INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING, VOL. 10, 603-621 (1996)

INDIRECT MODEL REFERENCE ADAPTIVE CONTROL WITH DYNAMIC ADJUSTMENT OF PARAMETERS

MANUEL A. DUARTE
Department of Electrical Engineering, University of Chile, Santiago, Chile

AND

KUMPATI S. NARENDRA
Center for Systems Science, Yale University, New haven, CT, U.S.A.

SUMMARY

The paper discusses in detail a new method for indirect model reference adaptive control (MRAC) of linear time-invariant continuous-time plants with unknown parameters. The method involves not only dynamic adjustment of plant parameter estimates but also those of the controller parameters. Hence the overall system can be described by a set of non-linear differential equations as in the case of direct control. Many of the difficulties encountered in the conventional indirect approach, where an algebraic equation is solved to determine the control parameters, are consequently bypassed in this method.

The proof of stability of the equilibrium state of the overall system is found to be different from that used in direct control. Using Lyapunov's theory, it is first shown that the parameter errors between the parameter estimates of the identifier and the true parameters of the plant, as well as those between the actual parameters of the controller and their desired values, are bounded. Following this, using growth rates of signals in the adaptive loop as well as order arguments, it is shown that the error equations are globally uniformly stable and that the tracking (control) error tends to zero asymptotically. This in turn establishes the fact that both direct and indirect model reference adaptive schemes require the same amount of prior information to achieve stable adaptive control.

KEY WORDS model reference adaptive control; indirect adaptive control; dynamic adaptive control; robust adaptive control

1. INTRODUCTION

Indirect adaptive control, which is one of two distinct approaches for the control of dynamical plants with unknown parameters, as it is commonly used, consists of two stages. In the first stage, the parameters of the plant are estimated dynamically on-line using input-output information. At every instant of time, assuming that the estimates represent the true values of the plant parameters, the control parameters are computed to achieve desired overall system characteristics.[1-3] In contrast with this, in direct adaptive control the control parameters are adjusted continuously based on the error between the output of the plant and the output of the reference model. The latter results in the overall system being described by a set of non-linear differential equations. This in turn makes the formulation of the stability problems of such systems

This paper was recommended for publication by editor E. Mosca

CCC 0890-6327/96/060603-19

Received 3 November 1994
Revised 17 November 1995

relatively straightforward. While the two methods have continued to flourish for over two decades, very little is known about the precise relation between them. In particular, it is not clear which approach would be preferable in a specific situation and whether identical results can be obtained by the two methods using the same prior information. In this paper a new dynamical method of adjusting the control parameters is suggested for the indirect adaptive control of a linear system with unknown parameters. As in direct control, this results in the overall system being described by a set of non-linear differential equations. The stability of the new indirect MRAC is established using the same prior information that is generally assumed for direct control.[4]

The adjustment of the control parameters in the conventional indirect method involves the solution of a matrix algebraic equation. This poses both implementational and analytic problems. Since the algebraic equation can be solved only at discrete instants of time (assuming that a solution exists), the method is more suited to discrete rather than continuous-time systems. Even for discrete-time systems it is known that a solution to the algebraic equation may not exist,[5] since parameter estimates rather than the true values are used. In References 6 and 7, where the adaptive control of an unknown continuous-time plant is discussed, the synthesis of a feedback matrix is realized asymptotically. The proof of convergence of such a system depends upon the persistent excitation of an external command signal. In Reference 8 there is an interesting method of adaptive regulation for nth-order plants based on the idea of the so-called 'identification mismatch error'. In contrast with the above approaches, in the method suggested in this paper the identification (output) error between the plant output and the output of the identification model and the control error are shown to be bounded and to tend to zero asymptotically. While the former is shown by the existence of a Lyapunov function, the latter is established using arguments based on growth rates of signals in the system. It is also shown that all the signals of the system remain bounded. Hence the method, while circumventing the theoretical difficulties encountered in the conventional indirect approach, also results in global uniform stability even when the reference input is not persistently exciting.

Recently a method of combining the direct and indirect approaches for adaptively controlling a linear time-invariant plant was suggested by the authors.[9,10] The method introduced here, of dynamically adjusting control parameters using an indirect approach, can be considered as the precursor of the former. The combined MRAC presented in References 9 and 10 includes the direct approach and therefore the tracking error (or the augmented error in the case of $n^* > 1$) is used in the adaptive laws, whereas in the scheme proposed here, since it is indirect in nature, these errors are not needed. This fact make an important difference between the two approaches. On the other hand, by making suitable simplifications, the dynamical indirect MRAC can be obtained as a particular case of the combined MRAC. Nevertheless, stability analyses are quite different, since in the combined MRAC the convergence to zero of the tracking error is determined as part of a set of differential equations, while in the dynamical indirect MRAC this convergence is a consequence of the convergence of other error signals (identification error and closed-loop estimation errors).

The basic difference between the conventional indirect method and the new dynamical method is brought out in Section 2, where the adaptive control of a first-order plant is described in detail. The same approach can be shown to carry over directly to a general nth-order plant when its relative degree n^* is unity. For a plant with $n^* \geqslant 2$, it is shown in Section 3 that the same method also applies if a suitable parametrization of the plant is used. The principal requirement in all cases is that the plant parameters and desired control parameters are linearly related. Section 4 deals with the robustness of the indirect model reference adaptive control (IMRAC).

2. INDIRECT MRAC OF FIRST-ORDER PLANTS

To make the principal ideas clear, we first consider in detail the indirect MRAC of a first-order linear time-invariant plant described by the differential equation

$$\dot{x}_p(t) = -a_p x_p(t) + k_p u(t) \tag{1}$$

where $u(t)$, $x_p(t) \in \mathbb{R}$ are the input and output of the plant respectively. The plant parameters a_p and k_p are constant, real and unknown, with $k_p \neq 0$, i.e. the plant is completely controllable. It is also assumed that the sign of k_p is known. Let the reference model be described by the differential equation

$$\dot{x}_m(t) = -a_m x_m(t) + k_m r(t) \tag{2}$$

where $r(t)$, $x_m(t) \in \mathbb{R}$ are the input and output of the reference model respectively. The parameters a_m and k_m are known scalar constants, with $a_m > 0$, i.e. the model reference is asymptotically stable. The reference input $r(\cdot)$ is assumed to be a piecewise continuous bounded function of time. Our objective is to make the control error (tracking error) $e_c(t) \triangleq x_p(t) - x_m(t)$ tend to zero with time, using an indirect control scheme in which $e_c(t)$ is not directly used and the control parameters are computed based on the plant parameter estimates. The control input $u(t)$ is generated as in the direct control scheme and has the form

$$u(t) = \theta(t) x_p(t) + k(t) r(t), \tag{3}$$

where $\theta(\cdot)$, $k(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ are the control parameters adjusted according to some adaptive laws to be specified later. Replacing (3) in (1), it is easy to verify that there exist real constants θ^* and k^*, referred to as the desired control parameters, such that

$$\begin{aligned} -a_p + k_p \theta^* &= -a_m \\ k_p k^* &= k_m \end{aligned} \tag{4}$$

This implies that the transfer function of the plant together with the controller, when $\theta(t) \equiv \theta^*$ and $k(t) \equiv k^*$, matches exactly that of the reference model.

To generate the controller parameters based on the plant parameter estimates, we can use either algebraic or dynamical methods. The algebraic method, used in conventional indirect adaptive control methods, is relatively well known in the literature[11] and uses the relationship (4). Let $\hat{a}_p(\cdot)$, $\hat{k}_p(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ be the plant parameter estimates of a_p and k_p respectively, obtained from an identification procedure to be specified later. Replacing the true values of the plant parameters as well as the desired control values in equations (4) by their respective estimates, we can compute the controller parameters as

$$\begin{aligned} \theta(t) &= [\hat{a}(t) - a_m]/\hat{k}_p(t) \\ k(t) &= k_m/\hat{k}_p(t) \end{aligned} \tag{5}$$

This method involves division by the estimate of k_p. This in turn can introduce numerical difficulties, which have been well treated in the literature. One well-known method to circumvent this difficulty is to assume that a lower bound for $|k_p|$ is known.[11]

The analysis of the adaptive system using the methods described previously reveals that even in a simple adaptive system an asymmetry exists between the direct and indirect approaches. While only the sign of k_p is needed to implement the adaptive law for direct control, knowledge of a lower bound on $|k_p|$ is also needed for indirect control. In what follows, a stable indirect MRAC is derived without assuming a lower bound on $|k_p|$. The approach is based on adjusting the control parameters dynamically.

Let an identification model be described by the equation

$$\dot{\hat{x}}_p(t) = -a_m e_i(t) - \hat{a}_p(t)x_p(t) + \hat{k}_p(t)u(t) \tag{6}$$

where $\hat{x}_p(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ is a function of time that estimates the output of the plant $x_p(t)$ at every instant of time and $e_i(t) \triangleq \hat{x}_p(t) - x_p(t) \in \mathbb{R}$ is the identification error (output error). Subtracting (1) from (6), we obtain

$$\dot{\hat{x}}_p(t) - \dot{x}_p(t) = \dot{e}_i(t) = -a_m e_i(t) - [\hat{a}_p(t) - a_p]x_p(t) + [\hat{k}_p(t) - k_p]u(t) \tag{7}$$

which can be written as

$$\dot{e}_i(t) = -a_m e_i(t) - \eta_a(t)x_p(t) + \eta_k(t)u(t) \tag{8}$$

where $\eta_a(t) \triangleq \hat{a}_p(t) - a_p \in \mathbb{R}$ and $\eta_k(t) \triangleq k_p(t) - k_p \in \mathbb{R}$ are the plant parameter errors.

At any instant t our objective is to adjust $\theta(t)$ and $k(t)$ based on the estimates $\hat{a}_p(t)$ and $\hat{k}_p(t)$ of the plant parameters. Since the desired control parameters satisfy the constraint given by equations (4), we define closed-loop estimation errors $\varepsilon_\theta(t)$ and $\varepsilon_k(t)$ by replacing the true plant parameters and the desired control parameters in equations (4) by their respective estimates $\hat{a}_p(t)$, $\hat{k}_p(t)$, $\theta(t)$ and $k(t)$ to obtain

$$\begin{aligned} \varepsilon_\theta(t) &\triangleq -\hat{a}_p(t) + \hat{k}_p(t)\theta(t) + a_m \\ \varepsilon_k(t) &\triangleq \hat{k}_p(t)k(t) - k_m \end{aligned} \tag{9}$$

Subtracting (4) from (9) respectively, we can express the closed-loop estimation errors in terms of the control and plant parameter errors as

$$\begin{aligned} \varepsilon_\theta(t) &= -\eta_a(t) + k_p\phi_\theta(t) + \theta(t)\eta_k(t) \\ \varepsilon_k(t) &= k_p\phi_k(t) + k(t)\eta_k(t) \end{aligned} \tag{10}$$

where $\phi_\theta(t) \triangleq \theta(t) - \theta^* \in \mathbb{R}$ and $\phi_k(t) \triangleq k(t) - k^* \in \mathbb{R}$ are the control parameter errors.

The control parameters $\theta(t)$ and $k(t)$ are adjusted using the adaptive laws

$$\begin{aligned} \dot{\phi}_\theta(t) &= \dot{\theta}(t) = -\operatorname{sgn}(k_p)\varepsilon_\theta(t) \\ \dot{\phi}_k(t) &= \dot{k}(t) = -\operatorname{sgn}(k_p)\varepsilon_k(t) \end{aligned} \tag{11}$$

The rules for adjusting the estimation parameters are now modified as follows:

$$\begin{aligned} \dot{\eta}_a &= \dot{\hat{a}}_p(t) = e_i(t)x_p(t) + \varepsilon_\theta(t) \\ \dot{\eta}_k &= \dot{\hat{b}}_p(t) = -e_i(t)u(t) - \theta(t)\varepsilon_\theta(t) - k(t\varepsilon_k(t) \end{aligned} \tag{12}$$

The adaptive estimation laws (12) and the adaptive control laws (11), together with the identification error (8) and the closed-loop estimation errors (10), determine the modified indirect adaptive scheme. This scheme assures the boundedness of all the signals in the system and that $\lim_{t\to\infty} |x_p(t) - x_m(t)| = 0$.

Comment 1

In the indirect method proposed above, neither the reference model nor the control error $e_c(t)$ is explicitly used. The control parameters $\theta(t)$ and $k(t)$ are obtained from the plant parameter estimates by using equations (11).

Stability analysis

Let a function $V(e_i, \phi_\theta, \phi_k, \eta_a, \eta_k)$ be defined as

$$V = \tfrac{1}{2}[e_i^2 + |k_p|(\phi_\theta^2 + \phi_k^2) + \eta_a^2 + \eta_k^2]$$

The time derivative of V along the trajectories of the system described by equations (8) and (10)–(12) can be expressed as $\dot{V} = -a_m e_i^2 - \varepsilon_\theta^2 - \varepsilon_k^2 \leq 0$. Therefore V is a Lyapunov function and it follows that $e(t)$, $\hat{a}_p(t)$, $\hat{k}_p(t)$, $\theta(t)$ and $k(t)$ are uniformly bounded and also that e_i, ε_θ $\varepsilon_k \in \mathcal{L}^2 \triangleq \{f(t): \mathbb{R}^+ \to \mathbb{R}/\int_{t_0}^{t} |f(\tau)|^2 \, d\tau < \infty, \forall \in \mathbb{R}^+\}$.

However, boundedness of the signals $x_p(t)$, $\hat{x}_p(t)$ and $u(t)$ does not directly follow and hence the following analysis is used. Replacing the control input $u(t)$ given by (3) in the estimator (6) and using the definition of the closed-loop estimation errors (9), we can write

$$\dot{\hat{x}}_p(t) = -a_m \hat{x}_p(t) + \varepsilon_\theta(t) x_p(t) + [\varepsilon_k(t) + k_m] r(t)$$

Noting that $x_p(t) = \hat{x}_p(t) - e_i(t)$, the above equation can be rewritten as

$$\dot{\hat{x}}_p(t) = [-a_m + \varepsilon_\theta(t)]\hat{x}_p(t) - \varepsilon_\theta(t) e_i(t) + |\varepsilon_k(t) + k_m] r(t) \tag{13}$$

Equation (13) describing $\hat{x}_p(t)$ differs from equation (2) describing $x_m(t)$ owing to the presence of terms depending on $\varepsilon_\theta e(t)$ and $\varepsilon_k(t)$. Since $\varepsilon_\theta \in \mathcal{L}^2$ and $\varepsilon_\theta, e_i, \varepsilon_k, r \in \mathcal{L}^\infty$, it follows that $\hat{x}_p \in \mathcal{L}^\infty$. Hence $x_p = \hat{x}_p - e_i \in \mathcal{L}^\infty$ and therefore $u(t) = \theta(t)x_p(t) + k(t)r(t)$ is uniformly bounded. From (8) and (9) it can be seen that $\dot{e}_i$, $\dot{\varepsilon}_\theta$ and $\dot{\varepsilon}_k$ are uniformly bounded. Therefore we conclude that[12]

$$\lim_{t\to\infty} e_i(t) = 0, \quad \lim_{t\to\infty} \varepsilon_\theta(t) = 0, \quad \lim_{t\to\infty} \varepsilon_k(t) = 0$$

Further, from equations (13) and (2) it also follows that $\lim_{t\to\infty} |\hat{x}_p(t) - x_m(t)| = 0$ and hence $x_p(t)$ approaches $x_m(t)$ asymptotically, which in turn implies that $\lim_{t\to\infty} e_c(t) = 0$.

The results presented in this section indicate that for the case of first-order plants the prior information needed for stable adaptive control is the same whether a direct or indirect approach is used. The same is shown to be true in the following section for higher-order plants.

Comment 2

In contrast with the algebraic method, the control parameters are adjusted continuously in the dynamical method to reduce the closed-loop parameter estimates. Hence difficulties commonly encountered in indirect control, even when the zeros of the plant lie in the open left half of the complex plane, do not arise when the latter method is used. For non-minimum phase plants it is well known that steps have to be taken to avoid unstable pole–zero cancellations. This may be easier to accomplish using a dynamic approach.

Comment 3

The parameter errors ϕ_θ, ϕ_k, η_a and η_k lie in $\mathbb{R}^4$. However, it can be shown that if the reference input $r(t)$ is persistently exciting in only $\mathbb{R}^2$, then all the parameter errors will tend to zero asymptotically.

3. INDIRECT MRAC OF PLANTS OF ARBITRARY ORDER AND RELATIVE DEGREE

The new approach introduced in Section 2 for first-order plants was shown to be globally uniformly stable. This result is generalized in this section to the adaptive control of a general

nth-order plant of arbitrary relative degree, using the Indirect approach with dynamical adjustment of the control parameters.

3.1. *Statement of the problem*

Let an nth-order linear time-invariant plant with input–output pair $\{u(\cdot), y_p(\cdot)\}$ be described by the transfer function

$$W_p(s) = k_p \frac{Z_p(s)}{R_p(s)} \tag{14}$$

where $u(\cdot)$, $y_p(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$, $Z_p(s)$ is an mth-degree monic polynomial and $R_p(s)$ is an nth-degree monic polynomial, defined as $Z_p(s) \triangleq s^m + b_{m-1}s^{m-1} + \cdots + b_0$ and $R_p(s) \triangleq s^n + a_{n-1}s^{n-1} + \cdots + a_0$, where a_i, for $i = 0, 1, \ldots, n-1$, b_j, for $j = 0, 1, \ldots, m-1$, and k_p are real unknown constants. We define vectors $a = [a_0, a_1, \ldots, a_{n-2}]^T \in \mathbb{R}^{n-1}$ and $b = [b_0, b, \ldots, b_{m-1}]^T \in \mathbb{R}^m$. We will further assume that the plant is completely controllable and observable, or equivalently that the polynomials $Z_p(s)$ and $R_p(s)$ are coprime. We shall assume that the plant transfer function satisfies the four standard assumptions given by: (i) an upper bound on the order n of the plant is known; (ii) the exact relative degree $n^* = n - m$ of the plant is available; (iii) the sign of the high-frequency gain k_p of the plant is known; (iv) the zeros of the transfer function $W_p(s)$ lie in the open left half of the complex plane. For simplicity we will assume that n rather than an upper bound on the order of the plant is known to the designer.

The transfer function of the reference model has the form

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} \tag{14}$$

which relates the reference input $r(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ and the output of the reference model, $y_m(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$. The reference input $r(t)$ is assumed to be bounded and piecewise continuous. Further, the polynomials $Z_m(s)$ and $R_m(s)$ are monic and Hurwitz of degree m and n respectively, with $Z_m(s) \triangleq s^m + b^m_{m-1}s^{m-1} + \cdots + b^m_0$ and $R_m(s) \triangleq s^n + a^m_{n-1}s^{n-1} + \cdots + a^m_0$, where a^m_i, for $i = 0, 1, \ldots, n-1$, b^m_j, for $j = 0, 1, \ldots, m-1$, and k_m are real constants.

3.2. *Structure of the Controller*

The controller structure to be used in the proposed indirect MRAC scheme has the same form as that used in the direct MRAC scheme.[11] The input to the plant has the form

$$u(t) = \theta^T(t)w(t) \tag{15}$$

with $\theta(t), w(t) \in \mathbb{R}^{2n}$. The control parameter vector $\theta(t)$ and the auxiliary vector signals $w(t)$ are defined as $\theta(t) \triangleq [k(t), \theta_1^T(t), \theta_0(t), \theta_2^T(t)]^T$ and $w(t) \triangleq [r(t), w_1^T(t), y_p(t), w_2^T(t)]^T$. Both $k(t)$ and $\theta_0(t)$ as well as $r(t)$ and $y_p(t)$ are scalar functions of time, while $\theta_1(t), \theta_2(t), w_1(t), w_2(t) \in \mathbb{R}^{n-1}$ and their components are denoted as $\theta_1(t) \triangleq [\theta_0^1(t), \theta_1^1(t), \ldots, \theta_{n-2}^1(t)]^T$, $\theta_2(t) \triangleq [\theta_0^2(t), \theta_1^2(t), \ldots, \theta_{n-2}^2(t)]^T$, $w_1(t) \triangleq [w_0^1(t), w_1^1(t), \ldots, w_{n-2}^1(t)]^T$ and $w_2(t) \triangleq [w_0^2(t), w_1^2(t), \ldots, w_{n-2}^2(t)]^T$, with $\theta_i^1(t), \theta_i^2(t), w_i^1(t), w_i^2(t) \in \mathbb{R}$ for $i = 0, 1, \ldots, n-2$.

Here $w_1(t)$ and $w_2(t)$ will be referred to as the control sensitivity vectors and are generated using the differential equations

$$\begin{aligned} \dot{w}_1(t) &= \Lambda w_1(t) + lu(t) \\ \dot{w}_2(t) &= \Lambda w_2(t) + ly(t) \end{aligned} \tag{16}$$

where Λ is a stable $(n-1)\times(n-1)$ matrix, l is an $(n-1)$-vector and the pair (Λ, l) is controllable. It is well known[11] that there exists a constant parameter vector $\theta^* \in \mathbb{R}^{2n}$ such that the transfer function of the plant together with the controller, when $\theta(t) \equiv \theta^*$, is identical with that of the reference model, where $\theta^* \triangleq [k^*, \theta_1^{*\mathrm{T}}, \theta_0^*, \theta_2^{*\mathrm{T}}]^{\mathrm{T}}$. Here $k^*, \theta_0^* \in \mathbb{R}$, $\theta_1^* \triangleq [\theta_0^{1*}, \ldots, \theta_{n-2}^{1*}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ and $\theta_2^* \triangleq [\theta_1^{2*}, \ldots, \theta_{n-2}^{2*}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$, with $\theta_i^{1*}, \theta_i^{2*} \in \mathbb{R}$ for $i = 0, 1, \ldots, n-2$. The aim of adaptive control is to determine suitable rules for adjusting $\theta(t)$ using observed signals so that all the signals of the overall system remain bounded and the control error $e_c(t) = y_p(t) - y_m(t) \in \mathbb{R}$ is such that $\lim_{t\to\infty} e_c(t) = 0$.

So far nothing has been said about the structure of the matrix Λ and the vector l. Since our ultimate objective is to define closed-loop estimation errors, as in the first-order case, we need to relate the desired control parameters θ^* to the plant and reference model parameters. To make this relationship simple, we choose (Λ, l) to be in controllable canonical form such that

$$\Lambda = \begin{bmatrix} \begin{matrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} & \vline & I_{(n-2)} \\ \hline & -\lambda^{\mathrm{T}} & \end{bmatrix}, \qquad l = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

where $\lambda \triangleq [\lambda_0, \lambda_1, \ldots, \lambda_{n-2}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$, $\lambda_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n-2$, and $I_{(n-2)}$ is the $(n-2)\times(n-2)$ identity matrix.

3.3. *Identification model*

From the identification point of view the plant defined in (14) can be parametrized in any fashion. However, in practice the parametrization is chosen to make the relationship between the estimates given by the identification procedure and the plant parameter estimates, as well as the relationship between the plant parameters, the desired control parameters and reference model parameters, as simple as possible. In this study we parametrize the plant so that the reference model, which is not explicitly used, becomes part of the parametrization.[13] The reason for the choice of this parametrization lies in the fact that it results in a linear relationship between plant parameter estimates and desired controller parameters.

The output of the plant is expressed as

$$y_p(t) = W_m(s)\left(\frac{k_p}{k_m}[u(t) + \beta^{\mathrm{T}} w_1(t)] + \alpha_{n-1} y_p(t) + \alpha^{\mathrm{T}} w_2(t)\right) \tag{17}$$

The plant parameters α and β are defined as $\alpha \triangleq [\alpha_0, a_1, \ldots, \alpha_{n-2}]^{\mathrm{T}} \in {}^{n-1}$, $\beta \triangleq [\beta_0, \beta_1, \ldots, \beta_{n-2}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ and $\alpha_{n-1}, k_p \in \mathbb{R}$ are real constants. The auxiliary signals $w_1(t), w_2(t) \in \mathbb{R}^{n-1}$ are defined as in the case of the controller (see equations (16)).

Since $\beta^{\mathrm{T}}(sI - \Lambda)^{-1} l = \beta(s)/\Lambda(s)$ and $a^{\mathrm{T}}(sI - \Lambda)^{-1} l = \alpha(s)/\Lambda(s)$, where $\alpha(s) \triangleq a_{n-2}s^{n-2} + \alpha_{n-3}s^{n-3} + \cdots + \alpha_0$, $\beta(s) \triangleq b_{n-2}s^{n-2} + \beta_{n-3}n^{n-3} + \cdots + \beta_0$ and $\Lambda(s) \triangleq \lambda^{n-1} + \lambda_{n-2}s^{n-2} + \lambda_{n-3}s^{n-3} + \cdots + \lambda_0$, it can be shown that the transfer function between $y_p(t)$ and $u(t)$ in (17) is exactly that of the plant given by (14) provided that the following Bezout identity is satisfied between the parameters $\alpha, \beta, \alpha_{n-1}$ and a, b, a_{n-1}:

$$R_p(s)[\Lambda(s) + \alpha(s)] + k_m Z_p(s)[\beta(s) + \alpha_{n-1}\Lambda(s)] = Z_p(s)\Lambda_1(s)R_m(s) \tag{18}$$

$\Lambda_1(s)$ in equation (18) is any monic, stable polynomial of degree $n-m-1$ with the property that $\Lambda(s) = Z_m(s)\Lambda_1(s)$.

Since the identifier is at the discretion of the designer, there is an alternative way of parametrizing the plant so that simple stable adaptive identification laws can be generated. The procedure is based on the fact that equation (17) can be rewritten as

$$y_p(t) = \frac{k_p}{k_m}[\bar{u}(t) + \beta^T\bar{w}_1(t)] + \alpha_{n-1}\bar{y}_p(t) + \alpha^T\bar{w}_2(t) \tag{19}$$

where $\bar{y}_p(t) \triangleq W_m(s)y_p(t)$, $\bar{u}(t) \triangleq W_m(s)u(t)$, $\bar{w}_1(t) \triangleq W_m(s)\, I_{(n-1)}\, w_1(t)$ and $\bar{w}_2(t) \triangleq W_m(s)\, I_{(n-1)}\, w_2(t)$, with $I_{(n-1)}$ denoting the $(n-1)\times(n-1)$ identity matrix.

The corresponding series–parallel identification model associated with the parametrization of the plant, which delivers the estimate $\hat{y}_p(t)$ of the plant output $y_p(t)$, has the form

$$\hat{y}_p(t) = \frac{\hat{k}_p}{k_m}[\bar{u}(t) + \hat{\beta}^T(t)\bar{w}_1(t)] + \hat{\alpha}_{n-1}(t)\bar{y}_p(t) + \hat{\alpha}^T(t)\bar{w}_2(t) \tag{20}$$

except for exponentially decaying terms due to initial conditions. Since the latter do not affect the analysis, they are not included in the following. The plant parameters estimates $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ are denoted as $\hat{\alpha}(t) = [\hat{\alpha}_0(t), \hat{\alpha}_1(t), \ldots, \hat{\alpha}_{n-2}(t)]^T \in \mathbb{R}^{n-1}$ and $\hat{\beta}(t) = [\hat{\beta}_0(t), \hat{\beta}_1(t), \ldots, \hat{\beta}_{n-2}(t)]^T \in \mathbb{R}^{n-1}$, where $\hat{\alpha}_i(t) \in \mathbb{R}$ and $\hat{\beta}_j(t) \in \mathbb{R}$, for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, n-2$, are the estimate values of the plant parameters α_i and β_j respectively.

Therefore from equations (19) and (20) the identification error $e_i(t) \triangleq \hat{y}_p(t) - y_p(t) \in \mathbb{R}$ can be written as

$$e_i(t) = \frac{1}{k_m}[\bar{u}(t) + \hat{\beta}^T(t)\bar{w}_1(t)]\eta_{k_p}(t) + \frac{k_p}{k_m}\eta_\beta^T(t)\bar{w}_1(t) + \eta_{\alpha_{n-1}}(t)\bar{y}_p(t) + \eta_\alpha^T(t)\bar{w}_2(t) \tag{21}$$

where $\eta_\alpha(t) \triangleq \hat{\alpha}(t) - \alpha \in \mathbb{R}^{n-1}$, $\eta_\beta(t) \triangleq \hat{\beta}(t) - \beta \in \mathbb{R}^{n-1}$, $\eta_{\alpha_{n-1}}(t) \triangleq \hat{\alpha}_{n-1} - \alpha_{n-1} \in \mathbb{R}$ and $\eta_{k_p}(t) \triangleq \hat{k}_p(t) - k_p \in \mathbb{R}$ are the plant parameter errors. From now on, the plant parameter vector and its estimate will be denoted as $p \triangleq [k_p, \beta^T, \alpha_{n-1}, \alpha^T]^T \in \mathbb{R}^{2n}$ and $\hat{p}(t) \triangleq [\hat{k}_p(t), \hat{\beta}^T(t), \hat{\alpha}_{n-1}(t), \hat{\alpha}^T(t)]^T \in \mathbb{R}^{2n}$ respectively.

3.4. Closed-loop estimation errors

From equations (15) and (17) it can be seen that there exists a constant parameter vector $\theta^* \in \mathbb{R}^{2n}$ such that the transfer function of the plant together with the controller is identical with that of the reference model when $\theta(t) \equiv \theta^*$. Moreover, the desired controller parameter θ^* is related to the plant parameters $\alpha, \beta, \alpha_{n-1}$ and k_p as

$$\begin{aligned} \theta_1^* + \beta &= 0 \\ k_p\theta_0^* + k_m\alpha_{n-1} &= 0 \\ k_p\theta_2^* + k_m\alpha &= 0 \\ k^*k_p - k_m &= 0 \end{aligned} \tag{22}$$

From the above relationships it is possible to define the closed-loop estimation errors as

$$\begin{aligned} \varepsilon_{\theta 1}(t) &\triangleq \theta_1(t) + \hat{\beta}(t) \\ \varepsilon_{\theta 0}(t) &\triangleq \hat{k}_p(t)\theta_0(t) + k_m\hat{\alpha}_{n-1}(t) \\ \varepsilon_{\theta 2}(t) &\triangleq \hat{k}_p(t)\theta_2(t) + k_m\hat{\alpha}(t) \\ \varepsilon_k(t) &\triangleq k(t)\hat{k}_p(t) - k_m \end{aligned} \tag{23}$$

or equivalently, subtracting (22) from (23) respectively, we rewrite the closed-loop estimation errors in a form that is better for stability analysis purposes:

$$\begin{aligned}\varepsilon_{\theta 1}(t) &= \phi_{\theta 1}(t) + \eta_\beta(t)\\ \varepsilon_{\theta 0}(t) &= k_p\phi_{\theta 0}(t) + \theta_0(t)\eta_{k_p}(t) + k_m\eta_{a_{n-1}}(t)\\ \varepsilon_{\theta 2}(t) &= k_p\phi_{\theta 2}(t) + \theta_2(t)\eta_{k_p}(t) + k_m\eta_a(t)\\ \varepsilon_k(t) &= k_p\phi_k(t) + k(t)\eta_{k_p}(t)\end{aligned} \tag{24}$$

where $\phi_{\theta 1}(t) \triangleq \theta_1(t) - \theta_1^* \in \mathbb{R}^{n-1}$, $\phi_k(t) \triangleq k(t) - k^* \in \mathbb{R}$, $\phi_{\theta_0}(t) \triangleq \theta_0(t) - \theta_0^* \in \mathbb{R}$ and $\phi_{\theta_2}(t) \triangleq \theta_2(t) - \theta_0^* \in \mathbb{R}^{n-1}$ are the control parameter errors.

3.5. *Adaptive laws*

In order to guarantee global stability of the overall system, we choose the identification and control adaptive laws, based on error equations (21) and (24), as

$$\begin{aligned}\dot\phi_k(t) &= \dot k(t) = -\operatorname{sgn}(k_p)\varepsilon_k(t)\\ \dot\phi_{\theta_1}(t) &= \dot\theta_1(t) = -\varepsilon_{\theta_1}(t)\\ \dot\phi_{\theta_0}(t) &= \dot k_0(t) = -\operatorname{sgn}(k_p)\varepsilon_{\theta_0}(t)\\ \dot\phi_{\theta_2}(t) &= \dot k_2(t) = -\operatorname{sgn}(k_p)\varepsilon_{\theta_2}(t)\end{aligned} \tag{25}$$

$$\begin{aligned}\dot\eta_{k_p}(t) &= \dot{\hat k}_p(t) = -e_i(t)[\hat\beta^T(t)w_1(t) + \bar u(t)]/[N(t)k_m] - k(t)\varepsilon_k(t) - \theta_2^T(t)\varepsilon_{\theta_2}(t) - \theta_0(t)\varepsilon_0(t)\\ \dot\eta_\beta(t) &= \dot{\hat\beta}(t) = -\operatorname{sgn}(k_p)e_i(t)w_1(t)/[N(t)k_m] - \varepsilon_{\theta_1}(t)\\ \dot\eta_{a_{n-1}}(t) &= \dot{\hat a}_{n-1}(t) = -e_i(t)\bar y_p(t)/N(t) - k_m\varepsilon_{\theta_0}(t)\\ \dot\eta_a(t) &= \dot{\hat a}(t) = -e_i(t)\bar w_2(t)/N(t) - k_m\varepsilon_{\theta_2}(t)\end{aligned} \tag{26}$$

where $N(t)$ is a normalization factor defined as $N(t) \triangleq 1 + [\bar u(t) + \hat\beta^T(t)\bar w_1(t)]^2 + \bar w_1^T(t)\bar w_1(t) + \bar y_p^2(t) + \bar w_2^T(t)\bar w_2(t)$. The overall indirect MRAC scheme is shown in Figure 1.

3.6. *Proof of stability*

The proof of stability of the overall indirect adaptive MRAC scheme is carried out in two stages. In the first stage, as stated in Lemma 1, the boundedness of the parameter errors is established. In the second stage, as stated in Theorem 1, the boundedness of all the signals in the adaptive loop is proved as well as the fact that the control error, the identification error and the closed-loop estimation errors tend asymptotically to zero. The latter is done using growth rates of signals and order arguments.[14]

Lemma 1

Let the system defined by the identification error $e_i(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ given by (21), the closed-loop estimation errors $\varepsilon_{\theta_1}(\cdot)$, $\varepsilon_{\theta_2}(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^{n-1}$ and $\varepsilon_k(\cdot), \varepsilon_{\theta_0}(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$ given by (23), the control parameter errors $\phi(t) = \theta(t) - \theta^* = [\phi_k(t), \phi_{\theta_1}^T(t), \phi_{\theta_1}(t), \phi_{\theta_2}^T(t)]^T \in \mathbb{R}^{2n}$ with the control adaptive laws given by (25) and the estimation parameter errors $\eta(t) \triangleq \hat p(t) - p = [\eta_{k_p}(t), \eta_\beta^T(t), \eta_{a_{n-1}}(t), \eta_a^T(t)]^T \in \mathbb{R}^{2n}$ with the identification adaptive laws given by (26), be denoted by S. Then it can be shown that

$$\phi(\cdot), \eta(\cdot), \varepsilon_{\theta_1}(\cdot), \varepsilon_{\theta_2}(\cdot), \varepsilon_{\theta_0}(\cdot), \varepsilon_k(\cdot), \dot\phi(\cdot), \dot\eta(\cdot) \in \mathcal{L}^\infty$$

$$\frac{e_i(\cdot)}{\sqrt{N(\cdot)}}, \varepsilon_{\theta_1}(\cdot), \varepsilon_{\theta_2}(\cdot), \varepsilon_{\theta_0}(\cdot), \varepsilon_\kappa(\cdot), \dot\phi(\cdot), \dot\eta(\cdot) \in \mathcal{L}^2$$

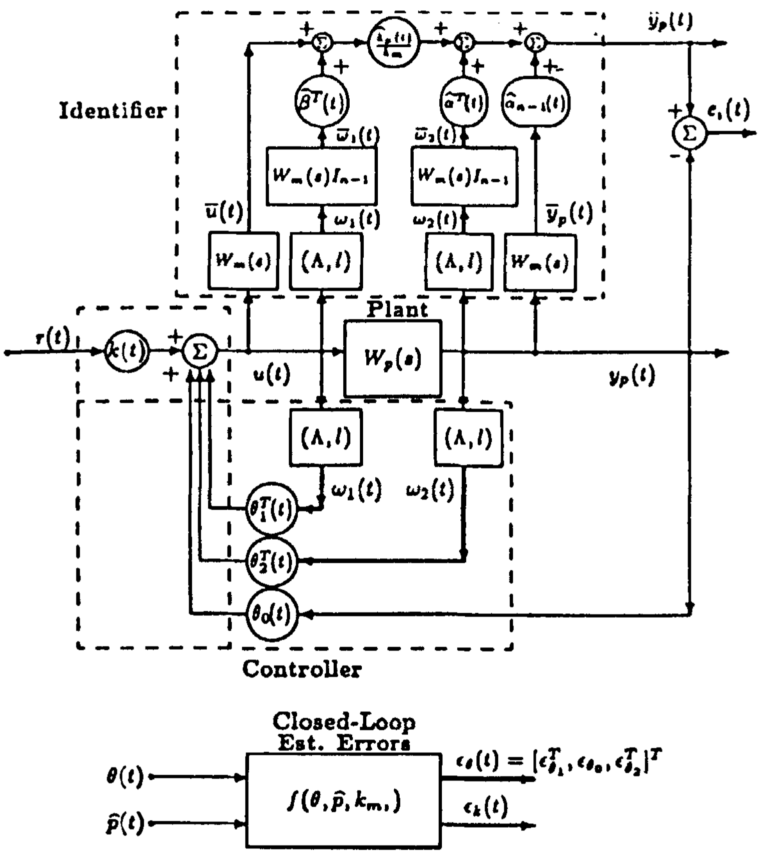

Figure 1. Indirect scheme for a general plant

Proof. Consider the Lyapunov function candidate $V = \frac{1}{2}(\eta_{k_p}^2 + |k_p|\,\eta_\beta^T\eta_\beta + \eta_{\alpha_{n-1}}^2 + \eta_\alpha^T\eta_\alpha + |k_p|\,\phi_k^2 + |k_p|\,\phi'^T_{\theta_1}\phi_{\theta_1} + |k_p|\,\phi_{\theta_0}^2 + |k_p|\,\phi_{\theta_2}^T\phi_{\theta_2})$. The time derivative along any trajectory of the system S defined by equations (21) and (24)–(26) yields $\dot{V} = e_i^2/N(t) - |k_p|\,\varepsilon_{\theta_1}^T\varepsilon_{\theta_1} - \varepsilon_{\theta_2}^T\varepsilon_{\theta_2} - \varepsilon_k^2 - \varepsilon_{\theta_0}^2 \geq 0$. Hence V is indeed a Lyapunov function for the system S and the signals $\theta_1(t)$, $\theta_2(t)$, $\theta_0(t)$, $k(t)$, $\hat{a}(t)$, $\hat{a}_{n-1}(t)$, $\hat{\beta}(t)$ and $\hat{k}_p(t)$ are uniformly bounded, which proves that $\phi(\cdot)$, $\eta(\cdot) \in \mathcal{L}^\infty$. Integrating $V(t)$ between t_0 and ∞, noting that $V(\infty) - V(t_0)$ is finite, we prove that $e_i(\cdot)/\sqrt{N(\cdot)}, \varepsilon_{\theta_1}(\cdot), \varepsilon_{\theta_2}(\cdot), \varepsilon_{\theta_0}(\cdot), \varepsilon_k(\cdot) \in \mathcal{L}^2$.

From equations (25) and (26) it follows that $\dot{\phi}(\cdot)$, $\dot{\eta}(\cdot) \in \mathcal{L}^2$. Then $\varepsilon_{\theta_1}(\cdot)$, $\varepsilon_{\theta_2}(\cdot)$, $\varepsilon_{\theta_0}(\cdot)$, $\varepsilon_k(\cdot)\mathcal{L}^\infty$ follows immediately from equation (23). From equation (25) it can be concluded that $\dot{\phi}(\cdot) \in \mathcal{L}^\infty$.

Defining the vectors $\zeta(t) \triangleq [u(t) + \hat{\beta}^T(t)w_1(t), w_1^T(t), y_p(t), w_2^T(t)]^T \in \mathbb{R}^{2n}$ and $\bar{\zeta}(t) \triangleq [\bar{u}(t) + \hat{\beta}^T(t)\bar{w}_1(t), \bar{w}_1^T(t), \bar{y}_p(t), \bar{w}_2^T(t)] \in \mathbb{R}^{2n}$, equation (21) can be written as

$$e_i(t) = \eta^T(t)P_1\bar{\zeta}(t) = \bar{\zeta}^T(t)P_1\eta(t) \tag{27}$$

where P_1 is the $2n \times 2n$ diagonal matrix defined as

$$P_1 \triangleq \begin{bmatrix} 1/k_m & & & \\ & (k_p/k_m)I_{(n-1)} & & \\ & & 1 & \\ & & & I_{(n-1)} \end{bmatrix}$$

Similarly, equations (26) can be written as

$$\dot{\eta}(t) = \dot{p}(t) = -\frac{e_i(t)}{N(t)} P_2\bar{\zeta}(t) - \xi(t) \tag{28}$$

where P_2 is the $2n \times 2n$ diagonal matrix defined as

$$P_2 \triangleq \begin{bmatrix} 1/k_m & & & \\ & [\operatorname{sgn}(k_p)/k_m]I_{(n-1)} & & \\ & & 1 & \\ & & & I_{(n-1)} \end{bmatrix}$$

with $N(t) = 1 + \bar{\zeta}^T(t)\bar{\zeta}(t)$ and $\xi(t) \triangleq [k(t)\varepsilon_k(t) + \theta_2^T(t)\varepsilon_{\theta_2}(t) + \theta_0(t)\varepsilon_{\theta_0}(t), \varepsilon_{\theta_1}^T(t), k_m\varepsilon_{\theta_0}(t), k_m\varepsilon_{\theta_2}^T(t)]^T \in \mathbb{R}^{2n}$. Replacing (27) in (28), we obtain

$$\dot{\eta}(t) = \dot{p}(t) = -\frac{1}{N(t)} P_2\bar{\zeta}(t)\bar{\zeta}^T(t)P_1\eta(t) - \xi(t)$$

which can be rewritten as

$$\dot{\eta}(t) = \dot{p}(t) = -P_2 \frac{\bar{\zeta}(t)}{\sqrt{[1 + \bar{\zeta}^T(t)\bar{\zeta}(t)]}} \frac{\bar{\zeta}^T(t)}{\sqrt{[1 + \bar{\zeta}^T(t)\bar{\zeta}(t)]}} P_1\eta(t) - \xi(t) \tag{29}$$

Since $\bar{\zeta}(t)/\sqrt{[1 + \bar{\zeta}^T(t)\bar{\zeta}(t)]}$ is bounded and $\eta(t($ and $\xi(t)$ were shown to be bounded, $\dot{\eta}(\cdot) \in \mathcal{L}^\infty$ follows from equation (29). This completes the proof.

In order to prove the boundedness of the other signals in the adaptive system, more involved arguments based on growth rates of unbounded signals are needed. This is stated in Theorem 1.

Theorem 1

The system S defined in Lemma 1 is globally uniformly stable. For uniformly bounded reference input $r(t)$ and arbitrary initial conditions,

$$\text{(i) } \hat{y}_p(t), w_1(t), w_2(t), y_p(t), e_i(t) \in \mathcal{L}^\infty$$

$$\text{(ii) } \lim_{t\to\infty} \{e_i(t), \varepsilon_{\theta 1}^T(t), \varepsilon_{\theta 2}^T(t), \varepsilon_{\theta 0}(t), \varepsilon_k(t)\} = 0$$

$$\text{(iii) } \lim_{t\to\infty} e_c(t) = 0$$

Proof. The proof of stability follows along similar lines as the proof of stability of the direct MRAC for the ideal case, although some fundamental differences can be found along the way. The proof will be sketched indicating the principal steps. For further details the reader is referred to the source references.[4,10,11,14]

From Lemma 1 it is possible to write

$$e_i(t) = \gamma(t)\sqrt{[1 + \bar{\zeta}^T(y)\bar{\zeta}(t)}, \quad \text{where } \gamma(\cdot) \in \mathcal{L}^2 \tag{30}$$

The closed-loop system consisting of plant and controller can be written as[11]

$$\dot{x}(t) = [A_{mn} + b_{mn}\phi'^T(t)C]x(t) + b_{mn}[\phi_k(t) + k^*]r(t) \tag{31}$$

where the $(2n-1)\times(3n-2)$ matrix C and the $(2n-1)$-vector $\phi'(t)$ have the form

$$C=\begin{bmatrix}0 & I_{(n-1)} & 0\\ h_p^T & 0 & 0\\ 0 & 0 & I_{(n-1)}\end{bmatrix},\qquad \phi'(t)=[\phi_{\theta_1}^T(t),\phi_{\theta_0}(t),\phi_{\theta_2}^T(t)]$$

with $h_{mn}^T(sI-A_{mn})^{-1}b_{mn}=(k_p/k_m)W_m(s)$. Here A_{mn} is a $(3n-2)\times(3n-2)$ matrix and b_{mn} and h_{mn} are $(3n-2)$-vectors defined as

$$A_{mn}\triangleq\begin{bmatrix}A_p-\theta_0^*h_pb_p^T & b_p\theta_1^{*T} & b_p\theta_2^{*T}\\ \theta_0^* lh_p^T & \Lambda+l\theta_1^{*T} & l\theta_2^{*T}\\ lh_p^T & 0 & \Lambda\end{bmatrix},\qquad b_{mn}\triangleq\begin{bmatrix}b_p\\ l\\ 0\end{bmatrix},\qquad h_{mn}\triangleq[h_p^T,0,0]^T \tag{32}$$

The expanded state vector is defined as $x(t)=[x_p^T(t),w_1^T(t),w_2^T(t)]^T\in\mathbb{R}^{3n-2}$ and A_p is the $n\times n$ evolution matrix of the plant whose state space representation is $\dot{x}_p(t)=A_px_p(t)+b_pu(t)$, $y_p(t)=h_p^Tx_p(t)$.

Since $\phi(\cdot)\in\mathscr{L}^\infty$, equation (31) can be considered as a linear time-varying differential equation with bounded coefficients. It follows that $\|x(t)\|$ can grow at most exponentially. Also, since $r(\cdot)$ is a piecewise continuous, uniformly bounded function and $\theta(t)$ and $\hat{\beta}(t)$ are uniformly bounded, all components of $x(t)$, $\zeta(t)$ and $\bar{\zeta}(t)$ belong to $PC_{[0,\infty)}$, the set of all real piecewise continuous functions defined on the interval $[0,\infty)$ which have bounded discontinuities.

Let the signals of the system grow in an unbounded fashion, i.e. $\lim_{t\to\infty}\sup_{\tau\le t}\|x(\tau)\|=\infty$. Following a similar reasoning as that used in Reference 11, we can prove that

$$|u(t)|,\|w_1(t)\|=O\Big[\sup_{\tau\le t}\|w(\tau)\|\Big] \tag{33}$$

$$\sup_{\tau\le t}|y_p(\tau)|\sim\sup_{\tau\le t}\|w_2(\tau)\|\sim\sup_{\tau\le t}\|w(\tau)\|\sim\sup_{\tau\le t}\|\bar{w}(\tau)\| \tag{34}$$

where $\bar{w}(t)\triangleq[\bar{r}(t),\bar{w}_1^T(t),\bar{y}_p(t),\bar{w}_2^T(t)]^T\in\mathbb{R}^{2n}$. The notation $g(t)\sim f(t)$ means that $g(t)$ and $f(t)$ grow at the same rate and reads as '$g(t)$ is equivalent to $f(t)$'. Also, $g(t)=O[f(t)]$ means that $g(t)$ does not grow more rapidly than $f(t)$ and reads as '$g(t)$ is large order $f(t)$'.

The output of the identification model (20) can be written as

$$\hat{y}_p(t)=\hat{p}^T(t)\hat{P}_1(y)\bar{w}(t)$$

where $P_1(t)$ is the $2n\times 2n$ diagonal matrix defined as

$$\hat{P}_1(t)\triangleq\begin{bmatrix}1/k_m & & & \\ & [\hat{k}_p(t)/k_m]I_{(n-1)} & & \\ & & 1 & \\ & & & I_{(n-1)}\end{bmatrix}$$

The previous equation can be rewritten as

$$\hat{y}_p(t)=\hat{p}^T(t)\hat{P}_1(t)W_m(s)I_{(2n)}[w(t)]-W_m(s)[\hat{p}^T(t)\hat{P}_1(t)w(t)]+W_m(s)[\hat{p}^T(t)\hat{P}_1(t)w(t)] \tag{35}$$

The term $\hat{p}^{T}(t)\hat{P}_1(t)w(t)$ can be expanded as

$$\hat{p}^{T}(t)\hat{P}_1(t)w(t) = \frac{\hat{k}_p(t)}{k_m} k(t)r(t) + \frac{\hat{k}_p(t)}{k_m}\theta_1^{T}(t)w_1(t) + \frac{\hat{k}_p(t)}{k_m}\theta_0(t)y_p(t)$$

$$+ \frac{\hat{k}_p(t)}{k_m}\theta_2^{T}(t)w_2(t) + \frac{\hat{k}_p(t)}{k_m}\hat{\beta}^{T}(t)w_1(t) + \hat{\alpha}_{n-1}(t)y_p(t) + \hat{\alpha}^{T}(t)w_2(t)$$

Using the definition of the closed-loop estimation errors (23), we have

$$\hat{p}^{T}(t)\hat{P}_1(t)w(t) = \varepsilon^{T}(t)\hat{P}'_1(t)w(t) + r(t) \tag{36}$$

where $\varepsilon(t) \triangleq [\varepsilon_k(t), \varepsilon^{T}{}_{\theta_1}(t), \varepsilon_{\theta_0}(t), \varepsilon^{T}{}_{\theta_2}(t)]^{T} \in \mathbb{R}^{2n}$ and

$$\hat{P}_1(t) \triangleq \begin{bmatrix} 1/k_m & & & \\ & [\hat{k}_p(t)/k_m]I_{(n-1)} & & \\ & & 1/k_m & \\ & & & (1/k_m)I_{(n-1)} \end{bmatrix}$$

Replacing (36) in (35), we can write

$$\hat{y}_p(t) = \hat{p}^{T}(t)\hat{P}_1(t)W_m(s)I_{(2n)}[w(t)] - W_m(s)[\hat{p}^{T}(t)\hat{P}_1(t)_1(t)w(t)]$$
$$+ W_m(s)([\varepsilon^{T}(t)\hat{P}'_1(t)w(t) + r(t)]$$

Since $r, \hat{P}_1, \hat{P}'_1 \in \mathscr{L}^{\infty}$ and $\varepsilon \in \mathscr{L}^2$, we can conclude that

$$\hat{y}_p(t) = o\Big[\sup_{\tau \le t} \|w(\tau)\|\Big] + \delta(t) \tag{37}$$

where $\delta(\cdot) = W_m(s)[r(\cdot)] \in \mathscr{L}^{\infty}$. On the other hand, the output of the plant can be expressed as

$$y_p(t) = \hat{y}_p(t) - e_i(t)$$

Using equations (30) and (37), we can conclude that

$$\hat{y}_p(t) = o\Big[\sup_{\tau \le t} \|w(\tau)\|\Big] + \delta(t) + \gamma(t)\sqrt{[1 + \xi^{T}(t)\xi(t)]}$$

Since $w_2(t) = (sI - \Lambda)^{-1} l y_p(t)$, we have

$$\|w_2(t)\| \le \|(sI - \Lambda)^{-1} l\delta(t)\| + o\Big[\sup_{\tau \le t} \|w(\tau)\|\Big] + o\Big[\sup_{\tau \le t} \|\xi(\tau)\|\Big]$$

Hence we can conclude that

$$\|w_2(t)\| = o\Big[\sup_{\tau \le t} \|w(\tau)\|\Big]$$

which contradicts equation (34), according to which $w_2(t)$ and $w(t)$ grow at the same rate if they grow in an unbounded fashion. Hence all the signals in the feedback loop are uniformly bounded and properties (i) immediately follow.

It was proved that $\varepsilon_{\theta_1}, \varepsilon_{\theta_2}, \varepsilon_{\theta_0}, \varepsilon_k \in \mathscr{L}^2$, and since it can be seen that $\dot{\varepsilon}_{\theta_1}, \dot{\varepsilon}_{\theta_2}, \dot{\varepsilon}_{\theta_0}, \dot{\varepsilon}_k \in \mathscr{L}^{\infty}$, $\lim_{t\to\infty}\{\varepsilon_{\theta}^{T}(t), \varepsilon_{\theta_2}^{T}, \varepsilon_{\theta_0}(t), \varepsilon_k(t),\} = 0$. Also, since it was shown that $\bar{\xi}$ and $\dot{\eta}$ are bounded, $\dot{e}_i$ is bounded. Finally, from equation (30), $\xi \in \mathscr{L}^{\infty}$ implies that $e_i \in \mathscr{L}^2$.

Hence $\lim_{t\to\infty} e_i(t) = 0$. From equation (37) it can be seen that $\lim_{t\to\infty} \hat{y}_p(t) = y_m(t)$, hence $\lim_{t\to\infty} e_c(t) = 0$, proving property (iii). This completes the proof of Theorem 1.

Comment 4

In this section the indirect method described in Section 2 for a first-order plant was extended to the general case of a plant of order n and arbitrary relative degree $1 \leqslant n^* \leqslant n$. As in direct control, a substantial simplification can be achieved when $n^* = 1$. In such a case the parametrization of the plant directly in terms of a_i $(i = 0, 1, \ldots, n-1)$ and b_j $(j = 0, 1, \ldots, n-2)$ can be used, since a linear relationship exists between the plant parameters and the desired control parameters;[15] In this parametrization, filtered signals $\bar{y}_p(t)$, $\bar{u}(t)$, $\bar{w}_1(t)$ and $\bar{w}_2(t)$ are not needed and signals $y_p(t)$, $u(t)$, $w_1(t)$ and $w_2(t)$ are used instead. The way in which identification and controller parameters are adjusted is basically that shown in (25) and (26), except for the normalization factor $N(t)$. The parametrizations used in Section 3 for the plant as well as the controller are needed to obtain a similar linear relationship even in the more general case of $n^* \geqslant 2$.

Comment 5

It can be shown that if the spectral measure of the reference input $r(t)$ is concentrated in at least $2n$ points, then the parameter errors $\phi(t) \in \mathbb{R}^{2n}$ and $\eta(t) \in \mathbb{R}^{2n}$ will also tend to zero asymptotically. Equivalently, if the reference input $r(t)$ is persistently exciting in $\mathbb{R}^{2n}$, the $4n$ elements of the parameter error vectors $\phi(t)$ and $\eta(t)$ will tend to zero as $t \to \infty$.

Comment 6

For the sake of simplicity the above stability analyses for first- and nth-order plants were carried out using unity adaptive gains. These results can be directly extended to cases where the adaptive gains are arbitrary positive constants, constant positive definite matrices or even time-varying positive definite matrices.

3.7. Simulation example

In this subsection a set of simulations of a second-order plant is presented to compare the behaviour of the direct MRAC and the dynamical indirect MRAC proposed here.

An unstable second-order plant was simulated to test the MRAC schemes under ideal conditions. The differential equation describing the plant is

$$\ddot{y}_p(t) + \dot{y}_p(t) - 2y_p(t) = 2u(t)$$

where $y_p(0) = 0{\cdot}5$. The model reference was chosen as

$$\ddot{y}_m(t) + 4\dot{y}_m(t) + 3y_m(t) = r(t)$$

where $y_m(0) = 0$. Simulation results are shown in Figures 2 and 3 for different types of reference inputs. All initial conditions were set to zero and all adaptive gains were chosen as 10. The value of λ was chosen as -1.

From the simulations it can be seen that the theoretical results are verified. In particular, in all simulations $e_c(t) \to 0$ as $t \to \infty$ without persistent excitation. The better transient behaviour of the proposed scheme is evident.

An interesting point is that parametric convergence (controller and identifier) is achieved if only parameter controller convergence is obtained. In fact,let us assume that persistent excitation is such that controller parameter errors are driven to zero; then, since closed-loop estimation errors given by (24) tend to zero, it can be concluded that plant parameter errors are

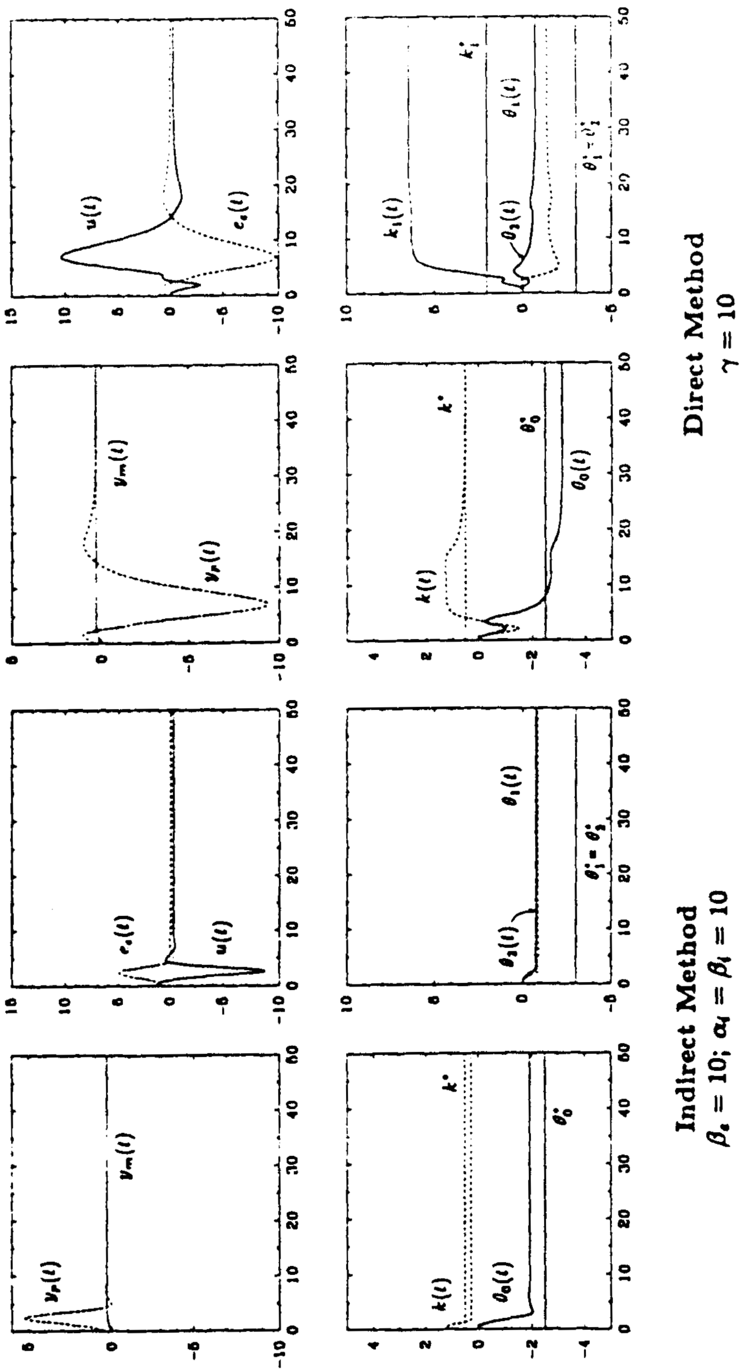

Figure 2. MRAC for second-order plant and constant reference $r(t) = 1$

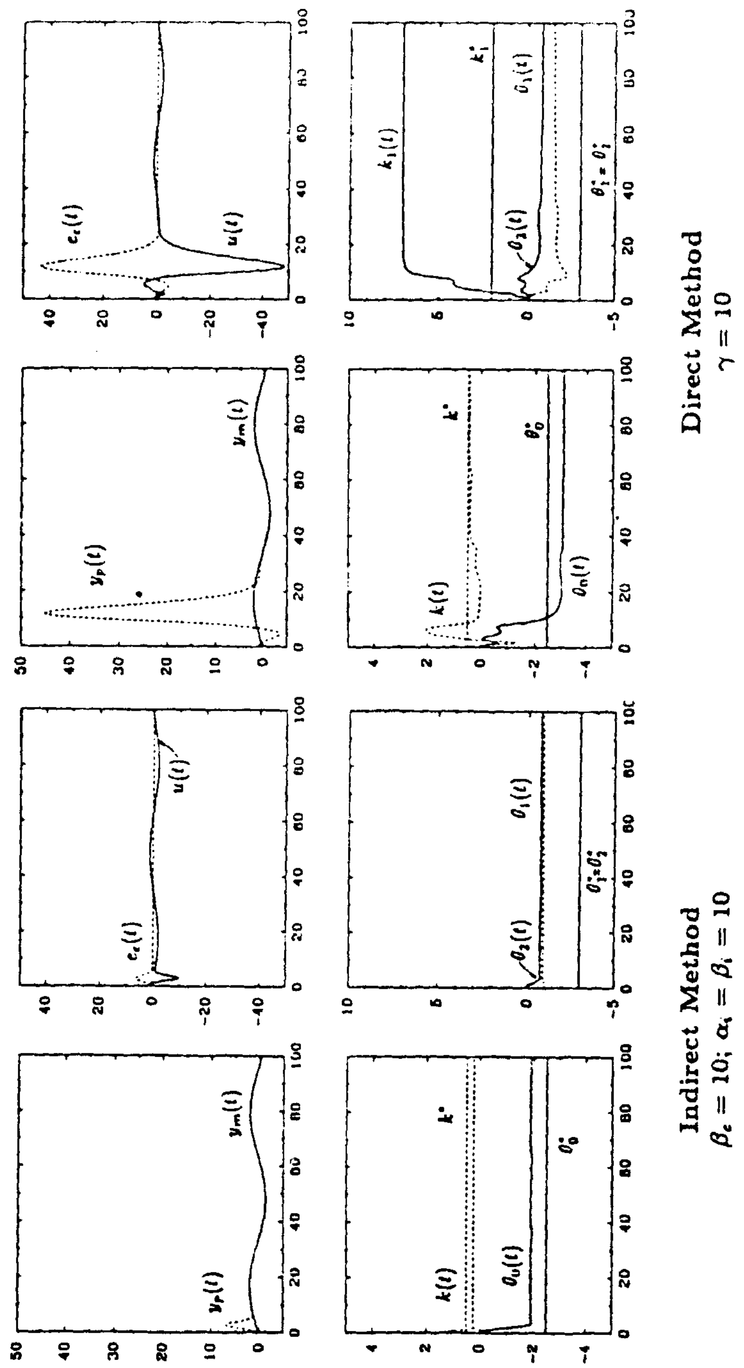

Figure 3. MRAC for second-order plant and sinusoidal reference $r(t) = 1 + 5\sin(0{\cdot}1t)$

also driven to zero. This means that from the persistent excitation viewpoint the dynamical indirect MRAC and direct MRAC schemes are equivalent.

3.8. *Robustness of dynamical IMRAC*

The dynamical indirect model reference adaptive controller presented in the previous subsection can be modified to account for bounded external perturbations. One way of doing this is to modify the adaptive laws in the same way as in the direct MRAC. We will consider the plant defined by

$$\dot{x}_p(t) = A_p x_p(t) + b_p u(t) + d_p p_1(t)$$
$$y_p(t) = c_p^T x_p(t) + p_2(t)$$

where $A_p \in \mathbb{R}^{n\times n}$ is the evolution matrix, $x_p, b_p, d_p, c_p \in \mathbb{R}^n$ are the state vector, command vectors and observation vector respectively, $u, y_p, p_1, p_2 \in \mathbb{R}$ are the plant input, plant output, input perturbation and output perturbation respectively and $p_1(\cdot), p_2(\cdot)$: $[0,\infty) \to \mathbb{R}$ are two bounded piecewise continuous functions of time. In addition, it is assumed that $p_1(t)$ is differentiable.

The structure of the adaptive laws now has the general form

$$\begin{aligned}
\dot{\eta}_{kp}(t) = \dot{\hat{k}}_p(t) &= \gamma\{e_i(t)[\hat{\beta}^T(t)\bar{w}_1(t) + \bar{u}(t)]/[N(t)k_m] \\
&\qquad - k(t)\varepsilon_k(t) - \theta_0(t)\varepsilon_{\theta_0}(t) - \theta_2^T(t)\varepsilon_{\theta_2}(t)\} - \hat{k}_p(t)f_i(\hat{k}_p) \\
\dot{\eta}_\beta(t) = \dot{\hat{\beta}}(t) &= \gamma\{-\mathrm{sgn}(k_p)e_i(t)\bar{w}_1(t)/[N(t)k_m] - \varepsilon_{\theta_1}(t)\} - \hat{\beta}(t)f_i(\hat{\beta}) \\
\dot{\eta}_{a_{n-1}}(t) = \dot{\hat{a}}_{n-1}(t) &= \gamma[-e_i(t)\bar{y}_p(t)/N(t) - k_m\varepsilon_{\theta_0}(t)] - \hat{a}_{n-1}(t)f_i(\hat{a}_{n-1}) \\
\dot{\eta}_a(t) = \dot{\hat{a}}(t) &= \gamma[-e_i(t)\bar{w}_2(t)/N(t) - k_m\varepsilon_{\theta_2}(t)] - \hat{a}(t)f_i(\hat{a})
\end{aligned} \tag{38}$$

$$\begin{aligned}
\dot{\phi}_k(t) = \dot{k}(t) &= \gamma[-\mathrm{sgn}(k_p)\varepsilon_k(t)] - k(t)f_c(k) \\
\dot{\phi}_{\theta_1}(t) = \dot{\theta}_1(t) &= \gamma[-\varepsilon_{\theta_1}(t)] - \theta_1(t)f_c(\theta_1) \\
\dot{\phi}_{\theta_0}(t) = \dot{\theta}_0(t) &= \gamma[-\mathrm{sgn}(k_p)\varepsilon_{\theta_0}(t)] - \theta_0(t)f_c(\theta_0) \\
\dot{\phi}_{\theta_2}(t) = \dot{\theta}_2(t) &= \gamma[-\mathrm{sgn}(k_p)\varepsilon_{\theta_2}(t)] - \theta_2(t)f_c(\theta_2)
\end{aligned} \tag{39}$$

where $N(t) \in \mathbb{R}$ is the normalization factor defined in (26). The functions $f_i(\cdot)$ and $f_c(\cdot)$ and the parameter γ have different meanings depending on the modifications introduced in the adaptive laws. It will be assumed that external perturbations are uniformly bounded, i.e. $|p_1(t)| \geqslant p_0$ and $|p_2(t)| \geqslant p_0$.

To provide global stability of the overall adaptive system, Kreisselmeier and Narendra[16] propose modified adaptive laws for the direct MRAC. In this case the adaptive laws (25) and (26) for adjusting controller and identifier parameters take the form indicated in (38) and (39) with $\gamma = 1$ and the functions $f_i(\cdot)$ and $f_c(\cdot)$ are defined as

$$f(x)\begin{cases}(1 - \|x\|/x_M^*)^2 & \text{if } \|x\| > x_M^* \\ 0 & \text{elsewhere}\end{cases} \tag{40}$$

where $\|x(t)\| \geqslant x_M^*$ for all $t \leqslant t_0$ (x_M^* is a known bound on the norm of $x(t)$). This scheme uses the knowledge of a bound on $\|\theta^*\|$ as well as on $\|p\|$.

In order to provide global stability of the adaptive system for the direct MRAC under bounded external perturbations, Peterson and Narendra[17] suggested modified adaptive laws including a dead zone. The corresponding adaptive laws for the IMRAC based on these results are those shown in (38) and (39) with $\gamma = 1$ and $f_i(\cdot) = f_c(\cdot) = 0$ if $|e_i(t)| > p_0$. Otherwise the updating procedure is stopped by making zero the time derivatives of the parameter estimates, i.e. $\gamma = 0$ and $f_i(\cdot) = f_c(\cdot) = 0$ if $|e_i(t)| \leq p_0$.

Following the ideas of Ioannou and Kokotovic,[18] the adaptive laws suggested here for adjusting controller and identifier parameters to provide global stability of the adaptive system under bounded external perturbations are those mentioned in (38) and (39) with $f_i(\cdot) = \sigma_i/N(t)$ and $f_c(\cdot) = \sigma_c/N(t)$, where $\sigma_i > 0$ and $\sigma_c > 0$.

If we now use the modification suggested by Narendra and Annaswamy[19] for the direct MRAC, in this case the adaptive laws are given by equations (38) and (39) with $f_i(\cdot) = \tau_i |e_i(t|/N(t)$ and $f_c(\cdot) = \tau_c |\varepsilon_c|/N(t)$.

4. CONCLUSIONS

Conventional methods of adaptive control based on the indirect approach involve the solution of algebraic equations for determining control parameters. In contrast, the method discussed in this paper is based on the dynamic adjustment of control parameters, which bypasses many of the difficulties of the conventional indirect MRAC. It is also shown that such an approach is equivalent to the direct method, in that the two achieve stable adaptation using identical prior information.

Since the overall adaptive system is described by a set of stable non-linear differential equations, many of the difficulties encountered when the algebraic method is used are avoided in this case. It has also been shown by the authors[9,10] that the approach suggested here can be readily combined with the direct MRAC method to improve overall performance.

Robustness of the dynamical MRAC can be achieved by modifying control as well as identification adaptive laws. This was presented in four schemes included in Section 3. With these modifications the resulting adaptive scheme is globally stable in the presence of bounded external perturbations.

ACKNOWLEDGEMENTS

The work reported here was supported by the National Science Foundation under Grant ECS-8520428 and by CONICYT under Project FONDECYT 1199/91.

REFERENCES

1. Kreisselmeier, G., 'An approach to stable indirect adaptive control', *Automatica*, **21**, 425–431 (1985).
2. Middleton, R., 'Indirect continuous time adaptive control', *Automatica*, **23**, 793–795 (1985).
3. Goodwin, G. C. and K. S. Sin, *Adaptive Filtering Prediction and Control*, Prentice-Hall, Englewood Cliffs, NJ, 1984.
4. Narendra, K. S., Y. H. Lin and L. S. Valavani, 'Stable adaptive controller design—Part II: Proof of stability', *IEEE Trans. Automatic Control*, **AC-25**, 440–448 (1980).
5. Morse, A. S., 'Towards a unified theory of parameter adaptive control: tunability', *IEEE Trans. Automatic Control*, **AC-35**, 1002–1012 (1990).
6. Kreisselmeier, G., 'Adaptive control via adaptive observation and asymptotic feedback matrix synthesis', *IEEE Trans. Automatic Control*, **AC-25**, 717–722 (1980).
7. Kreisselmeier, G., 'On adaptive state regulation', *IEEE Trans. Automatic Control*, **AC-27**, 3–17 (1982).
8. Kreisselmeier, G. and M. C. Smith, 'Stable adaptive regulation of arbitrary n-th order plants', *IEEE Trans. Automatic Control*, **AC-31**, 299–305 (1986).
9. Duarte, M. A. and K. S. Narendra, 'Combined direct and indirect approach to adaptive control', *IEEE Trans. Automatic Control*, **AC-34**, 1071–1075 (1989).
10. Duarte, M. A. and K. S. Narendra, 'A new approach to model reference adaptive control', *Int. j. adapt. control signal process.*, **3**, 53–73 (1989).
11. Narendra, K. S. and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1989.
12. Barbalat, I., 'Systemes d'équations différentielles d'oscillations non linéares', *Rev. Math. Pures Appl., Bucharest*, **IV**, 267–270 (1959).

13. Narendra, K. S. and L. S. Valavani, 'Direct and indirect model reference adaptive control', *Automatica*, **15**, 653–664 (1979).
14. Narendra, K. S., A. M. Annaswamy and R. P. Singh, 'A general approach to the stability analysis of adaptive systems', *Int. J. Control*, **41**, 193–216 (1985).
15. Duarte, M. A. 'Stable direct and indirect adaptive control', *Ph.D. Thesis*, Department of Electrical Engineering, Yale University, New Haven, CT, 1988.
16. Kreisselmeier, G. and K. S. Narendra, 'Stable model reference adaptive control in the presence of bounded disturbances', *IEEE Trans. Automatic Control*, **AC-27**, 1169–1175 (1982).
17. Peterson, B. B. and K. S. Narendra, 'Bounded error adaptive control', *IEEE Trans. Automatic Control*, **AC-27**, 1161 –1168 (1982).
18. Ioannou, P. A. and P. V. Kokotovic, *Adaptive Systems with Reduced Models*, Springer, New York, 1983.
19. Narendra, K. S. and A. M. Annaswamy, 'A new adaptive law for robust adaptive control without persistent excitation', *IEEE Trans. Automatic Control*, **AC-32**, 134–145 (1987).