ *(no change)*

(13) $\{x_3x_9, x_4x_8\} \rightarrow$ *(no change)*

One can check that subset on any main effect except x_5 *permits the semifoldover design to estimate as many two-factor interactions as the full foldover design, since they satisfy the conditions of Theorem 6.3.1. For instance, if we subset on* x_6*, then the indicator polynomial function of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$\begin{aligned} f_1(x) &= \frac{1}{16} + \frac{1}{16}x_1x_2x_6x_7 + \frac{1}{16}x_1x_2x_4x_5x_8 + \frac{1}{16}x_4x_5x_6x_7x_8 + \frac{1}{16}x_1x_2x_3x_5x_9 \\ &\quad + \frac{1}{16}x_3x_5x_6x_7x_9 + \frac{1}{16}x_3x_4x_8x_9 + \frac{1}{16}x_1x_2x_3x_4x_6x_7x_8x_9 + \frac{1}{16}x_6 \\ &\quad + \frac{1}{16}x_1x_2x_7 + \frac{1}{16}x_1x_2x_4x_5x_6x_8 + \frac{1}{16}x_4x_5x_7x_8 + \frac{1}{16}x_1x_2x_3x_5x_6x_9 \\ &\quad + \frac{1}{16}x_3x_5x_7x_9 + \frac{1}{16}x_3x_4x_6x_8x_9 + \frac{1}{16}x_1x_2x_3x_4x_7x_8x_9 \end{aligned}$$

and the indicator polynomial function of $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$ *is*

$$\begin{aligned}
f_2(x) &= \frac{1}{16} + \frac{1}{16}x_1x_2x_6x_7 + \frac{1}{16}x_1x_2x_4x_5x_8 + \frac{1}{16}x_4x_5x_6x_7x_8 + \frac{1}{16}x_1x_2x_3x_5x_9 \\
&\quad + \frac{1}{16}x_3x_5x_6x_7x_9 + \frac{1}{16}x_3x_4x_8x_9 + \frac{1}{16}x_1x_2x_3x_4x_6x_7x_8x_9 + \frac{1}{16}x_2x_3x_4x_5 \\
&\quad + \frac{1}{16}x_1x_3x_4x_5x_6x_7 + \frac{1}{16}x_1x_3x_8 + \frac{1}{16}x_2x_3x_6x_7x_8 + \frac{1}{16}x_1x_4x_9 \\
&\quad + \frac{1}{16}x_2x_4x_6x_7x_9 + \frac{1}{16}x_2x_5x_8x_9 + \frac{1}{16}x_1x_5x_6x_7x_8x_9.
\end{aligned}$$

The four and five-letter words which contain x_6 *in* $\mathcal{W}_e$ *are* $x_1x_2x_6x_7$, $x_4x_5x_6x_7x_8$ *and* $x_3x_5x_6x_7x_9$, *and in* $\mathcal{W}_o$ *are* $x_1x_3x_6x_8$, $x_1x_4x_6x_9$, $x_2x_3x_4x_5x_6$ *and* $x_2x_5x_6x_8x_9$. *Since* x_1x_3, x_1x_4, x_1x_8, x_1x_9, x_2x_3, x_2x_4, x_2x_8 *and* x_2x_9 *are not in any four and five-letter words in* $\mathcal{W}_e$, *they can be de-aliased with other two-factor interactions in the fraction* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$. *Since* x_1x_6 *contains* x_1 *and* x_2x_7 *is not in any four and five-letter words in* $\mathcal{W}_o$, *they can be de-aliased with* x_3x_8 *and* x_4x_9 *in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$. *One can check this from the indicator polynomial functions of* $\mathcal{F}^{(1)} \cup \mathcal{F}_o^{(1)}$ *and* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$.

Note that x_6 *appears in the first four sets, and so from the proof of Theorem 6.3.1, the two-factor interactions which contain* x_6 *in the first four sets can also be de-aliased with other two-factor interactions in the fraction* $\mathcal{F}^{(-1)} \cup \mathcal{F}_o^{(1)}$.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this thesis, we have studied some properties of indicator polynomial functions. Using indicator polynomial functions, we have extended some existing results of regular designs to non-regular designs and also established some general results which did not exist even for regular designs.

In Chapter 2, we have considered some properties of an indicator polynomial function with all its words are odd words or even words. We have established that in the case without replicates, an indicator polynomial function with certain property must represent a half fraction and a factorial design which is not a half fraction must has at least three words in its indicator polynomial function. We have also proved that there is no $(2l+1)$-factor design of resolution $(2l-1).^{*}x$ when the run size of the design is not equal to 2^{2l}. After proving that an indicator polynomial function with one word is a regular design or replicates of a regular design, we have shown that there is no indicator polynomial function which contains only two words. Moreover, we have investigated indicator polynomial functions with three words and gotten that indicator polynomial functions with three words must contain one or three even words. The forms of the indicator polynomial functions with three words have also been obtained.

In Chapter 3, we have proved that a m-factor resolution $(2l-1).^{*}x$ design can be converted into a $(m-1)$-factor resolution $(2l+1).x$ design in the same number of runs and any m-factor design with resolution equal or bigger than V can be converted into a $(m+1)$-factor resolution $III^{*}.x$ design in the same number of runs. We have also shown that a m-factor resolution $(2l-1)$-factor design can be converted into $(m-1)$-factor resolution $(2l).x$ design in the same number of runs and a m-factor design with resolution $(2l).x$ can be converted into a $(m+1)$-factor resolution $(2l-1).\ x$ design in the same number of runs.

After obtaining the indicator polynomial functions of semifoldover designs, we considered the addition of a smaller fraction to the original design and provided a way to find the indicator polynomial functions of partial foldover designs in Chapter 4. Especially, we have obtained the indicator polynomial functions of the partial foldover design which is obtained by adding a $\frac{1}{4}$ runs of the original design.

In Chapter 5, we have studied various semifoldover resolution $III.x$ designs. When subsetting on a main effect, we have established that the semifoldover design obtained by folding over on one or more, but not all, the factors can not de-alias more main effects that the semifoldover design obtained by folding over on all the factors. When subsetting on a two-factor interaction, we have provided necessary and sufficient conditions for a semifoldover design to de-alias a main effect. Some illustrative examples are also provided in this chapter.

In Chapter 6, we have studied semifoldover designs obtained from general two-level resolution $IV.x$ designs. When folding over on one factor, we have proved that a semifoldover design, obtained by subset on a main effect, can estimate as many two-factor interactions as the full foldover design; the necessary and sufficient conditions for a semifoldover design, obtained by subsetting on a two or three-factor interaction, to de-alias all the two-factor interactions which contain the foldover factor are also presented. We have also provided a sufficient condition for a semifoldover design, obtained by folding over on two or more factors and subsetting on a main effect, to estimate as many two-factor interactions as the full foldover design. Finally, we have presented some illustrative examples.

7.2 Future Work

Indicator polynomial functions are new and powerful tools. Their applications in factorial designs need to be explored further. There are several interesting problems for future research:

In Chapter 2, we have studied some properties of indictor polynomial functions. Specifically, we have shown that some forms of indicator polynomial functions must be a half fraction and we have also studied indicator polynomial functions with only one, two or three words. The following problems will be of further interest to study in the future:

1. If a factorial design is a half fraction, what can we say about its indicator polynomial function ?
2. When the indicator polynomial function contains four or more words, what are the possible forms of the indicator polynomial function ?

In Chapter 3, we have studied the connections between two-level factorial designs of resolution $III^*.x$ and resolution $V.x$ using the transformations proposed by Draper and Lin [14]. If we use different transformations, we may get different connections between resolutions of two-level factorial designs. This certain is worth examining in a future work.

In Chapter 4, we have provided the indicator polynomial functions of partial foldover designs. This allows us to study partial foldover designs obtained by adding smaller fractions, such as $\frac{1}{4}$ fractions, to original designs. One possible future work is to consider alias structures in such partial foldover designs and explore when an effect can be de-aliased in the partial foldover designs.

Chapters 5 and 6 have examined when a main effect or a two factor interaction can be de-aliased in a semifoldover design. But, this consideration is under the condition that there are no blocks. One possible future work is to study alias structures of blocked semifoldover designs and consider when an effect can be de-aliased in such a blocked semifoldover design.

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