INDIRECT ABELIAN THEOREMS AND A LINEAR VOLTERRA EQUATION

BY
KENNETH B. HANNSGEN

1. **Introduction and summary.** We study asymptotic behavior of solutions of

$$(1.1) \qquad x'(t) = k - \int_0^t [a(t-\tau)+c]x(\tau)\,d\tau, \qquad x(0) = x_0, \qquad \left(' = \frac{d}{dt}\right)$$

where k and x_0 are real, $c \geqq 0$, and $a(t)$ satisfies

(H1) $a(t) \in C(0, \infty) \cap L_1(0, 1)$. *$a(t)$ is nonnegative and nonincreasing*, $\lim_{t\to\infty} a(t) = 0$, *and* $0 < a(0+) \leqq \infty$;

(H2) $a(t)$ *is convex downward*; *i.e., for* $0<\varepsilon<1$ *and* $0<t_1<t_3<\infty$, $\varepsilon a(t_1) + (1-\varepsilon)a(t_3) \geqq a(t_2)$, *where* $t_2 = \varepsilon t_1 + (1-\varepsilon)t_3$.

By a familiar theorem on Volterra equations, (1.1) has a unique solution in $C^1[0, \infty)$. We define $u(t)$ as the solution of (1.1) with $k=0$, $x_0=1$, and we let $w(t) = \int_0^t u(\tau)\,d\tau$. It is easily checked that the solution of (1.1) is given by $x_0 u(t) + kw(t)$.

Treating (1.1) as a special case of a nonlinear equation, Levin proved in [5] that if $a(t) \in C[0, \infty)$, $a(t) \not\equiv a(0)$, and $(-1)^k a^{(k)}(t) \geqq 0$ for $0<t<\infty$, $k=0, 1, 2, 3$, then

$$(1.2) \qquad \lim_{t\to\infty} u(t) = 0$$

and

(1.3).(i) If $c + a(t) \in L_1(0, \infty)$ (in particular, $c=0$), then

$$\lim_{t\to\infty} w(t) = \left(\int_0^\infty a(t)\,dt\right)^{-1}$$

(ii) If $c > 0$, then $\lim_{t\to\infty} w(t) = 0$.

Levin also conjectured that

(1.4) If $c=0$, $a(t) \notin L_1(0, \infty)$, then $\lim_{t\to\infty} w(t) = 0$.

The present theorem shows that (H1) and (H2) together are nearly sufficient for (1.2), (1.3), and (1.4); in particular, Levin's conjecture is proved. The theorem also exhibits a class of kernels satisfying (H1) and (H2) but for which a different asymptotic behavior from (1.2), (1.3), and (1.4) can be established.

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More specifically, note that if $a(t)$ is given by

$$(1.5)\quad \begin{aligned} &\text{(a)} && a(t) = \sum_{k=1}^{\infty} \delta_k\left(1-\frac{\min\{t, kt_0\}}{kt_0}\right), \qquad t_0 = \frac{2\pi}{\tau_0} > 0 \\ &\text{(b)} && \delta_k \geqq 0, \qquad 0 < \delta \equiv \sum_{k=1}^{\infty} \delta_k = a(0) < \infty \\ &\text{(c)} && \Omega \equiv \{k \mid \delta_k > 0\} \text{ has no common divisor } > 1, \end{aligned}$$

then $a(t)=\sum_{j=k}^{\infty} \delta_j - t\sum_{j=k}^{\infty} (\delta_j/jt_0)$, $(k-1)t_0 \leqq t \leqq kt_0$. It follows that $a(t)$ is continuous, and on each interval $(k-1)t_0 \leqq t \leqq kt_0$ it is linear with slope $-\sum_{k=j}^{\infty}(\delta_j/jt_0)$. Then $a(t)$ satisfies (H1) and (H2). When (1.5) holds, we may also have

$$(1.6)\qquad \omega \equiv \sqrt{(\delta+c)} = j\tau_0, \qquad j = \text{positive integer}.$$

We will establish (1.2), (1.3), and (1.4) when $a(t)$ satisfies (H1), (H2), and

(H3) *$a(t)$ admits no representation* (1.5) *such that* (1.6) *holds.*

Complementary to (H3) is

(H4) *$a(t)$ satisfies* (1.5) *and* (1.6).

When (H4) holds, we define $\gamma=(3\delta+2c)/(\delta+c)$ and let

$$u_1(t) = u(t)-2\gamma^{-1}\cos \omega t$$

and

$$w_1(t) = w(t)-2(\gamma\omega)^{-1}\sin \omega t.$$

We prove

THEOREM. *Let $c \geqq 0$, and let $a(t)$ satisfy* (H1) *and* (H2). *Then*

(i)
$$\lim_{t\to\infty} u(t) = 0, \quad \textit{if } \text{(H3)} \textit{ holds},$$
$$\lim_{t\to\infty} u_1(t) = 0, \quad \textit{if } \text{(H4)} \textit{ holds}.$$

(ii) *If $c+a(t) \notin L_1(0,\infty)$,*

$$\lim_{t\to\infty} w(t) = 0, \quad \textit{if } \text{(H3)} \textit{ holds},$$
$$\lim_{t\to\infty} w_1(t) = 0, \quad \textit{if } \text{(H4)} \textit{ holds}.$$

(iii) *If $c+a(t) \in L_1(0,\infty)$,*

$$\lim_{t\to\infty} w(t) = \left(\int_0^{\infty} a(t)\,dt\right)^{-1}, \quad \textit{if } \text{(H3)} \textit{ holds},$$
$$\lim_{t\to\infty} w_1(t) = \left(\int_0^{\infty} a(t)\,dt\right)^{-1}, \quad \textit{if } \text{(H4)} \textit{ holds}.$$

Generalizations of the results in [5] to nonlinear versions of (1.1) are given by Levin and Nohel in [7]. A result of Friedman (Theorem C of [3]) implies (1.2) and (1.4) for $c+a(t)=t^{-\alpha}$, $0<\alpha<1$. Halanay [4] studied a nonlinear equation including (1.1) with $k=0$ when $c+a(s-\tau)-\varepsilon_0 e^{-\alpha|s-\tau|}$ is a positive kernel on $\{0 \leqq s \leqq t, 0 \leqq \tau \leqq t\}$ for all $t \geqq 0$ and some $\varepsilon_0 > 0$, $\alpha > 0$.

The Laplace transform argument of our proof resembles the proofs of the "indirect abelian" theorems in [2, pp. 265–275]. Such theorems were used by Levin and Nohel in [6] to find an asymptotic expansion as $t \to \infty$ of solutions of an equation similar to (1.1) but where the kernel is, among other things, completely monotonic on $[0, \infty)$.

Throughout the discussion S denotes the subset of the complex plane given by

$$S = \{s \mid \operatorname{Re} s \geqq 0, s \neq 0\}.$$

We define

$$A(s) = \lim_{T \to \infty} \int_0^T e^{-st} a(t)\, dt, \qquad s \in S.$$

Then $A(s)$ is the Laplace transform of $a(t)$; similarly let $X(s)$ be the Laplace transform of $x(t)$. Taking Laplace transforms formally in (1.1), we obtain $X(s)p(s) = x_0 + (k/s)$, where $p(s) = (c/s) + A(s) + s$. In Lemma 5 we show that when (H3) holds, $p(s) \neq 0$ for $s \in S$. Then

$$X(s) = (x_0 + (k/s))/p(s), \qquad s \in S.$$

The complex inversion formula for Laplace transforms, together with contour integration and some estimates on $A(s)$, yields

$$x(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right] e^{i\tau t} X(i\tau)\, d\tau, \qquad t > 0, \tag{1.7}$$

where for each $\varepsilon > 0$ the integrals are uniformly convergent in $t \geqq T > 0$. The Riemann-Lebesgue theorem and other abelian arguments then yield our results.

When (H4) holds, we find that $p(s)$ has exactly the two zeros $s = \pm i\omega$ in S. A formula similar to (1.7) but with principal values taken also at $\tau = \pm\omega$ is used in this case.

§2 presents a sequence of lemmas concerning $a(t)$ and $A(s)$. The theorem is proved in §3.

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2. **The Laplace transform of $a(t)$.**

LEMMA 1. *Let $\sigma > 0$, and let $a(t)$ satisfy* (H1). *Then*

(i) $e^{-\sigma t} a(t)$ *satisfies* (H1).

(ii) *If $a(t)$ satisfies* (H2), *so does $e^{-\sigma t} a(t)$.*

Proof. (i) Obvious.

(ii) Since $a(t)$ is continuous, it suffices to prove the convexity relation in (H2) with $\varepsilon=1/2$. Set $a_i=a(t_i)$ and $b_i=\exp(-\sigma t_i)$, $i=1, 2, 3$. Then $a_i, b_i \geqq 0$, $a_1-a_2\geqq a_2-a_3\geqq 0$, $b_1-b_2>b_2-b_3>0$. Hence $a_1b_1-a_2b_2=a_1(b_1-b_2)+b_2(a_1-a_2)\geqq a_2(b_2-b_3)+b_3(a_2-a_3)=a_2b_2-a_3b_3$, and (ii) is proved.

We state without proof the following easy consequence of convexity:

LEMMA 2. *If $a(t)$ satisfies* (H2), *then for any $\delta>0$, the function $a(t)-a(t+\delta)$ is nonincreasing.*

LEMMA 3. *Let $a(t)$ satisfy* (H1). *Then*

(i) *$A(s)$ is defined, finite, and continuous in S. $A(s)$ is holomorphic in* $\{\operatorname{Re} s>0\}$.

(ii) *For $\sigma+i\tau \in S$, $\tau\neq 0$,*

$$(2.1)\qquad \begin{aligned} |\operatorname{Im} A(\sigma+i\tau)| &\leqq \int_0^{\pi/|\tau|} a(t)\sin|\tau|t\,dt \\ &\leqq \int_0^{\pi/|\tau|} a(t)\,dt, \end{aligned}$$

and

$$(2.2)\qquad |\operatorname{Re} A(\sigma+i\tau)| \leqq \int_0^{\pi/2|\tau|} a(t)\,dt,$$

so that $|A(\sigma+i\tau)|\to 0$ as $|\tau|\to\infty$, uniformly in $0\leqq\sigma<\infty$.

(iii) *If $a(t)$ also satisfies* (H2) *and $\sigma+i\tau\in S$, then*

$$(2.3)\qquad \begin{aligned} |A(\sigma+i\tau)| &\geqq \frac{1}{\sqrt{2}}\int_0^{\pi/2|\tau|} \cos\tau t\, e^{-\sigma t}a(t)\,dt \\ &\geqq \frac{1}{2\sqrt{2}}\int_0^{\pi/3|\tau|} e^{-\sigma t}a(t)\,dt \end{aligned}$$

(*the case $\tau=0$, $\tau^{-1}=\infty$, is included*); *and if $\tau>0$,*

$$(2.4)\qquad \lim_{T\to\infty}\int_0^T a(t)\sin\tau t\,dt+\lim_{T\to\infty}\int_{\pi/2\tau}^T a(t)\cos\tau t\,dt \geqq 0.$$

Proof. (i) For $s=\sigma+i\tau\in S$, $\tau>0$, $T>0$, define

$$\phi(T,s) = \int_0^T a(t)e^{-\sigma t}\cos\tau t\,dt$$

$$\psi(T,s) = \int_0^T a(t)e^{-\sigma t}\sin\tau t\,dt.$$

Since $a(t)\to 0$ as $t\to\infty$, $A(s)$ has a nonpositive abscissa of convergence (see e.g. [8, Chapter II]) so that $A(s)$ is holomorphic in $\{\operatorname{Re} s>0\}$ with

$$(2.5)\qquad [A(s)]^* = A(s^*) = \lim_{T\to\infty}[\phi(T,s)+i\psi(T,s)]$$

for $\operatorname{Im} s>0$, $*=$complex conjugate.

For any $T>0$, ϕ and ψ are continuous functions of s. (H1) and Lemma 1(i) show that $e^{-\sigma t}a(t)$ is nonnegative and nonincreasing on $(0,\infty)$. For $s=\sigma+i\tau$, $\tau>0$, $T_1, T_2 \geqq (n+\frac{1}{2})\pi/\tau$, $n=$nonnegative integer,

$$|\phi(T_1, s)-\phi(T_2, s)| \leqq \int_{(n+1/2)\pi/\tau}^{(n+3/2)\pi/\tau} a(t)\,dt$$

and similarly for ψ. Since $a(t)\to 0$ as $t\to\infty$, $\phi(T,s)$ and $\psi(T,s)$ converge as $T\to\infty$, uniformly in any set of the form

$$S\cap\{\sigma+i\tau \mid 0<\tau_0\leqq\tau\leqq\tau_1<\infty\}$$

to continuous functions $\phi(s)$ and $\psi(s)$. Comparing this with (2.5), we see that (i) is proved.

(ii) Since $\operatorname{Re} A(\sigma+i\tau)$ and $\operatorname{Im} A(\sigma+i\tau)$ are respectively even and odd in τ, we may assume $\tau>0$. Since for each $T>0$ we have

$$|\phi(T, \sigma+i\tau)| \leqq \int_0^{\pi/2\tau} a(t)e^{-\sigma t}\cos\tau t\,dt$$
$$\leqq \int_0^{\pi/2\tau} a(t)\,dt,$$

(2.2) holds. (2.1) is obtained similarly.

(iii) The case $\tau=0$ is trivial, and by symmetry of $\operatorname{Re} A(\sigma+i\tau)$ and $\operatorname{Im} A(\sigma+i\tau)$ in τ, we may assume $\tau>0$. Note first that since $A(i\tau)=\phi(i\tau)-i\psi(i\tau)$,

$$\sqrt{2}|A(i\tau)| \geqq |\phi(i\tau)|+|\psi(i\tau)| \geqq \phi(i\tau)+\psi(i\tau). \tag{2.6}$$

But

$$\phi(i\tau) = \int_0^{\pi/2\tau} a(t)\cos\tau t\,dt+\lim_{n\to\infty}\sum_{k=1}^{n}\int_{2(k-1)\pi/\tau}^{2k\pi/\tau} a\left(t+\frac{\pi}{2\tau}\right)\cos\left[\tau\left(t+\frac{\pi}{2\tau}\right)\right]dt$$

and

$$\psi(i\tau) = \lim_{n\to\infty}\sum_{k=1}^{n}\int_{2(k-1)\pi/\tau}^{2k\pi/\tau} a(t)\sin\tau t\,dt,$$

so (2.6) becomes

$$\begin{aligned} &|\sqrt{2}A(i\tau)|-\int_0^{\pi/2\tau} a(t)\cos\tau t\,dt \\ &\qquad \geqq \lim_{n\to\infty}\sum_{k=1}^{n}\int_{2(k-1)\pi/\tau}^{2k\pi/\tau}\left\{a(t)\sin\tau t+a\left(t+\frac{\pi}{2\tau}\right)\cos\left[\tau\left(t+\frac{\pi}{2\tau}\right)\right]\right\}dt. \end{aligned} \tag{2.7}$$

We note that the right-hand side of (2.7) equals the left-hand side of (2.4) and that both (2.3) for $\sigma=0$ and (2.4) will follow if we show that this right-hand side is nonnegative. But for any integer $k\geqq 1$,

$$\begin{aligned} &\int_{2(k-1)\pi/\tau}^{2k\pi/\tau}\left\{a(t)\sin\tau t+a\left(t+\frac{\pi}{2\tau}\right)\cos\left[\tau\left(t+\frac{\pi}{2\tau}\right)\right]\right\}dt \\ &\qquad = \int_0^{\pi/\tau}\sin\tau t\,[a(t+x_0)-a(t+x_1)-a(t+x_2)+a(t+x_3)]\,dt, \end{aligned}$$

where $x_j = 2(k-1)\pi/\tau + j\pi/2\tau$, $j=0, 1, 2, 3$, and Lemma 2 with $\delta = \pi/2\tau$ shows that the integrand is nonnegative. For (2.3) with $\sigma>0$, we apply (2.3) with $\sigma=0$ to the function $b(t)=e^{-\sigma t}a(t)$, which satisfies (H1) and (H2) by Lemma 1, and which has Laplace transform $B(s)=A(s+\sigma)$. This completes the proof of Lemma 3.

COROLLARY 3.1. *Let $a(t)$ satisfy* (H1). *Then $|sA(s)| \to 0$ as $s \to 0$, $s \in S$.*

Proof. We let $s=\sigma+i\tau$. Applying (2.1) and (2.2) to the function $b(t)=e^{-\sigma t}a(t)$, we have

$$|sA(s)| = |s|\,|B(i\tau)| \leqq \sqrt{2}|s| \int_0^{\pi/|\tau|} e^{-\sigma t}a(t)\,dt,$$

when $\tau\neq 0$ and, trivially, also when $\tau=0$, $\sigma>0$. Thus if $\sigma \geqq |\tau|$, $|sA(s)| \leqq 2\sigma \int_0^\infty e^{-\sigma t} \cdot a(t)\,dt = 2\sigma A(\sigma)$, while if $|\tau|>\sigma$, we have $|sA(s)| \leqq 2|\tau| \int_0^{\pi/|\tau|} a(t)\,dt$; one of these estimates is valid for each $s\in S$. But since $a(t)\to 0$ as $t\to\infty$,

$$\int_0^x a(t)\,dt = o(x) \qquad (x\to\infty). \tag{2.8}$$

It follows that $|\tau| \int_0^{\pi/|\tau|} a(t)\,dt \to 0$ as $\tau\to 0$; and (2.8) together with an elementary abelian theorem for Laplace transforms [8, p. 182] gives $\sigma A(\sigma)\to 0$ as $\sigma\to 0+$. In view of our estimates for $|sA(s)|$, the corollary is proved.

COROLLARY 3.2. *Let $a(t)$ satisfy* (H1) *and* (H2), *and suppose that $a(t)\notin L_1(0,\infty)$. Then $[s+A(s)]^{-1}\to 0$ as $s\to 0$, $s\in S$.*

Proof. By Lemma 3(iii),

$$\begin{aligned} |A(s)| = |A(\sigma+i\tau)| &\geqq \frac{1}{2\sqrt{2}} \int_0^{\pi/3|\tau|} e^{-\sigma t}a(t)\,dt \\ &\geqq m \int_0^{\pi/3|s|} a(t)\,dt, \end{aligned}$$

where $m^{-1} = 2\sqrt{2}e^{\pi/3}$. Then for sufficiently small $|s|$,

$$|[s+A(s)]^{-1}| \leqq 2\left(m\int_0^{\pi/3|s|} a(t)\,dt\right)^{-1} = o(1) \qquad (|s|\to 0).$$

LEMMA 4. *Suppose $a(t)$ satisfies* (H1) *and* (H2). *Then exactly one of the following two cases holds*:

I. *Either* (i) $a(0+)=\infty$ *or* (ii) $a(0)\equiv a(0+)<\infty$ *and* $\forall\tau>0$ *there exists an integer* $k=k(\tau)>0$ *such that*

$$a\left(\frac{2(k-1)\pi}{\tau}\right) - 2a\left(\frac{2(k-1)\pi+\pi}{\tau}\right) + a\left(\frac{2k\pi}{\tau}\right) > 0. \tag{2.9}$$

II. (i) *There exists a positive number τ_0 and a sequence $\{\delta_k\}_{k=1}^\infty$ such that* (1.5) *holds.*

(ii) *The numbers* τ_0, t_0, *and* δ *and the sequence* $\{\delta_k\}$ *are determined uniquely by* (1.5). *All positive* τ *such that*

$$a\left(\frac{2(k-1)\pi}{\tau}\right)-2a\left(\frac{2(k-1)\pi+\pi}{\tau}\right)+a\left(\frac{2k\pi}{\tau}\right) = 0, \qquad k = 1, 2, \dots \tag{2.10}$$

are integral multiples of τ_0.

(iii) *The Laplace transform of* $a(t)$ *is given by*

$$A(s) = \frac{\delta}{s}+\frac{1}{s^2}\sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\exp[-skt_0]-1), \qquad s \in S. \tag{2.11}$$

Proof. First we note that a representation (1.5) implies (2.10) with $\tau=\tau_0$, so cases I and II exclude each other.

Now suppose we are not in case I, so that (2.10) holds for some $\tau=\tau_1$. Let J denote the set of positive integers j such that (2.10) holds when $\tau=\tau_1/j$. Then $1 \in J$. Also, J is a finite set; for (H1), (H2), and (2.10) with $k=1$, $\tau=\tau_1/j, j\in J$, show that $a(t)$ is linear with negative slope on $[0, 2j\pi/\tau_1]$ whenever $j \in J$. We let j_0 be the largest $j \in J$, and set $\tau_0=\tau_1/j_0$. Then for any integer $j>1$, (2.10) does *not* hold with $\tau=\tau_0/j$.

By convexity, (2.10) with $\tau=\tau_0$ shows that $a(t)$ is linear on each interval $2(k-1)\pi/\tau_0 \leqq t \leqq 2k\pi/\tau_0$; we let $-\lambda_k$ be its slope there. By (H1) and (H2),

$$\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots \geqq 0, \quad \text{and} \quad \lim_{k\to\infty}\lambda_k = 0.$$

We define $t_0=2\pi/\tau_0$ and $\delta_k=kt_0(\lambda_k-\lambda_{k+1})\geqq 0$, $k=1, 2, 3, \dots$. Then on the interval $(k-1)t_0 \leqq t \leqq kt_0$, the function defined by the right-hand side of (1.5a) has the value

$$\begin{aligned}\sum_{j=k}^{\infty}\delta_j\left(1-\frac{t}{jt_0}\right) &= \sum_{j=k}^{\infty}(\lambda_j-\lambda_{j+1})(jt_0-t)\\ &= \lambda_k(kt_0-t)+\sum_{j=k+1}^{\infty}\lambda_j t_0\\ &= \int_{\infty}^{t} da(\tau) = a(t).\end{aligned}$$

This proves (1.5a) and (1.5b).

For (1.5c), we note that if $j>1$ divides all k in Ω, then

$$a(t) = \sum_{k=1}^{\infty}\delta_{jk}\left(1-\frac{\min\{t, jkt_0\}}{jkt_0}\right),$$

and as in the proof of (1.5a), $a(t)$ is linear on $2j(k-1)\pi/\tau_0 \leqq t \leqq 2jk\pi/\tau_0$, $k=1, 2, \dots$. But then (2.10) holds with $\tau=\tau_0/j$, and we chose τ_0 so as to make this impossible. This completes the proof that II(i) holds when I does not hold.

To prove (ii), suppose

$$a(t) = \sum_{k=1}^{\infty}\delta_k'\left(1-\frac{\min\{t, kt_0'\}}{kt_0'}\right), \qquad t_0'= 2\pi/\tau_0' \tag{1.5a$'$}$$

with corresponding (1.5b′), (1.5c′). Let $k_1<k_2<k_3<\cdots$ be all the elements of Ω, and $k_1'<k_2'<\cdots$ all the elements of Ω'. By (1.5a, a′), for each i

(2.12) $$k_i t_0 = k_i' t_0' = \max\{x \mid \text{slope of } a(t) \text{ has exactly } i \text{ different values on } (0, x)\}.$$

In particular, $t_0/t_0'=k_1'/k_1=$ rational number, so $t_0/t_0'=p/q$, where p and q are relatively prime positive integers. Then for each i, by (2.12), $k_i p/q=k_i'=$ integer. By (1.5c), $q=1$, so by (1.5c′) also $p=1$ and $t_0=t_0'$. By (1.5a, a′), $\tau_0=\tau_0'$. By (1.5a′), the slope $-\lambda_k$ of $a(t)$ in $[(k-1)t_0, kt_0]$ is $-\sum_{j=k}^{\infty}(\delta_j'/jt_0)$, so by the definition of δ_k in the proof of (i), $\delta_k'=(\lambda_k-\lambda_{k+1})kt_0=\delta_k$. (1.5b, b′) give $\delta=\delta'$, and uniqueness is proved.

Any τ satisfying (2.10) leads, as in (i), to a representation (1.5′) with $\tau_0'=\tau/j$. By uniqueness $j\tau_0=j\tau_0'=\tau$, and (ii) is proved.

(iii) This follows from (1.5) by direct computation. This completes the proof of Lemma 4.

LEMMA 5. *Let $a(t)$ satisfy* (H1) *and* (H2), *and let $c\geqq 0$. Define*

$$p(s) = (c/s)+A(s)+s, \qquad s\in S.$$

Then

(i) *$p(s)$ has no zeros in S if* (H3) *holds. If* (H4) *holds $p(s)$ has exactly the two zeros $s=\pm i\omega$ in S.*

(ii) *When* (H4) *holds,*

(2.13) $$|p(s)-\gamma(s-i\omega)| = o(|s-i\omega|) \qquad (s\to i\omega,\ s\in S),$$

where $\gamma=(3\delta+2c)/(\delta+c)$.

Proof. (i) First, (2.11) shows that $p(\pm i\omega)=0$ when (H4) holds.

For $s=i\tau\neq 0$,

$$\operatorname{Re} A(i\tau) = \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^{2\pi/|\tau|} a\left(\frac{2(k-1)\pi}{|\tau|}+t\right)\cos\tau t\,dt.$$

But for each k, the integral in the sum is equal to

$$\int_0^{\pi/2|\tau|}[a(x_0+t)-a(x_1-t)-a(x_1+t)+a(x_2-t)]\cos\tau t\,dt,$$

where $x_j=[2(k-1)+j]\pi/|\tau|$, $j=0, 1, 2$. Lemma 2 with $\delta=(\pi/|\tau|)-2t$ shows that the integrand is nonnegative. Furthermore, Lemma 4 shows that if $a(t)$ is in case I of Lemma 4 or in case II with $j\tau_0\neq|\tau|$, $j=1, 2, 3, \ldots$, then there exists k such that the integrand is positive at $t=0$, and by continuity on an interval $0\leqq t<\varepsilon$; for this k the integral is positive. We conclude that $\operatorname{Re} A(i\tau)\geqq 0$, and if $\operatorname{Re} A(i\tau)=0$, then $a(t)$ is in case II of Lemma 4 with $\tau=j\tau_0$, $j=$ integer.

By Lemma 1, if $\sigma>0$, the function $e^{-\sigma t}a(t)$ satisfies (H1) and (H2). Thus for $\sigma>0$ (by the preceding paragraph for $\tau\neq 0$ and trivially for $\tau=0$), $\operatorname{Re} A(\sigma+i\tau) =\int_0^\infty [e^{-\sigma t}a(t)]\cos\tau t\,dt\geqq 0$.

To apply these remarks, we suppose that $p(s)=0$, $s=\sigma+i\tau \in S$. Then

$$0 = \operatorname{Re} p(s) = c\sigma/(\sigma^2+\tau^2)+\operatorname{Re} A(\sigma+i\tau)+\sigma.$$

Therefore $\sigma=0$ and $a(t)$ is in case II with $\tau=j\tau_0$, $j=$integer. But then, using (2.11),

$$0 = \operatorname{Im} p(s) = -(c+\delta)/j\tau_0+j\tau_0;$$

i.e., $c+\delta=(j\tau_0)^2=\tau^2$, so that (H4) holds and $s=\pm i\omega$. This proves (i).

(ii) By (2.11),

$$p(s)-s+\frac{\omega^2}{s}+\frac{\delta}{s^2}(s-i\omega) = \left[\frac{1}{s^2}\sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\exp[-skt_0]-1)\right]-\frac{\delta}{s^2}(s-i\omega).$$

On the left-hand side we expand s, ω^2/s, and δ/s^2 in Taylor series about $i\omega$; on the right we rearrange terms. Then

$$p(s)-\gamma(s-i\omega)+O(|s-i\omega|^2) = s^{-2}\sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\exp[-skt_0]-1+skt_0-i\omega kt_0) \quad (s\to i\omega,\ s\in S). \tag{2.14}$$

For $\operatorname{Re} s \geqq 0$,

$$\begin{aligned}|(\exp[-skt_0]-1+skt_0-i\omega kt_0)/kt_0| &\leqq |\exp[-skt_0]-1|/kt_0+|s-i\omega| \\ &\leqq 2/kt_0+|s-i\omega|.\end{aligned}$$

On the other hand, the power series expansion of $\exp(-skt_0)$ about $s=i\omega$ yields for $k|s-i\omega|\leqq 1$

$$\begin{aligned}|(\exp[-skt_0]-1+skt_0-i\omega kt_0)/kt_0| &= \left|\sum_{j=2}^{\infty}[(s-i\omega)(-kt_0)]^j/j!\right|/kt_0 \\ &= k|s-i\omega|^2\left|\sum_{j=0}^{\infty}t_0^{j+2}/(j+2)!\right|/t_0 \\ &\leqq (\exp[t_0]k/t_0)|s-i\omega|^2.\end{aligned}$$

Using these two estimates, we have for $\operatorname{Re} s\geqq 0$

$$\begin{aligned}&\left|\sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\exp[-skt_0]-1+skt_0-i\omega kt_0)\right| \\ &\quad\leqq \frac{\exp[t_0]}{t_0}|s-i\omega|^2\sum_{k=1}^{n(s)}k\delta_k+|s-i\omega|\sum_{k=n(s)+1}^{\infty}\delta_k+\frac{2}{t_0}\sum_{k=n(s)+1}^{\infty}\delta_k/k,\end{aligned} \tag{2.15}$$

where $n(s)$ is the greatest integer such that $n(s)|s-i\omega|\leqq 1$.

By (1.5a), $a(t)\geqq\sum_{k=1}^{m}\delta_k[1-(\min\{t, kt_0\})/kt_0]$ for $m=1, 2, 3, \ldots$. Since

$$\int_0^{kt_0}\delta_k(1-t/kt_0)\,dt = kt_0\delta_k/2,$$

we have $2\int_0^{mt_0}a(t)\,dt\geqq t_0\sum_{k=1}^{m}k\delta_k$. Hence

$$|s-i\omega|\sum_{k=1}^{n(s)}k\delta_k \leqq \frac{2}{t_0 n(s)}\int_0^{t_0 n(s)}a(t)\,dt\to 0 \quad \text{as } |s-i\omega|\to 0, \tag{2.16}$$

since $a(t) \to 0$. Also, since $\sum \delta_k < \infty$,

$$\sum_{k=n(s)+1}^{\infty} \delta_k + \frac{2}{t_0|s-i\omega|} \sum_{k=n(s)+1}^{\infty} \delta_k/k \leqq (1+2/t_0) \sum_{k=n(s)+1}^{\infty} \delta_k \to 0 \qquad \text{as } |s-i\omega| \to 0.$$

Combining this with (2.14), (2.15), and (2.16), we have (2.13), and Lemma 5 is proved.

3. **Proof of theorem.** Integrating (1.1), we obtain

$$x(t) = x_0 + kt - \int_0^t f(t-\tau)x(\tau)\, d\tau,$$

where $0 \leqq f(t) \equiv \int_0^t [a(\tau)+c]\, d\tau \leqq \int_0^1 a(\tau)\, d\tau + c + [a(1)+c]t$. By a standard result on Volterra equations [1, §7.6], $x(t)$ satisfies an inequality

$$|x(t)| \leqq B_1 e^{bt}, \qquad b, B_1 > 0.$$

Substituting in (1.1), we have

$$|x'(t)| \leqq k + B_1 e^{bt} \int_0^1 [a(\tau)+c]\, d\tau + [a(1)+c] \int_0^t B_1 e^{b\tau}\, d\tau \leqq B_2 e^{bt}.$$

Taking Laplace transforms in (1.1), we obtain $X(s)p(s) = x_0 + (k/s)$, $\operatorname{Re} s > b$, with $p(s) = c/s + A(s) + s$, as in Lemma 5. By Lemma 5(i),

$$X(s) = (x_0 + k/s)/p(s), \qquad \operatorname{Re} s > b, \tag{3.1}$$

and (3.1) defines $X(s)$ as a function holomorphic in $\{\operatorname{Re} s > 0\}$ and continuous in S (for H3) or in $S - \{\pm i\omega\}$ (for H4). Also note that by (2.5) we have $[X(s)]^* = X(s^*)$; and by Lemma 3(ii), $X(\sigma + i\tau) \to 0$ as $|\tau| \to \infty$, uniformly in $0 \leqq \sigma < \infty$.

(i) We set $x_0 = 1$, $k = 0$, $X = U$ in (3.1). Then

$$U(s) = \frac{1}{s} - \frac{(c/s) + A(s)}{c + sA(s) + s^2}, \qquad s = \sigma + i\tau \in S.$$

For any $\sigma \geqq 0$ and sufficiently large $R > 0$, the second term is in $L_1\{(-\infty, -R) \cup (R, \infty)\}$ as a function of τ, by Lemma 3(ii); and integration by parts shows that for any $T > 0$

$$\left[\int_{-\infty}^{-R} + \int_R^{\infty}\right] e^{i\tau t}(\sigma + i\tau)^{-1}\, d\tau$$

converges uniformly for $t \geqq T$. Then the exponential bound on $u(t)$ and $u(t) \in C'$ justify the inversion formula

$$2\pi u(t) = e^{\sigma t} \int_{-\infty}^{\infty} e^{i\tau t} U(\sigma + i\tau)\, d\tau, \qquad \sigma > b,\ t > 0. \tag{3.2}$$

If $c + a(t) \in L_1(0, \infty)$, $A(s)$ has limit $A(0) = \int_0^\infty a(t)\, dt$ at $s = 0$, so $U(s)$ is continuous with $U(0) = 1/A(0)$. If $c + a(t) \notin L_1(0, \infty)$, Corollaries 3.1 (if $c > 0$) and 3.2 (if $c = 0$) show that $U(s) \to 0$ as $s \to 0$, $s \in S$, and again $U(s)$ is continuous at $s = 0$.

Thus if (H3) holds, Cauchy's theorem and the fact that $U(\sigma+i\tau)\to 0$ as $|\tau|\to\infty$ uniformly in σ yield

$$2\pi u(t) = \int_{-\infty}^{\infty} e^{i\tau t}U(i\tau)\,d\tau. \tag{3.3}$$

The Riemann-Lebesgue theorem for finite intervals and the uniform convergence of the integral in (3.3) yield $u(t)\to 0$ as $t\to\infty$.

On the other hand, if (H4) holds, we set

$$U_1(s) = U(s)-2s/\gamma(s^2+\omega^2). \tag{3.4}$$

Since $2s/\gamma(s^2+\omega^2)$ is the Laplace transform of $2\gamma^{-1}\cos\omega t$, (3.2) holds with u_1 and U_1 in place of u and U. Using Cauchy's theorem as before, we have for $0<\rho<\omega$,

$$2\pi u_1(t) = \left[\int_{-\infty}^{-\omega-\rho}+\int_{-\omega+\rho}^{\omega-\rho}+\int_{\omega+\rho}^{\infty}\right]e^{i\tau t}U_1(i\tau)\,d\tau$$
$$+\frac{1}{i}\left[\int_{C_\rho^-}+\int_{C_\rho^+}\right]e^{st}U_1(s)\,ds, \qquad t>0,$$

where $C_\rho^\pm$ is the semicircle $\{\pm i\omega+\rho e^{i\theta},\ -\pi/2\leqq\theta\leqq\pi/2\}$. Since $U_1(s^*)=[U_1(s)]^*$, this may also be written

$$\pi u_1(t) = \left[\int_0^{\omega-\rho}+\int_{\omega+\rho}^{\infty}\right]\operatorname{Re}\{e^{i\tau t}U_1(i\tau)\}\,d\tau+\frac{1}{i}\int_{C_\rho^+}\operatorname{Re}\{e^{st}U_1(s)\}\,ds, \quad t>0. \tag{3.5}$$

Note that

$$U_1(s) = \frac{1}{p(s)}-\frac{1}{\gamma(s-i\omega)}+O(1) \qquad (s\to i\omega,\ s\in S).$$

Writing $p(s)=\gamma(s-i\omega)+[p(s)-\gamma(s-i\omega)]$, we find that

$$U_1(s)=\frac{p(s)-\gamma(s-i\omega)}{\gamma(s-i\omega)^2}\left[\frac{-1}{\gamma+(p(s)-\gamma(s-i\omega))/(s-i\omega)}\right]+O(1) \quad (s\to i\omega, s\in S). \tag{3.6}$$

Thus, for $s\in C_\rho^+$, (2.13) yields $e^{st}U_1(s)=o(\rho^{-1})$, $(\rho\to 0)$. Since $|C_\rho^+|=\pi\rho$, we may let $\rho\to 0$ in (3.5) and obtain for $0<\eta<\omega$

$$\pi u_1(t) = \left[\int_0^{\omega-\eta}+\int_{\omega+\eta}^{\infty}\right]\operatorname{Re}\{e^{i\tau t}U_1(i\tau)\}\,d\tau+\lim_{\varepsilon\to 0}\left[\int_{\omega-\eta}^{\omega-\varepsilon}+\int_{\omega+\varepsilon}^{\omega+\eta}\right]\operatorname{Re}\{e^{i\tau t}U_1(i\tau)\}\,d\tau.$$

Treating the first term as for (H3) (3.4 and integration by parts show that $\int_{\omega+\eta}^{\infty}$ converges uniformly), we have

$$\pi u_1(t) = \lim_{\varepsilon\to 0}\left[\int_{\omega-\eta}^{\omega-\varepsilon}+\int_{\omega+\varepsilon}^{\omega+\eta}\right]\{[\operatorname{Re} U_1(i\tau)]\cos\tau t-[\operatorname{Im} U_1(i\tau)]\sin\tau t\}\,d\tau+o(1), \tag{3.7}$$
$$(t\to\infty),\ 0<\eta<\omega.$$

For real λ define

$$S(\lambda) = \sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\sin kt_0\lambda-kt_0\lambda), \qquad C(\lambda) = \sum_{k=1}^{\infty}\frac{\delta_k}{kt_0}(\cos kt_0\lambda-1).$$

Note that

$$(3.8)\qquad C(\lambda) = C(-\lambda) \leqq 0, \qquad \lambda S(\lambda) = -\lambda S(-\lambda) \leqq 0,$$

and $C(\lambda + j\omega) = C(\lambda)$, for any integer j. By (2.14),

$$p(i\tau) - i\gamma(\tau-\omega) = [-C(\tau) + iS(\tau-\omega)]/\tau^2 + O(|\tau-\omega|^2) \qquad (\tau \to \omega).$$

(2.15) and the argument following it show that

$$(3.9)\qquad |C(\lambda)| + |S(\lambda)| = o(\lambda), \qquad \lambda \to 0.$$

Using these facts in (3.6) one computes

$$(3.10)\qquad \operatorname{Re} U_1(i\tau) = \frac{-C(\tau)}{(\tau-\omega)^2\tau^2[\gamma^2 + o(1)]} + O(1) \qquad (\tau \to \omega),$$

and

$$(3.11)\qquad \operatorname{Im} U_1(i\tau) = R(\tau-\omega) + o\left(\frac{C(\tau)}{(\tau-\omega)^2}\right) + O(1) \qquad (\tau \to \omega),$$

where

$$R(\lambda) = \frac{S(\lambda)[\gamma + S(\lambda)/\omega^2\lambda]}{\gamma\lambda^2\omega^2[(\gamma + S(\lambda)/\lambda\omega^2)^2 + C^2(\lambda)/\lambda^2\omega^4]}.$$

Now let $t = t^* = 2\pi/\omega$ in (3.7). Since $\sin \tau t^* = \sin(\tau-\omega)t^* = O(\tau-\omega)$, $(\tau \to \omega)$, we see from (3.9) and (3.11) that with $t = t^*$ the second term in the integrand in (3.7) is bounded on $(\omega-\eta, \omega+\eta)$. It follows from (3.7) and (3.10) that

$$(3.12)\qquad \lim_{\varepsilon\to 0}\left[\int_{\omega-\eta}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\omega+\eta}\right] \frac{C(\tau)\cos \tau t^*\, d\tau}{(\tau-\omega)^2\tau^2[\gamma^2 + o(1)]}$$

exists and is finite. But by the choice of t^*, the integrand in (3.12) is $\leqq C(\tau)/2(\tau-\omega)^2\gamma^2\omega^2$ for $|\tau-\omega|$ sufficiently small. Since $C(\tau) \leqq 0$ we conclude

$$(3.13)\qquad C(\tau)/(\tau-\omega)^2 \in L_1(\omega-\eta, \omega+\eta).$$

In view of (3.10), (3.11), and (3.13), an application of the Riemann-Lebesgue theorem to (3.7) yields

$$(3.14)\qquad -\pi u_1(t) = \lim_{\varepsilon\to 0}\left[\int_{\omega-\eta}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\omega+\eta}\right] R(\tau-\omega)\sin \tau t\, d\tau + o(1), \qquad t \to \infty.$$

Note that $R(-\lambda) = -R(\lambda)$. The change of variables $\lambda = \tau-\omega$ in (3.14) shows that to complete the proof of (i) we need only show that

$$(3.15)\qquad r(t) \equiv \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\eta} [\sin(\omega+\lambda)t - \sin(\omega-\lambda)t] R(\lambda)\, d\lambda = o(1) \qquad (t \to \infty).$$

A trigonometric identity and $|(\sin \lambda t)/\lambda| \leqq t$ permit us to write

$$(3.16)\qquad r(t) = 2\cos(\omega t)\int_0^{\eta} R(\lambda)\sin \lambda t\, d\lambda.$$

To prove (3.15), note first that

$$|S'(\lambda)| = \left|\sum_{k=1}^{\infty} \delta_k(\cos kt_0\lambda - 1)\right| \leqq 2\delta.$$

Similarly, $|C'(\lambda)| \leqq \delta$. Straightforward computations and estimates then show that $|R'(\lambda)| \leqq M/\lambda^2$ for some constant $M<\infty$, $0<\lambda<\eta$. Also note that $|\lambda R(\lambda)| \leqq K_1|S(\lambda)/\lambda| \leqq K_2$, $K_1, K_2<\infty$.

Now let $\varepsilon>0$ be given; pick $\mu>0$ so that $(M+K_2)<\mu\varepsilon/3$, and pick $T>0$ so that $(M+K_2)/\eta T \leqq \varepsilon/3$ and $|K_1 S(\lambda)/\lambda| \leqq \varepsilon/3\mu$ for $0<\lambda<\mu/T$. Then for $t \geqq T$, integration by parts yields

$$\left|\int_{\mu/t}^{\eta} R(\lambda) \sin \lambda t \, d\lambda\right| \leqq (M+K_2)\left(\frac{1}{\eta T}+\frac{1}{\mu}\right) \leqq 2\varepsilon/3,$$

while

$$\left|\int_0^{\mu/t} R(\lambda) \sin \lambda t \, d\lambda\right| \leqq \left|\int_0^{\mu/t} [\lambda R(\lambda) \sin \lambda t]/\lambda \, d\lambda\right| \leqq \varepsilon/3.$$

These estimates, together with (3.16), prove (3.15) and complete the proof of (i).

(ii) We set $x_0=0$, $k=1$, $X=W$ in (3.1). Note that $W(\sigma+i\tau)=O(\tau^{-2})$, $|\tau| \to \infty$, so $W(\sigma+i\tau) \in L_1\{(-\infty, R) \cup (R, \infty)\}$ for R sufficiently large and $0 \leqq \sigma < \infty$.

If (H3) holds, we use Cauchy's theorem as in (i) to obtain, for each $\varepsilon>0$ sufficiently small,

$$(3.17) \qquad 2\pi w(t) = \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right] e^{i\tau t} W(i\tau)\, d\tau + \frac{1}{i}\int_{D_\varepsilon} e^{st} W(s)\, ds,$$

where D_ε is the semicircle $\{\varepsilon e^{i\theta}, -\pi/2 \leqq \theta \leqq \pi/2\}$. Corollaries 3.1 and 3.2 show that $W(\varepsilon e^{i\theta})=o(1/\varepsilon)$, $(\varepsilon \to 0)$, uniformly in $-\pi/2 \leqq \theta \leqq \pi/2$. Since $|D_\varepsilon|=\varepsilon\pi$, we may let $\varepsilon \to 0$ in (3.17) and obtain

$$2\pi w(t) = \lim_{\varepsilon \to 0}\left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right] e^{i\tau t} W(i\tau)\, d\tau, \qquad t > 0.$$

Now the symmetry of $W(i\tau)$ in τ and the Riemann-Lebesgue theorem imply that for $\Delta>0$,

$$(3.18) \qquad \pi w(t) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\Delta} \operatorname{Re}\{e^{i\tau t} W(i\tau)\}\, d\tau + o(1) \qquad (t \to \infty).$$

Next we derive a formula similar to (3.18) for $w_1(t)$ in the (H4) case. Define

$$W_1(s) = W(s) - 2/\gamma(s^2+\omega^2).$$

Note that $2/\gamma(s^2+\omega^2)$ is the Laplace transform of $2(\gamma\omega)^{-1} \sin \omega t$. Note also that $W_1(s)=U_1(s)/s=(U_1(s)/\omega)+O(1)$, $(s \to \omega, s \in S)$. Continuing as with $U_1(s)$ in (i) and using $W_1(i\tau) \in L_1(\omega+\eta, \infty)$, we obtain

$$\pi w_1(t) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\omega-\eta} \operatorname{Re}\{e^{i\tau t} W_1(i\tau)\}\, d\tau$$
$$+\omega^{-1} \lim_{\rho \to 0}\left[\int_{\omega-\eta}^{\omega-\rho} + \int_{\omega+\rho}^{\omega+\eta}\right] \operatorname{Re}\{e^{i\tau t} U_1(i\tau)\}\, d\tau + o(1),$$
$$t \to \infty, \ 0 < \eta < \omega.$$

Since $W_1(s)-W(s)$ is bounded on $(0, \omega-\eta)$, we may replace W_1 by W in the first term above. By (3.7) and conclusion (i) of the theorem, the second term is $o(1)$, $(t\to\infty)$. Thus for $0<\Delta<\omega$,

$$\pi w_1(t) = \lim_{\varepsilon\to 0}\int_\varepsilon^\Delta \operatorname{Re}\{e^{i\tau t}W_1(i\tau)\}\,d\tau + o(1) \qquad (t\to\infty). \tag{3.19}$$

Now set $\Delta=1$ when (H3) holds, $\Delta=\omega/2$ when (H4) holds. By (3.18) and (3.19) we can complete the proof by showing that

$$\lim_{t\to\infty}\left(\lim_{\varepsilon\to\infty}\int_\varepsilon^\Delta \operatorname{Re}\{e^{i\tau t}W(i\tau)\}\,d\tau\right) = 0. \tag{3.20}$$

If $c>0$, $W(i\tau)$ is continuous on $[0, \Delta]$, by Corollary 3.1, so that (3.20) holds. It remains to prove (3.20) with $c=0$.

For $0<\tau\leqq\Delta$, define

$$\begin{aligned}
&\text{(a)}\quad \phi_1(i\tau) = \int_0^{\pi/2\tau} a(t)\cos\tau t\,dt\\
&\text{(b)}\quad \phi_2(i\tau) = \lim_{T\to\infty}\int_{\pi/2\tau}^{T} a(t)\cos\tau t\,dt\\
&\text{(c)}\quad \phi(i\tau) = \phi_1(i\tau)+\phi_2(i\tau)\\
&\text{(d)}\quad \psi(i\tau) = \lim_{T\to\infty}\int_0^T a(t)\sin\tau t\,dt.
\end{aligned} \tag{3.21}$$

As in the proof of Lemma 3,

$$\operatorname{Re}A(i\tau) = \phi(i\tau) \quad\text{and}\quad \operatorname{Im}A(i\tau) = -\psi(i\tau). \tag{3.22}$$

Lemma 3 gives some useful facts about these functions. In addition, since $0\leqq a(t)\downarrow$, $\phi_2(i\tau)\leqq 0$, so that (2.4) implies

$$\psi(i\tau) \geqq |\phi_2(i\tau)|, \qquad 0<\tau\leqq\Delta. \tag{3.23}$$

Also, (2.1), (3.22), and $0\leqq a(t)\downarrow$ yield

$$\begin{aligned}
0 \leqq \psi(i\tau) &\leqq \int_0^{\pi/\tau} a(t)\sin\tau t\,dt\\
&\leqq 4\int_0^{\pi/4\tau} a(t)\cos\tau t\,dt\\
&\leqq 4\int_0^{\pi/2\tau} a(t)\cos\tau t\,dt,
\end{aligned}$$

i.e.,

$$0 \leqq \psi(i\tau) \leqq 4\phi_1(i\tau), \qquad 0<\tau\leqq\Delta. \tag{3.24}$$

The choice of Δ insures that $W(i\tau)$ is continuous on $(0, \Delta]$. Using (3.22) and our assumption $c=0$, we compute

$$W(i\tau) = \frac{(\psi(i\tau)-\tau)}{\tau|A(i\tau)+i\tau|^2} - i\,\frac{\phi(i\tau)}{\tau|A(i\tau)+i\tau|^2}.$$

Defining $y(t)=\lim_{\varepsilon\to 0}\int_\varepsilon^\Delta \operatorname{Re}\{e^{i\tau t}W(i\tau)\}\,d\tau$, we have

$$y(t) = \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\Delta \cos\tau t\,\frac{\psi(i\tau)\,d\tau}{\tau|A(i\tau)+i\tau|^2}-\int_\varepsilon^\Delta \frac{\cos\tau t\,d\tau}{|A(i\tau)+i\tau|^2}+\int_\varepsilon^\Delta\frac{\sin\tau t}{\tau}\,\frac{\phi(i\tau)d\tau}{|A(i\tau)+i\tau|^2}\right].$$

By Corollary 3.2, the middle integrand is continuous on $[0, \Delta]$. Since $|\sin\tau t/\tau|\leqq t$ for $t>0$ and

$$\frac{|\phi(i\tau)|}{|A(i\tau)+i\tau|}=\frac{|\phi(i\tau)|}{|\phi(i\tau)+i(\tau-\psi(i\tau))|}\leqq 1,$$

Corollary (3.2) also shows that the third integrand is continuous in $[0, \Delta]$ for each t. Therefore

$$\begin{aligned}y(t) &= \lim_{\varepsilon\to 0}\int_\varepsilon^\Delta \cos\tau t\,\frac{\psi(i\tau)\,d\tau}{\tau|A(i\tau)+i\tau|^2}-\int_0^\Delta \frac{\cos\tau t\,d\tau}{|A(i\tau)+i\tau|^2}\\ &\quad+\int_0^\Delta\frac{\sin\tau t}{\tau}\,\frac{\phi(i\tau)\,d\tau}{|A(i\tau)+i\tau|^2},\qquad t>0.\end{aligned}\tag{3.25}$$

But for $0<\tau<\pi/3t$ we have $\cos\tau t>1/2$ and $\psi(i\tau)\geqq 0$; hence the existence of the limit in (3.25) shows that

$$\psi(i\tau)/\tau|A(i\tau)+i\tau|^2\in L_1(0, \Delta).\tag{3.26}$$

Applying the Riemann-Lebesgue theorem to the first two integrals in (3.25), we obtain

$$y(t) = \int_0^\Delta\frac{\sin\tau t}{\tau}\,\frac{\phi(i\tau)\,d\tau}{|A(i\tau)+i\tau|^2}+o(1)\qquad(t\to\infty).\tag{3.27}$$

Another consequence of (3.26), together with (3.23), is

$$\phi_2(i\tau)/\tau|A(i\tau)+i\tau|^2\in L_1(0, \Delta).\tag{3.28}$$

We rewrite (3.27) as

$$\begin{aligned}y(t) &= \int_0^\Delta\frac{\sin\tau t}{\tau}\,\frac{d\tau}{\phi_1(i\tau)}+\int_0^\Delta\frac{\sin\tau t}{\tau}\,\frac{\phi_1(i\tau)\phi(i\tau)-\phi^2(i\tau)-(\psi(i\tau)-\tau)^2}{\phi_1(i\tau)|A(i\tau)+i\tau|^2}\,d\tau+o(1)\\ &\qquad\qquad(t\to\infty)\\ &= y_1(t)+y_2(t)+o(1)\qquad(t\to\infty).\end{aligned}\tag{3.29}$$

Then

$$y_2(t) = -\int_0^\Delta \sin\tau t\left[\frac{\phi_1(i\tau)+\phi_2(i\tau)}{\phi_1(i\tau)}\,\phi_2(i\tau)+\frac{\psi(i\tau)-\tau}{\phi_1(i\tau)}\,(\psi(i\tau)-\tau)\right]\frac{d\tau}{\tau|A(i\tau)+i\tau|^2}.\tag{3.30}$$

Now, (3.23) and (3.24) show that $|(\phi_1(i\tau)+\phi_2(i\tau))/\phi_1(i\tau)|$ and $|\psi(i\tau)/\phi_1(i\tau)|$ are bounded on $(0, \Delta)$. Since $a(t)\notin L_1(0, \infty)$, (3.21a) gives

$$\phi_1(i\tau)\geqq\frac{1}{2}\int_0^{\pi/3\tau}a(t)\,dt\to\infty,\quad\text{as }\tau\to 0,\tag{3.31}$$

so that $|\tau/\phi_1(i\tau)|$ is also bounded. Then by (3.26), (3.28), and Corollary 3.2, the coefficient of $\sin \tau t$ in (3.30) is in $L_1(0, \Delta)$. Using the Riemann-Lebesgue theorem once more, we have

$$\lim_{t\to\infty} y_2(t) = 0. \tag{3.32}$$

Finally, to treat $y_1(t)$, we note that for $\tau > 0$

$$\frac{d}{d\tau}\phi_1(i\tau) = -\int_0^{\pi/2\tau} ta(t)\sin \tau t\, dt \leqq 0.$$

Thus by (3.31), $1/\phi_1(i\tau) \downarrow 0$ as $\tau \downarrow 0$; in particular, $1/\phi_1(i\tau)$ is of bounded variation on $[0, \Delta]$, and a familiar theorem concerning the kernel $(\sin \tau t)/\tau$ [8, p. 64] yields $y_1(t) \to 0$ as $t \to \infty$. Combining this with (3.29) and (3.32), we have (3.20), and (ii) is proved.

(iii) A proof similar to that for (ii) can be obtained by considering $W_2(s) \equiv W(s) - 1/A(0)s$. We obtain (iii) from (i) as done in the proof of Theorem 2(i) of [5].

By the definition of $w(t)$, we have $u(t) = w'(t)$, where the asymptotic behavior of $u(t)$ is given by conclusion (i). (1.1) for w becomes

$$u(t) - 1 = -\int_0^t a(t-\tau)w(\tau)\, d\tau, \tag{3.33}$$

since $c + a(t) \in L_1(0, \infty)$ implies $c = 0$.

When (H3) holds, Levin's proof of Theorem 2(i) in [5] applies word for word to give $w(t) \to \left(\int_0^\infty a(t)\, dt\right)^{-1}$ as $t \to \infty$.

If (H4) holds, we use $A(i\omega) = -i\omega$ (from Lemma 5) and $a(t) \in L_1(0, \infty)$ to compute

$$\begin{aligned} -\int_0^t a(t-\tau)\frac{2\sin \omega\tau\, d\tau}{3\omega} &= \frac{-2}{3\omega}\int_0^t a(\tau)\sin[\omega(t-\tau)]\, d\tau \\ &= \operatorname{Im}\left\{\frac{-2e^{i\omega t}}{3\omega}\left[A(i\omega) - \int_t^\infty a(\tau)e^{-i\omega\tau}\, d\tau\right]\right\} \\ &= \frac{2\cos \omega t}{3} + o(1) \qquad (t \to \infty). \end{aligned}$$

Then by (3.33) and conclusion (i),

$$\int_0^t a(t-\tau)w_1(\tau)\, d\tau = 1 + o(1) \qquad (t \to \infty),$$

and $w_1'(t) \to 0$ as $t \to \infty$. The proof of (iii) can now be completed by the method of Theorem 2(i) of [5], which gives $w_1(t) \to \left(\int_0^\infty a(t)\, dt\right)^{-1}$ as $t \to \infty$. This completes the proof of the theorem.

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University of California,
Los Angeles, California