Hindawi
Scientific Programming
Volume 2022, Article ID 5077968, 11 pages
https://doi.org/10.1155/2022/5077968

Research Article

Indiscernibility and Discernibility Relations Attribute Reduction with Variable Precision

Xu Li,[1] **Jianguo Tang**,[1] **Bing Hu**,[1] **and Yi Li**[2]

[1]*Artificial Intelligence and Big Data College, Chongqing College of Electronic Engineering, No. 76 Daxuecheng East Road, Shapingba District, Chongqing 401331, China*
[2]*School of Information Science, Beijing Language and Culture University, No. 15 Xueyuan Road, Haidian District, Beijing 100083, China*

Correspondence should be addressed to Xu Li; lixufe12@163.com

Received 21 December 2021; Revised 4 March 2022; Accepted 18 March 2022; Published 25 April 2022

Academic Editor: Jianping Gou

Attribute reduction is a popular topic in research on rough sets. In the classical model, much progress has been made in the study of the attribute reduction of indiscernibility and discernibility relations. To enhance the fault tolerance of the model, concepts of both indiscernibility and discernibility relations involving uncertain or imprecise information are proposed in this paper. The attribute reductions of the relative β-indiscernibility relation and relative β-discernibility relation and their algorithms are proposed. When the precision satisfies certain conditions, the reduction of two relation concepts can be converted into a positive region reduction. Therefore, the discernibility matrix is used to construct the reductions of the two relation concepts and the positive region. Furthermore, the corresponding algorithm of the relative β-indiscernibility (discernibility) relation reduction can be optimized when the precision is greater than 0.5, and this is used to develop an optimization algorithm that constructs the discernibility matrix more efficiently. Experiments show the feasibility of the two relation reduction algorithms. More importantly, the reduction algorithms of the two relations and the optimization algorithm are compared to demonstrate the feasibility of the optimization algorithm proposed in this paper.

1. Introduction

The rough set (RS) theory [1, 2], proposed by Polish mathematician Zdzislaw Pawlak, is a useful data processing method for dealing with incomplete and inconsistent problems. The investigation of indiscernibility and discernibility relations between objects is an important task in RS theory. In classical RS theory, an indiscernibility relation between objects exists when all of the objects have the same attribute values. A discernibility relation exists between the objects if and only if not every attribute value of the objects is the same. Existing research has studied in-depth indiscernibility and discernibility relations in RSs [3–5] in certain research fields. For instance, in political sociology, if one side emphasizes the difference between countries, this can cause conflict to expand; on the contrary, if one side emphasizes the commonality between countries, this can provide better conditions for negotiation [6]. Because classical RS theory is sensitive to classification error, its application is quite limited, and its classification ability is poor. To overcome this limitation, the variable precision (VP) model [7, 8] is constructed using precision, which processes the data more effectively, thus advancing the development of RS theory and broadening its application to other fields.

Even more in-depth research has been performed on attribute reduction, which forms the minimum subset of knowledge classification by deleting redundant or unrelated attributes according to specific rules. Attribute reduction has been studied in reference [9–14] and has been applied to fault diagnosis, risk assessment, and other fields [15]. In reference [16], a VP model that is more fault-tolerant than the RS model was proposed, and seven forms of reduction in the VP model were discussed. In reference [17–19], the concept of attribute reduction in the VP model was proposed, and through a strict proof of the corresponding discernibility matrix, a reduction algorithm based on that matrix was given. Beynon [20] studied the VP reduction by comparing different thresholds. The influence of parameters on classification was studied in the VP model [21]. Chen et al. [22] defined a fuzzy relation on condition attributes, and then, a local attribute reduction algorithm was proposed using the corresponding discernibility matrix. In addition, fuzzy relations [23–25] have been studied in [26–28]. The aim of most research on attribute reduction is to solve practical problems in different scenarios and has led to various forms of attribute reduction, such as covering reduction [29, 30], cost-sensitive reduction [31–33], local attribute reduction [22, 34, 35], incremental attribute reduction [36–38], and information entropy reduction [39, 40].

Skowron et al. [41, 42] were the first to propose using the discernibility matrix to reduce attributes in information systems. This method can determine all the reductions by transforming the function in conjunctive normal form (CNF) into a function in disjunctive normal form (DNF). This method is called the discernibility matrix-based reduction method and is supported by mathematical proofs. Although its calculation complexity is high, it remains the main method to obtain all reductions.

In this paper, the relative β-indiscernibility relation (β–IR) and relative β-discernibility relation (β –DR) of the VP model are proposed, in which VP model, probabilistic model [43], and decision-theoretic model [44] all extend RS. However, the differences among them lie in the selection of parameters. Because all the results can be obtained using the discernibility matrix, this paper is based on the discernibility matrix. Moreover, the relationships among relative β–IR reduction, relative β-DR reduction, and positive region reduction when the precision is greater than 0.5 are investigated after modifying the decision values of some objects.

The structure of this paper is divided into six parts. In Section 2, there are reviews of upper approximation, lower approximation, and β-approximation. In Section 3, the definition and reduction of the relative β–IR are proposed, and then, an algorithm for relative β–IR reduction is given. In Section 4, the definition and reduction of the relative β-DR are proposed, and then, an algorithm for relative β-DR relation reduction is also given. In Section 5, it is shown that when the precision is greater than 0.5, the relative β–IR (relative β-DR) reduction in a decision table is equivalent to the positive region reduction in a new decision table in which some objects' decision values have been updated. Then, an optimization algorithm for converting the relative β–IR (relative β-DR) reduction to a positive region reduction is proposed. In Section 6, we used a naive Bayes (NB) classifier, support vector machine (SVM), and decision tree (DT) on UCI datasets to compare the accuracy after reduction and evaluate the effectiveness of the proposed algorithms. Section 7 concludes the paper.

2. Preliminary

An information table [1, 2, 5] is also called an information system. It is assumed that the knowledge in an information table is described and represented by a set of rows and a set of columns, in which rows and columns denote objects and attributes, respectively. The information table is a tuple $S = (U, At, \{V_a|a \in At\}), \{I_a|a \in At\}$, where U is a universal set, At is a attribute set, V_a is nonempty set of values for an attribute a, and $I_a: U \longrightarrow V_a$ is a function such that each $x \in U$ takes a value $I_a(x)$ on attribute a. Given $A \subseteq At$, the equivalence relation is defined by $R_A = \{(x, y)|x, y \in U \times U, I_a(x) = I_a(y)\}$ for each $a \in A$. Moreover, an equivalence class for x on a set of attribute A is denoted as $[x]_A = \{y|(x, y) \in R_A\}$. It can be easily seen that this equivalence relation is reflexive, symmetric, and transitive. For $A \subseteq At$, the quotient set is denoted by $U/A = \{A_1, A_2, \ldots, A_l\}$, where $[x]_A \in U/A$.

In $S = (U, At)$, if C is a set of condition attributes and D is a set of decision attributes, where $At = C \cup D$ and $C \cap D = \varnothing$, then an information table is called a *decision table*. In this paper, a decision table S is written as $(U, C \cup D)$, where $C = \{a_1, a_2, \ldots, a_m\}$, $D = \{d1, d2, \ldots, dt\}$, and $R_d = \cap_{i=1}^{t} R_{di}$. For the convenience of proof, a set of decision attributes is written as a singleton set, namely $D = \{d\}$.

Definition 1 (see [1, 2]). Let (U, C) be an information system and X be a subset of U, $B \subseteq C$. The lower and upper approximations of X are as follows:

$$\begin{aligned} \underline{R}(X) &= \{x|[x]_R \subseteq X\}, \\ \overline{R}(X) &= \{x|[x]_R \cap X \neq \varnothing\}. \end{aligned} \tag{1}$$

Definition 2 (see [1, 2]). Let $(U, C \cup D)$ be a decision table in which U/D is a quotient set induced by D. The positive region is defined as follows:

$$Pos_C(D) = \cup_{i=1}^{l} \left(\underline{R_C}(D_i)\right), \tag{2}$$

where $D_i \in U/D$.

That is, for each $[x]_C$, if $[x]_C \subseteq Pos_C(D)$, then $[x]_C$ is contained in $[x]_D$. Conversely, if $[x]_C$ is contained in $[x]_D$, then $[x]_C \subseteq Pos_C(D)$.

Definition 3 (see [19]). Let $(U, C \cup D)$ be a decision table, where $B \subseteq C$. If it satisfies the following conditions,

$$\begin{aligned} &Pos_C(D) = Pos_B(D), \\ &\text{for any } B' \subset B, Pos_B(D) \neq Pos_{B'}(D). \end{aligned} \tag{3}$$

then B is called the positive region reduction of C.

Given a decision table, its corresponding discernibility matrix [11] of positive region reduction $MP = (mp_{ij})_{s \times n}$ is as follows:

$$mp_{ij} = \begin{cases} a|a \in C, (x_i, x_j) \notin R_a, & x_i \in Pos_C(D), (x_i, x_j) \notin R_D, \\ \varnothing, & \text{otherwise}. \end{cases} \quad (4)$$

In a matrix F, s is equal to $|Pos_C(D)|$, n is the cardinality of U, and $|Pos_C(D)|$ is the cardinality of the positive region.

Definition 4. (see [34]). Given $X \subseteq U$, for each $x \in U$, the characteristic function $\lambda_X(x)$ is defined as follows:

$$\lambda_X(x) = \begin{cases} 1, x \in X, \\ 0, x \notin X. \end{cases} \quad (5)$$

Lemma 1 (see [7, 34]). *For positive integers $i \in \{1, 2, \ldots, n\}$, where $[x_i]_R$ is an equivalence class on relation R and $X \subseteq U$, $W_R \lambda_X$ is expressed as follows:*

$$W_R \lambda_X = \left[\frac{|[x_1]_R \cap X|}{|[x_1]_R|}, \frac{|[x_2]_R \cap X|}{|[x_2]_R|}, \ldots, \frac{|[x_n]_R \cap X|}{|[x_n]_R|} \right]^T, \quad (6)$$

where T denotes the transpose.

For $x_i \in U$, $|[x_i]_R \cap X| / |[x_i]_R| = [(\lambda_R (x_i, x_1) / |[x_i]_R| \lambda_X (x_1)) + (\lambda_R (x_i, x_2) / |[x_i]_R| \lambda_X (x_2)) + \ldots + (\lambda_R (x_i, x_n) / |[x_i]_R| \lambda_X (x_n))]$, where $\lambda_R (x_i, x_j) = 1$ if and only if $(x_i, x_j) \in R$.

Definition 5 (see [7, 34]). Let R be an equivalence relation on U, $\beta \in (0, 1)$. Then, the β -approximation of X is defined as

$$R^{\beta}(X) = \{x | P(X | [x]_R) \geq \beta\}, \quad (7)$$

where $P(X|[x]_R) = |[x]_R \cap X| / |[x]_R|$.

Theorem 1 (see [3, 4]). *Let R be an equivalence relation on U. Then, for any subset $X \subseteq U$, then $\lambda_{R(\beta)}(X) = (W_R \lambda_X)_\beta$, where X_β denotes the β-cut set of fuzzy set X.*

Lemma 2 (see [3, 4]). *For a decision table, suppose that $\mu_{CD} = (W_{R_C} \lambda_{D_1}, W_{R_C} \lambda_{D_2}, \ldots, W_{R_C} \lambda_{D_l})$, where D_i is a decision class of objects with a decision value that is equal to $f_d(i)$. Then, $(b_{ij})_{n \times l}$ is a fuzzy matrix denoted as*

$$\begin{pmatrix} \mu_{CD}(x_1) \\ \mu_{CD}(x_2) \\ \vdots \\ \mu_{CD}(x_n) \end{pmatrix} = \begin{pmatrix} \frac{|[x_1]_C \cap D_1|}{|[x_1]_C|} & \frac{|[x_1]_C \cap D_2|}{|[x_1]_C|} & \cdots & \frac{|[x_1]_C \cap D_l|}{|[x_1]_C|} \\ \frac{|[x_2]_C \cap D_1|}{|[x_2]_C|} & \frac{|[x_2]_C \cap D_2|}{|[x_2]_C|} & \cdots & \frac{|[x_2]_C \cap D_l|}{|[x_2]_C|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{|[x_n]_C \cap D_1|}{|[x_n]_C|} & \frac{|[x_n]_C \cap D_2|}{|[x_n]_C|} & \cdots & \frac{|[x_n]_C \cap D_l|}{|[x_n]_C|} \end{pmatrix}, \quad (8)$$

where $b_{ij} = |[x_i]_C \cap D_j| / |[x_i]_C| = P(D_j | [x_i]_C)$, i is a positive integer, and j is a positive integer.

Theorem 2 (see [7, 34]). *Given $(U, C \cup D)$, $\forall x \in U$, for $\beta \in [0, 1)$,*

$$(\mu_{CD})_\beta = \lambda_{(R_C)^{(\beta)}(D_1)}(x), \lambda_{(R_C)^{(\beta)}(D_2)}(x), \ldots, \lambda_{(R_C)^{(\beta)}(D_l)}(x). \quad (9)$$

The proof of Theorem 2 is shown in reference [34]. In this section, $(\mu_{CD})_\beta$ was introduced using the β-cut set of fuzzy set X. In Section 3, the relative β-indiscernibility relation is proposed for decision tables, and then, the corresponding discernibility matrix is proposed.

3. Relative β-Indiscernibility Relation Reduction

Given $(U, C \cup D)$, with the quotient set $U/D = \{D_1, D_2, \ldots, D_l\}$ induced by the equivalence relation R_D, we first define the relative β–indiscernibility relation for $\beta \in (0, 1)$.

Definition 6. Given a set B, the relative β-indiscernibility relation is defined as follows:

$$Ind_B^{\beta}(D) = \{(x, y) | x, y \in U, (\mu_{BD}(x))_\beta = (\mu_{BD}(y))_\beta\}, \quad (10)$$

where $(\mu_{BD}(x))_\beta = (|[x]_B \cap D_1| / |[x]_B|, |[x]_B \cap D_2| / |[x]_B|, \ldots, |[x]_B \cap D_l| / |[x]_B|)_\beta$ and $\beta \in (0, 1)$.

For a binary relation $Ind_B^{\beta}(D)$ on U, because $\forall (x, x) \in Ind_B^{\beta}(D)$, it is reflexive. If $\forall (x, y) \in Ind_B^{\beta}(D)$, then $(y, x) \in Ind_B^{\beta}(D)$, and it is symmetric. $\forall x, y, z \in U$, $\forall (x, y) \in Ind_B^{\beta}(D)$ and $(y, z) \in Ind_B^{\beta}(D)$, then $(x, z) \in Ind_B^{\beta}(D)$, and hence, it is transitive. Because relation $Ind_B^{\beta}(D)$ satisfies the above three properties, it is an equivalence relation.

In the decision table presented in Table 1, the condition set C contains a_1, a_2, a_3, and D is a decision attribute set. Here, $U/C = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \{x_7\}\}$ and $U/D = \{D_1, D_2, D_3\}$, where $D_1 = \{x_1, x_3\}$, $D_2 = \{x_2 x_4, x_5, x_6\}$, and $D_3 = \{x_7\}$. For different values of the threshold β, $Ind_B^{\beta}(D)$ may be different.

If $\beta = 0.3$, then by Theorem 2, we have $(\mu_{CD}(x_1))_{0.3} = (\mu_{CD}(x_2))_{0.3} = (1, 1, 0)$, $(\mu_{CD}(x_3))_{0.3}$

Table 1: Decision table.

U	C			D
	a_1	a_2	a_3	
x_1	1	1	1	0
x_2	1	1	1	0
x_3	0	1	0	0
x_4	0	1	0	1
x_5	0	1	0	1
x_6	1	0	0	1
x_7	0	1	1	2

$= (\mu_{CD}(x_4))_{0.3} = (\mu_{CD}(x_5))_{0.3} = (1,1,0)$, $(\mu_{CD}(x_6))_{0.3} = (0,1,0)$, and $(\mu_{CD}(x_7))_{0.3} = (0,0,1)$.

If $\beta = 0.6$, then by Theorem 2, we have $(\mu_{CD}(x_1))_{0.6} = (\mu_{CD}(x_2))_{0.6} = (0,0,0)$, $(\mu_{CD}(x_3))_{0.6} = (\mu_{CD}(x_4))_{0.6} = (\mu_{CD}(x_5))_{0.6} = (0,1,0)$, $(\mu_{CD}(x_6))_{0.6} = (0,1,0)$, and $(\mu_{CD}(x_7))_{0.6} = (0,0,1)$.

According to Definition 6, we have, for example, $(x_1,x_3) \in Ind_C^{0.3}(D)$ and $(x_3,x_6) \notin Ind_C^{0.3}(D)$; however, $(x_1,x_3) \notin Ind_C^{0.6}(D)$, $(x_3,x_6) \in Ind_C^{0.6}(D)$.

Definition 7. Given $(U, C \cup D)$, if $B \subseteq C$ satisfies the following two conditions:

(a) $Ind_C^{\beta}(D) = Ind_B^{\beta}(D)$

(b) for any $B' \subset B$, $Ind_{B'}^{\beta}(D) \neq Ind_C^{\beta}(D)$

then B is the reduction of the relative β–IR.

Given a decision table, a corresponding discernibility matrix $M^{\beta} = (m_{ij}^{\beta})_{n\times n}$ for the relative β–IR is as follows:

$$m_{ij}^{\beta} = \begin{cases} \{a \mid a \in C, (x_i, x_j) \notin R_a\}, x_i \in Pos_C(D), & if (x_i, x_j) \notin Ind_C^{\beta}(D) \\ \varnothing, & \text{otherwise} \end{cases}, \tag{11}$$

where n is $|U|$.

Lemma 3. *Let $(U, C \cup D)$ be a decision table and $\forall(x,y) \in U \times U$. If $(x,y) \notin Ind_C^{\beta}(D)$, then $m_{ij}^{\beta} \neq \varnothing$.*

Proof. If $(x,y) \notin Ind_C^{\beta}(D)$, then $\exists D_k \in U/D$ s.t. $(|[x_i]_C \cap D_k|/|[x_i]_C|)_\beta \neq (|[x_j]_C \cap D_k|/|[x_j]_C|)_\beta$. Thus, $(x_i, x_j) \notin R_C$. Hence, there exists $a_s \in C$ such that $(x_i, x_j) \notin R_s$. Then, $m_{ij}^{\beta} \neq \varnothing$. □

Theorem 3. *Let $(U, C \cup D)$ be a decision table. For $B \subseteq C$, the following three conditions are equivalent:*

(a) $Ind_C^{\beta}(D) = Ind_B^{\beta}(D)$

(b) if $m_{ij}^{\beta} \neq \varnothing$, $m_{ij}^{\beta} \cap B \neq \varnothing$

(c) if $(x,y) \in R_B$, then $(x,y) \in Ind_C^{\beta}(D)$

Proof. (a)⇒(b) Let $m_{ij}^{\beta} \neq \varnothing$. Then, there exists $(x_i, x_j) \notin Ind_C^{\beta}(D)$. Using proof by contradiction, suppose $m_{ij}^{\beta} \cap B = \varnothing$. Then, $(x_i, x_j) \in R_B$. Thus, by Definition 6, $(x_i, x_j) \in Ind_B^{\beta}(D)$, which leads to a contradiction.

(b)⇒(c) Using proof by contradiction, assume $(x_i, x_j) \notin Ind_C^{\beta}(D)$. By Lemma 3, $m_{ij}^{\beta} \neq \varnothing$. From condition (b), $m_{ij}^{\beta} \cap B \neq \varnothing$, and hence, $a_S \in m_{ij}^{\beta} \cap B$, namely, $(x_i, x_j) \notin R_B$, which is a contradiction.

(c)⇒(a) In this case, $(x_i, x_j) \in Ind_C^{\beta}(D)$ holds if and only if $(\mu_{CD}(x_i))_\beta = (\mu_{CD}(x_j))_\beta$, namely, for each $D_k \in U/D$, $(|[x_i]_C \cap D_k|/|[x_i]_C|)_\beta \neq (|[x_j]_C \cap D_k|/|[x_j]_C|)_\beta$, $B \subseteq C, \forall x \in U$. Then, $[x]_B = \cup_{s=1}^{l}[y_s]_C$, which is the partition of $[x]_B$ with respect to R_C. We have two cases:

(i) If $|[y_s]_C \cap D_k|/|[y_s]_C| \geq \beta$, then $|[x]_B \cap D_k|/|[x]_B| = \sum_{s=1}^{l}|[y_s]_C \cap D_k|/|[y_s]_C||[y_s]_C|/|[x]_B| \geq \beta \sum_{s=1}^{l}|[y_s]_C|/|[x]_B| = \beta$

(ii) If $|[y_s]_C \cap D_k|/|[y_s]_C| < \beta$, then $|[x]_B \cap D_k|/|[x]_B| = \sum_{s=1}^{l}|[y_s]_C \cap D_k|/|[y_s]_C|$ $|[y_s]_C|/|[x]_B| \geq \beta \sum_{s=1}^{l}|[y_s]_C|/|[x]_B| = \beta$

By (I) and (II), $(\mu_{CD}(x_i))_\beta = (\mu_{BD}(x_i))_\beta$ and $(\mu_{CD}(x_j))_\beta = (\mu_{BD}(x_j))_\beta$ if and only if $(x_i, x_j) \in Ind_B^{\beta}(D)$. Thus, Theorem 3 is proved. □

Corollary 1. *Let $(U, C \cup D)$ be a decision table. If $\varnothing \neq B \subseteq C$ and precision $\beta \in (0,1)]$, then B is the reduction of the relative β–IR if and only if it is a minimal subset, which satisfies $m_{ij}^{\beta} \cap B \neq \varnothing$ for any $m_{ij}^{\beta} \neq \varnothing$.*

Using Corollary 1, we present the following algorithm for the relative β-IR reduction for $(U, C \cup D)$.

Considering the decision table given in Table 1, when $\beta = 0.6$, the discernibility matrix is constructed as follows:

$$\begin{bmatrix} \varnothing & \varnothing & \{a_1,a_3\} & \{a_1,a_3\} & \{a_1,a_3\} & \{a_2,a_3\} & \{a_1\} \\ \varnothing & \varnothing & \{a_1,a_3\} & \{a_1,a_3\} & \{a_1,a_3\} & \{a_2,a_3\} & \{a_1\} \\ \{a_1,a_3\} & \{a_1,a_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \{a_3\} \\ \{a_1,a_3\} & \{a_1,a_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \{a_3\} \\ \{a_1,a_3\} & \{a_1,a_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \{a_3\} \\ \{a_2,a_3\} & \{a_2,a_3\} & \varnothing & \varnothing & \varnothing & \varnothing & C \\ \{a_1\} & \{a_1\} & \{a_3\} & \{a_3\} & \{a_3\} & C & \varnothing \end{bmatrix}. \tag{12}$$

The discernibility function in CNF is $f = (a_1 + a_3)(a_2 + a_3)a_1a_3$. From CNF to DNF, we have $f = a_1a_3$. Thus, the subset $\{a_1a_3\}$ is the unique attribute reduction of the relative β–IR.

4. Relative β-Discernibility Relation Reduction

The relative β-indiscernibility relation was proposed in Section 3, and the concept of its complementary relation is proposed in this section. In contrast to the discernibility

```
Input: (U, C ∪ D)
Output: reduction results of C
(1) m^β_ij = ∅, B_i = ∅;
(2) for each x in U do
(3)     compute (μ_CD(x))_β;
(4) endfor
    or in U dofx
    or in U do   fx
(7)         if (x_i, x_j) ∉ Ind^β_C(D) then
    m^β_ij = m^β_ij ∪ {a_i}, a_i ∈ C.
(8)         endif
(9)     endfor
(10) endfor
(11)    execute CNF to DNF function
(12) return all B_i.// B_i (i = 1, 2, ..., k) is one of the attribute reductions.
```

Algorithm 1: The algorithm of the relative β-lR reduction.

matrix of the relative β–IR(β-DR) reduction, in $(U, C \cup D)$, when the precision is greater than 0.5, the decision values of some objects can be modified, and then, the relative β-discernibility relation is calculated using a positive region reduction.

Definition 8. Let $(U, C \cup D)$ be a decision table and B be a subset of C. The relative β-discernibility relation is defined as follows:

$$Dis_B^\beta(D) = \left\{(x, y) | x, y \in U, \left(\mu_{BD}(x)\right)_\beta \neq \left(\mu_{BD}(y)\right)_\beta\right\}, \quad (13)$$

where $(\mu_{BD}(x))_\beta = (|[x]_B \cap D_1|/\ |[x]_B|, |[x]_B \cap D_2|/ |[x]_B|, \ldots, |[x]_B \cap D_l|/|[x]_B|)_\beta$.

For $(U, C \cup D)$, $\beta \in (0, 1]$, relation $Ind_C^\beta(D)$ and relation $Dis_C^\beta(D)$ are complementary to each other. For a binary relation $Dis_C^\beta(D)$, if $(\mathrm{x}, \mathrm{y}) \in Dis_C^\beta(D)$, then $(\mathrm{y}, \mathrm{x}) \in Dis_C^\beta(D)$, and hence, it is symmetric. It is not reflexive because $(\mu_{CD}(x))_\beta = (\mu_{CD}(x))_\beta$ for all $\mathrm{x} \in \mathrm{U}$. Moreover, it is not necessarily transitive.

For different β, the set of $Dis_C^\beta(D)$ may also be different. In Table 1, for example, let $\beta\ = 0.3$. Then, $(x_2, x_5) \notin Dis_C^{0.3}(D)$, and $(x_5, x_6) \in Dis_C^{0.3}(D)$. However, let $\beta = 0.6$. We then have $(x_2, x_5) \in Dis_C^{0.6}(D)$, and $(x_5, x_6) \notin Dis_C^{0.6}(D)$.

Definition 9. Let $(U, C \cup D)$ be a decision table, $B \subseteq C$. If B satisfies the following conditions:

(a) $Dis_C^\beta(D) = Dis_B^\beta(D)$

(b) for any $B^{'} \subset \mathrm{B}$, $Dis_{B^{'}}^\beta(D) \neq Dis_C^\beta(D)$

then B is the reduction of the relative β-DR.

Given a decision table, its corresponding discernibility matrix is $M' = (m^{'})_{n \times n}$ for the relative β-DR, where n is the cardinality of U. Each element $m^{'}$ is defined as follows:

$$m'^{\beta}_{ij} = \begin{cases} \left\{a | a \in C, \left(x_i, x_j\right) \notin R_a\right\}, & if\left(x_i, x_j\right) \in Dis_C^\beta(D), \\ \varnothing, & \text{otherwise}. \end{cases} \quad (14)$$

Lemma 4. *Let $(U, C \cup D)$ be a decision table. $\forall (x_i, x_j) \in U \times U$, if $(x_i, x_j) \in Dis_C^\beta(D)$, then $m'^{\beta}_{ij} \neq \varnothing$.*

Proof. The proof is similar to that of Lemma 3 in Section 3. □

Theorem 4. *Let $(U, C \cup D)$ be a decision table. If $B \subseteq C$, then the following conditions (a), (b), and (c) are equivalent:*

(a) $Dis_C^\beta(D) = Dis_B^\beta(D)$

(b) If $m'^{\beta}_{ij} \neq \varnothing$, $m'^{\beta}_{ij} \cap B \neq \varnothing$

(c) If $(x, y) \in R_B$, then $(x, y) \notin Dis_C^\beta(D)$

Proof. The proof is similar to that of Theorem 3 in Section 3. □

Lemma 5. *Let $(U, C \cup D)$ be a decision table. If $(x_i, x_j) \in U \times U$, then $m^\beta_{ij} = m'^{\beta}_{ij}$.*

Proof. For all $(x_i, x_j) \in U \times U$, then $(x_i, x_j) \in Dis_C^\beta(D)$ if and only if $(x_i, x_j) \notin Ind_C^\beta(D)$, and hence, $m^\beta_{ij} = m'^{\beta}_{ij}$.

Because $(x_i, x_j) \in Dis_C^{\beta}(D)$ is the complement of a set $Ind_C^{\beta}(D)$, it is known from equations (12) and (14) that their discernibility matrices are the same. □

Corollary 2. *Let $(U, C \cup D)$ be a decision table. If and precision $\beta \in (0,1]$, then B is the reduction of the relative β-DR if and only if it is a minimal subset, which satisfies $m_{ij}^{'\beta} \cap B \neq \varnothing$ for any $m_{ij}^{'\beta} \neq \varnothing$.*

Using Corollary 2, the algorithm of the relative β-DR reduction for $(U, C \cup D)$ is proposed as follows.

Considering the decision table given in Table 1, letting $\beta = 0.3$, the discernibility matrix is constructed as follows:

$$\begin{bmatrix} \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{a_2,a_3\} & \{a_1\} \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{a_2,a_3\} & \{a_1\} \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{a_1,a_2\} & \{a_3\} \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{a_1,a_2\} & \{a_3\} \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{a_1,a_2\} & \{a_3\} \\ \{a_2,a_3\} & \{a_2,a_3\} & \{a_1,a_2\} & \{a_1,a_2\} & \{a_1,a_2\} & \varnothing & C \\ \{a_1\} & \{a_1\} & \{a_3\} & \{a_3\} & \{a_3\} & C & \varnothing \end{bmatrix}. \tag{15}$$

The discernibility function in CNF is $f = (a_1 + a_3)(a_2 + a_3)a_1a_3$. From CNF to DNF, it is $f = a_1a_3$. Thus, according to Definition 9, $\{a_1a_3\}$ is the unique result in the decision table.

5. Optimization of the Reduction Algorithm

Let $(U, C \cup D)$ be a decision table with $D_k \in U/D$. For all $x \in D_k$ and value $d_k \in V_d$, let $I_d(\mathrm{x}) = d_k$, which means that the object x takes the value d_k in V_d. Obviously, every x in D_k has the same decision value d_k for the decision attribute. Given precision $\beta \in (0.5,1]$, for any $[x]_C$ satisfying $\beta \leq |[x]_C \cap D_k|/|[x]_C| < 1$, suppose $x_i \in [x]_C$ and $I_d(x_i) \neq d_k$. Then, change the values that objects x_i have for decision attribute d such that $I_d(x_i) = d_k$, that is, $[x]_C \subseteq D_k'$. This constructs a new quotient set U/D', denoted by $U/D' = \{D_1', D_2', \ldots, D_l'\}$. The decision table that changes the decision values of some objects using this method is called the new decision table and is denoted by $(U, C \cup D')$. Indeed, we note that D_i' may be the empty set for $\exists D_i' \in U/D'$.

Theorem 5. *Let $(U, C \cup D)$ be a decision table with quotient set U/D. Given precision $\beta \in (0.5,1]$, an updated new decision table $(U, C \cup D')$ is constructed with a new quotient set $U/D' = \{D_1', D_2', \ldots, D_l'\}$. Then, $(\mu_{CD}(x))_\beta = (\mu_{CD'}(x))_1$.*

Proof. $\forall D_k \in U/D$ and for any $[x]_C$ satisfying $|[x]_C \cap D_k|/|[x]_C| \geq \beta > 0.5$, by Lemma 2, we have $|[x]_C \cap D_i|/|[x]_C| < 0.5\,(i \neq k)$. Thus, $|[x]_C \cap D_k|/|[x]_C| = 1$ for the corresponding $D_k' \in U/D'$, and then, by Theorem 2, $(\mu_{CD}(x))_\beta = (\mu_{CD'}(x))_1$. If $|[x]_C \cap D_k|/|[x]_C| < \beta$, then $|[x]_C \cap D_k'|/|[x]_C| < \beta$ for the corresponding $D_k' \in U/D'$. Then, $(\mu_{CD}(x))_\beta = (\mu_{CD'}(x))_1$. □

Proposition 1. *Given a new decision table $(U, C \cup D')$, where U/D' is a new quotient set by D' and B⊆C, B is the positive region reduction of C if and only if B is the reduction of the relative β–IR (β-DR) for $\beta = 1$.*

Proof. (⇒) If B is the attribute reduction of C, and $Pos_C(D') = Pos_B(D')$, then $[x]_C \subseteq D_k'$ if and only if $[x]_B \subseteq D_k'$ for $\forall D_k' \in U/D'$, that is, $|[x]_C \cap D_k'|/|[x]_C| = |[x]_B \cap D_k'|/|[x]_B|$. Thus, $(\mu_{CD'}(x))_1 = (\mu_{BD'}(x))_1$, and hence, $Ind_C^1(D') = Ind_B^1(D')$. Suppose $B' \subset \mathrm{B}$, $Pos_C(D') \neq Pos_{B'}(D')$. Then, there exists $D_k' \in U/D'$, s.t. $[x]_C \subseteq D_k'$ and $[x]_{B'} \nsubseteq D_k'$. Therefore, $Ind_C^1(D') \neq Ind_{B'}^1(D')$.

(⇐) If $Ind_C^1(D') = Ind_B^1(D')$, for $\forall D_k' \in U/D'$ and $[x]_C \subseteq D_k'$, if and only if $[x]_B \subseteq D_k'$, then $Pos_C(D') = Pos_B(D')$. Suppose $B' \subset \mathrm{B}$. Then, $Ind_C^1(D') \neq Ind_{B'}^1(D')$, and there exists $x \in Pos_C(D')$ and $x \notin Pos_{B'}(D')$. Hence, $Pos_C(D') \neq Pos_{B'}(D')$.

From discernibility matrices M^β (equations (13) and (15)), the corresponding matrices of the two above concepts are equivalent. For $(U, C \cup D')$, B is the reduction of the relative β–IR for $\beta = 1$ if and only if B is the reduction of the relative β-DR for $\beta = 1$. □

Corollary 3. *Given a decision table $(U, C \cup D)$ with precision $\beta \in (0.5,1]$, a new decision table $(U, C \cup D')$ is constructed. Then, B is the reduction of the relative β–IR(β-DR) with respect to $(U, C \cup D)$ if and only if B is the positive region for $(U, C \cup D')$.*

Proof. By Theorem 5 and Proposition 1, Corollary 3 clearly holds.

The above shows that the relative β–IR (β-DR) reduction with a precision greater than 0.5 is equivalent to positive region reduction, after modifying the decision values of some objects that satisfy the condition. When the precision of β–IR (β-DR) reduction is greater than 0.5, the decision values of some objects can be modified, and a new decision table can be constructed. In the new decision table, many heuristic algorithms can be used to calculate the positive region (Definition 3) reduction.

According to Corollary 3, an optimization algorithm that converts the relative β–IR (β-DR) reduction to a positive region reduction (CRRPRR) is as follows.

When comparing the relative β–IR (β-DR) reduction algorithm with the Algorithm 3, the time complexity of both algorithms is $O((|C| + |D|) \times |U|)$, where C represents the number of condition attributes and D represents the number of decision attributes. For Algorithm 1 and 2, the time complexity of constructing the discernibility matrix is $O(|U|^2 \times |C|)$ and the space complexity of constructing discernibility matrix is $O(|U|^2 \times |U/D|)$ where $|U/D|$ is the number of equivalent classifications induced by decision set D. For the Algorithm 3, the time complexity of calculating the positive region in the new decision table is $O(|U| \times |U/D| \times ((|[x]_C| \times |D_j|) + |[x]_C|))$, where $|U/D|$ is the number of equivalent classifications induced by decision set D and $|D_j|$ is the basis of the equivalent class D_j. Moreover, the time complexity of constructing the discernibility matrix is $O(|Pos_C(D')| \times |U| \times |C|)$, and the

Input: $(U, C \cup D)$
Output: reduction results of C
(1) $m_{ij}^{\beta} = \varnothing$, $B_i = \varnothing$;
(2) **for** x in U **do**
(3) compute $(\mu_{CD}(x))_{\beta}$;
(4) **endfor**
forin U **do**x
forin U **do** x
(5) **if** $(x_i, x_j) \in \mathrm{Dis}_C^{\beta}(D)$ **then**
$m_{ij}^{\prime\beta} = m_{ij}^{\prime\beta} \cup \{a_i\}, a_i \in C.$
(6) **endif**
(7) **endfor**
(8) **endfor**
(9) execute CNF to DNF function
(10) return all B_i.// $B_i (i = 1, 2, \ldots, k)$ is one of the attribute reductions

ALGORITHM 2: The algorithm of the relative β-DR reduction.

Input: decision table $(U, C \cup D)$, given $\beta \in (0.5, 1]$
Output: reduction results of C
(1) $mp_{ij} = \varnothing$, $B_i = \varnothing$;
(2) **for** x in U **do**
(3) **if** $\beta \leq |[x]_C \cap D_k| / |[x]_C| < 1$ **then**
(4) $I_D(x)$ is modified;
(5) **endif**
(6) **endfor**
new decision table (U,C∪D′) is constructed //a
for in **do**$x|Pos_C(D)|$
for in U do x
(7) **if** $x_i \in Pos_C(D)$, $(x_i, x_j) \notin R_D$ **then**
$mp_{ij} = mp_{ij} \cup \{a_i\}, a_i \in C.$
(8) **endif**
(9) **endfor**
(10) **endfor**
(11) execute CNF to DNF function
(12) return all B_i.// $B_i (i = 1, 2, \ldots, k)$ is one of the attribute reductions

ALGORITHM 3: An optimization algorithm CRRPRR.

space complexity of constructing discernibility matrix is $O(|U|^2)$. The time and space complexities of the discernibility function transformation of both algorithms are the same. Obviously, the CRRPRR algorithm is better than the original algorithm.

In Table 1, $[x_1]_C = \{x_1, x_2\}$, $[x_3]_C = \{x_3, x_4, x_5\}$. Given precision $\beta = 0.6$, $|[x_3]_C \cap D_2| / |[x_3]_C| \geq \beta$, with $D_2 = \{x_2 x_4, x_5, x_6\}$. Thus, $I_d(x_3) = 1$ in a new decision table. For $|[x_1]_C \cap D_1| / |[x_1]_C| < \beta$, with $D_1 = \{x_1, x_3\}$, the decision values of any objects in $[x_1]_C$ are not changed. Hence, the new decision table $(U, C \cup D')$ is shown in Table 2.

According to the CRRPRR optimization algorithm, $Pos_C(D') = \{x_3, x_4, x_5, x_6, x_7\}$ by Definition 2. Then, the discernibility matrix is constructed as follows:

$$\begin{bmatrix} \{a_1, a_3\} & \{a_1, a_3\} & \varnothing & \varnothing & \varnothing & \{a_1, a_2\} & \{a_3\} \\ \{a_1, a_3\} & \{a_1, a_3\} & \varnothing & \varnothing & \varnothing & \{a_1, a_2\} & \{a_3\} \\ \{a_1, a_3\} & \{a_1, a_3\} & \varnothing & \varnothing & \varnothing & \{a_1, a_2\} & \{a_3\} \\ \{a_2, a_3\} & \{a_2, a_3\} & \{a_1, a_2\} & \{a_1, a_2\} & \{a_1, a_2\} & \varnothing & C \\ \{a_1\} & \{a_1\} & \{a_3\} & \{a_3\} & \{a_3\} & C & \varnothing \end{bmatrix}. \quad (16)$$

The discernibility function in CNF is $f = (a_1 + a_3)(a_1 + a_2)(a_2 + a_3) a_1 a_3$. From CNF into DNF, we have $f = a_1 a_3$. Thus, (a_1, a_3) is the unique attribute reduction of C in $(U, C \cup D')$.

It must be explained that the basic condition for the conversion of the relative β –IR (β –DR) reduction into

Table 2: New decision table.

U	C			D'
	a_1	a_2	a_3	
x_1	1	1	1	0
x_2	1	1	1	1
x_3	0	1	0	1
x_4	0	1	0	1
x_5	0	1	0	1
x_6	1	0	0	1
x_7	0	1	1	2

Table 3: Brief description of the data sets.

ID	Datasets	Abbreviation	$\|U\|$	$\|C\|$	$\|U/D\|$
1	Iris	Iris	150	4	3
2	Statlog (Heart)	Sta.	270	13	2
3	Wine	Wine	178	13	3
4	SPECT heart	S.H.	167	22	2
5	Zoo	Zoo	101	16	7
6	Flags	Flags	194	28	7
7	Acute inflammations	A.I.	120	6	4
8	Computer hardware	C.H.	209	9	9
9	Lenses	Len.	24	4	3
10	Soybean	Soy.	47	34	4
11	E.coli	E.coli	336	7	8
12	Haberman's survival	H.S.	306	3	2
13	Balance scale	B.S.	625	4	3
14	Servo	Ser.	167	4	8
15	Teaching assistant evaluation	T.A.E.	151	5	3
16	Cryotherapy	Cr.	90	6	2
17	Ionosphere	Io.	351	34	2
18	Tic-Tac-Toe	T.T.T.	958	9	2
19	Thoracic surgery	T.S.	470	16	2
20	Hayes-Roth	H.R.	160	4	3

positive region reduction is that the precision must be greater than 0.5. When the precision is greater than 0.5, this not only improves the fault tolerance of the model but also ensures its credibility *l*. If $\beta = 0.5$ is given (e.g., as in Table 1), for $[x_1]_C = \{x_1, x_2\}$, because $I_d(x_1) = 0$, $I_d(x_2) = 1$, the decision values of any objects in $\{x_1, x_2\}$ cannot be modified according to the conversion method in this paper. Another case occurs when $\beta = 0.8$. In this case, none of the decision values of any objects can be modified in Table 1, and the above conversion method is not feasible. □

6. Experimental Analysis

In this section, we evaluate the performances of the proposed algorithms through some comparison experiments. In our experiments, twenty datasets from the UCI were used. All the information is shown in Table 3, in which $|U|$ is the cardinality of U, and $|C|$ and $|U/D|$ are the numbers of condition attributes and decision classes, respectively. Using tenfold cross-validation, three classifiers (kernel NB, fine Gaussian SVM, and a DT) were used to test the classification accuracies of the results after reduction. Kernel NB is a classifier that uses estimated kernel densities. The classification accuracy of fine Gaussian SVM is higher than that of SVM after the Gaussian kernel is introduced. DT is a supervised learning model that learns decision rules. The algorithms were implemented on a MacBook Pro (early 2015) with an Intel(R) Core(TM) i5 CPU at 2.7 GHz and Intel Iris Graphics 6100 GPU. The algorithms in this paper were coded in Python 3.6.8 using scikit-learn 0.20.3.

Because of the requirements of Algorithm 3, the accuracy of the tables in this experiment must be greater than 0.5. The runtime results for different precisions (0.7, 0.8, and 0.9) are shown in Figure 1. In this figure, the yellow line represents the average time for Algorithm 3 to run for each precision. The experimental results show that Algorithm 1, Algorithm 2, and Algorithm 3 obtain the same results, but the runtime of the latter is less than that of the former. Although the cost of converting CNF to DNF is high, the advantage of the Algorithm 3 is obvious when the number of objects is large and the number of attributes is relatively small.

Because the proposed algorithms can obtain all the results, the mean classification accuracy is reported. For all precision values, the classification accuracies obtained by kernel NB, fine Gaussian SVM, and DT are shown in Figures 2–4, respectively.

In Figure 2, in most cases, the classification accuracy of the kernel NB classifier is low for the reduction results obtained from a low precision value. For example, for the HS

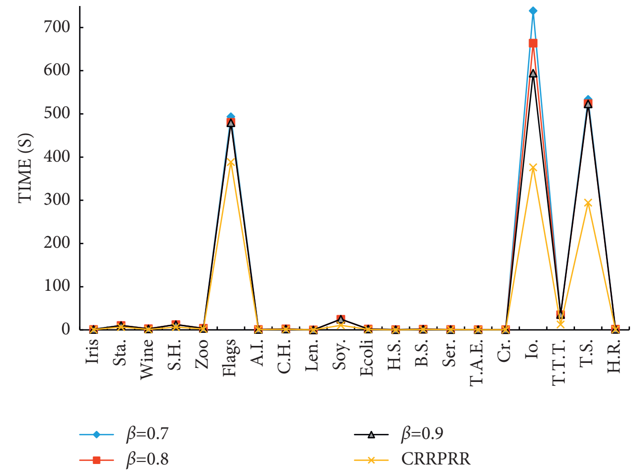

Figure 1: Comparison of runtime (in seconds).

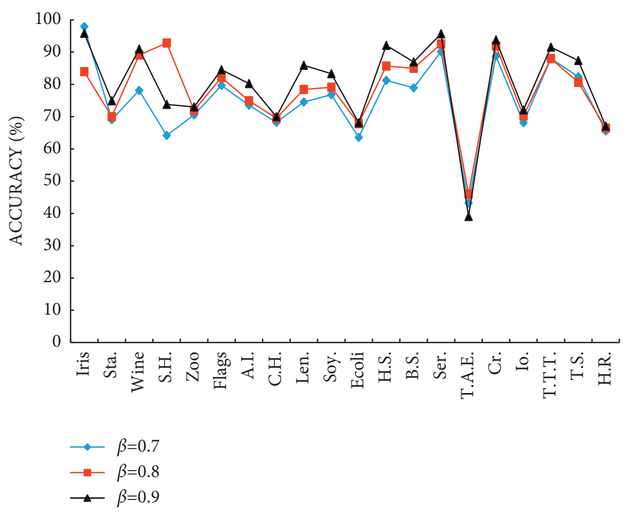

Figure 2: Classification accuracy results of kernel NB.

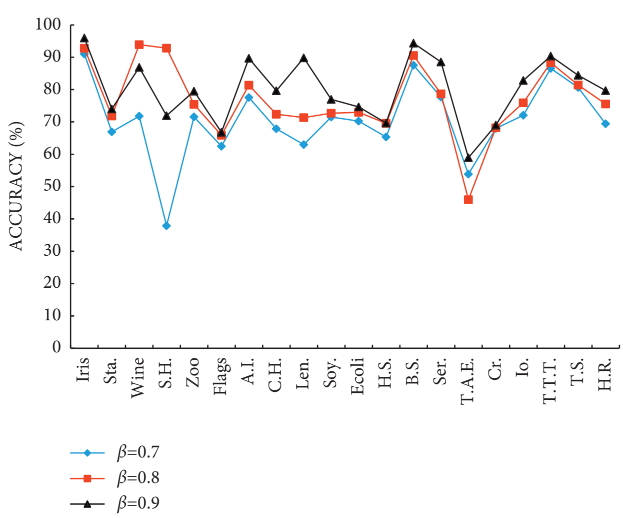

Figure 3: Classification accuracy results of fine Gaussian SVM.

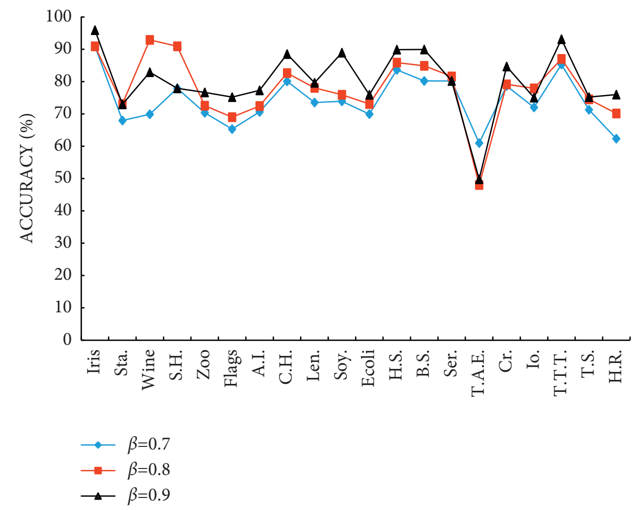

Figure 4: Classification accuracy results of DT.

dataset, given precision values of 0.7, 0.8, and 0.9, the accuracy results are 81.25%, 85.66%, and 92.11%, respectively.

For all data sets except for Wine, SH, and TAE, a higher precision value leads to a higher classification accuracy for the fine Gaussian SVM classifier in Figure 3. However, the classification accuracy is relatively low. In most cases, when the precision is high, the reduction results lead to higher classification accuracies for the DT classifier, but the classification accuracy is low. For example, on the HR dataset, given precision values of 0.7, 0.8, and 0.9, the accuracy results are 62.33%, 70.15%, and 76.01%, respectively.

This experiment demonstrates the feasibility of the concept proposed in this paper. When the precision is less than 0.5, its reliability is not high. Moreover, the CRRPRR optimization algorithm and the relative β–IR (β-DR) reduction algorithm could not be linked. Therefore, the precision of the algorithms must be greater than 0.5. Figures 2–4 show that when the precision is larger, the classification accuracy is higher when the results are obtained by the proposed algorithms. On the contrary, it is lower when the precision is lower.

7. Conclusions

This study extends the work of [5, 22, 34, 41] to investigate the relative β–IR (β-DR) for the first time using a discernibility matrix-based method. Under certain conditions, precision $\beta > 0.5$, and the relationship between the relative β –IR (β –DR) reduction and positive region reduction was found by modifying some decision values, which is of a certain significance to the study of VP RSs. The corresponding optimization algorithm was then proposed. The discernibility matrix corresponding to the relative β –IR (β –DR) reduction has a high time complexity of $O(|U|^2 \times |C|)$, whereas the time complexity for constructing a positive region reduction is much less. Therefore, when the precision $\beta > 0.5$, although the reduction results are the same, the optimization CRRPRR algorithm reduces the computational complexity. In addition, because the attribute importance cannot be calculated in the relative β –IR (β –DR) reduction to obtain the results, we modified some of the decision values that satisfy the condition so that the attribute importance can also be calculated in the new decision table. The feasibility of the algorithm proposed in this paper was demonstrated by experimental analysis. In future, we will attempt to remove the restriction of equivalence relation and further study problems such as precision reduction and the relative β –IR(β –DR) reduction.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 61972052) and the Tang Jiyong Teaching Studio.

References

[1] Z. Pawlak, *Rough Sets: Theoretical Aspect of Reasoning about Data*, Kluwer Academic Publishers, London, 1991.

[2] P. Zdzisaw, "Rough set," *International Journal of Computer & Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[3] J. Dai, H. Hu, W.-Z. Wu, Y. Qian, and D. Huang, "Maximal-discernibility-pair-based approach to attribute reduction in fuzzy rough sets," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2174–2187, 2018.

[4] K. Qin and S. H. Jing, "The attribute reductions based on indiscernibility and discernibility relations," in *Proceedings of the International Joint Conference on Rough Sets*, pp. 306–316, Cham, Switzerland, July 2017.

[5] Y. Zhao, Y. Yao, and F. Luo, "Data analysis based on discernibility and indiscernibility," *Information Sciences*, vol. 177, no. 22, pp. 4959–4976, 2007.

[6] Z. Pawlak, "Conflicts and negotations," in *Proceedings of the International Conference on Rough Sets and Knowledge Technology*, pp. 12–27, Chongqing, China, 2006.

[7] G. L. Liu, "Matrix approaches for variable precision rough approximations," in *Proceedings of the International Conference on Rough Sets and Knowledge Technology*, pp. 214–221, Tianjin, China, July 2015.

[8] W. Ziarko, "Variable precision rough set model," *Journal of Computer and System Sciences*, vol. 46, no. 1, pp. 39–59, 1993.

[9] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa, "Attribute reduction in multi-adjoint concept lattices," *Information Sciences*, vol. 294, pp. 41–56, 2015.

[10] G. Liu, Z. Hua, and J. Zou, "A unified reduction algorithm based on invariant matrices for decision tables," *Knowledge-Based Systems*, vol. 109, pp. 84–89, 2016.

[11] G. Liu, Z. Hua, and Z. Chen, "A general reduction algorithm for relation decision systems and its applications," *Knowledge-Based Systems*, vol. 119, pp. 87–93, 2017.

[12] Z. Pawlak and R. Sowinski, "Rough set approach to multi-attribute decision analysis," *European Journal of Operational Research*, vol. 72, no. 3, pp. 443–459, 1994.

[13] A. K. Tiwari, S. Shreevastava, T. Som, and K. K. Shukla, "Tolerance-based intuitionistic fuzzy-rough set approach for attribute reduction," *Expert Systems with Applications*, vol. 101, pp. 205–212, 2018.

[14] Y. Yao and Y. Zhao, "Attribute reduction in decision-theoretic rough set models," *Information Sciences*, vol. 178, no. 17, pp. 3356–3373, 2008.

[15] M. Sarwar, M. Akram, and P. Liu, "An integrated rough ELECTRE II approach for risk evaluation and effects analysis in automatic manufacturing process," *Artificial Intelligence Review*, vol. 54, no. 6, pp. 4449–4481, 2021.

[16] I. Masahiro, "Several approaches to attribute reduction in variable precision rough set model," in *Modeling Decision for Artificial Intelligence*, pp. 215–226, Tsukuba, Japan, 2005.

[17] G. Liu, "Assignment reduction of relation DecisionSystems," in *Proceedings of the International Joint Conference on Rough Sets*, pp. 384–391, Olsztyn, Poland, 2017.

[18] J. S. Mi, W. Z. Wu, and W. X. Zhang, "Approaches to knowledge reduction based on variable precision rough set model," *Information Sciences*, vol. 159, no. 3-4, pp. 255–272, 2004.

[19] W. X. Zhang, Y. Liang, and W. Z. Wu, *Information System and Knowledge Discovery*, Science Press, Beijing, 2003.

[20] M. Beynon, "Reducts within the variable precision rough sets model: a further investigation," *European Journal of Operational Research*, vol. 134, no. 3, pp. 592–605, 2001.

[21] T.-P. Hong, T.-T. Wang, and S.-L. Wang, "Mining fuzzy β-certain and β-possible rules from quantitative data based on the variable precision rough-set model," *Expert Systems with Applications*, vol. 32, no. 1, pp. 223–232, 2007.

[22] D. G. Chen and S. Y. Zhao, "Local reduction of decision system with fuzzy rough sets," *Fuzzy Sets and Systems*, vol. 161, no. 13, pp. 1871–1883, 2010.

[23] M. Akram, S. Siddique, and M. G. Alharbi, "Clustering algorithm with strength of connectedness for m-polar fuzzy network models," *Mathematical Biosciences and Engineering*, vol. 19, no. 1, pp. 420–455, 2022.

[24] M. Akram and F. Zafar, "Multi-criteria decision-making methods under soft rough fuzzy knowledge," *Journal of Intelligent and Fuzzy Systems*, vol. 35, no. 3, pp. 3507–3528, 2018.

[25] F. Zafar and M. Akram, "A novel decision-making method based on rough fuzzy information," *International Journal of Fuzzy Systems*, vol. 20, no. 3, pp. 1000–1014, 2018.

[26] C. Wang, Y. Qian, W. Ding, and X. Feng, "Feature selection with fuzzy-rough minimum classification error criterion," *IEEE Transactions on Fuzzy Systems*, vol. 2021, p. 1, 2021.

[27] C. Wang, Y. Huang, W. Ding, and Z. Cao, "Attribute reduction with fuzzy rough self-information measures," *Information Sciences*, vol. 549, pp. 68–86, 2021.

[28] C. Wang, Y. Huang, M. Shao, Q. Hu, and D. Chen, "Feature selection based on neighborhood self-information," *IEEE Transactions on Cybernetics*, vol. 50, no. 9, pp. 4031–4042, 2020.

[29] D. G. Chen, C. Z. Wang, and Q. H. Hu, "A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets," *Information Sciences*, vol. 177, no. 17, pp. 3500–3518, 2007.

[30] C. Wang, M. Shao, B. Sun, and Q. Hu, "An improved attribute reduction scheme with covering based rough sets," *Applied Soft Computing*, vol. 26, pp. 235–243, 2015.

[31] X. Jia, W. Liao, Z. Tang, and L. Shang, "Minimum cost attribute reduction in decision-theoretic rough set models," *Information Sciences*, vol. 219, pp. 151–167, 2013.

[32] V. López, S. del Río, J. M. Benítez, and F. Herrera, "Cost-sensitive linguistic fuzzy rule based classification systems under the MapReduce framework for imbalanced big data," *Fuzzy Sets and Systems*, vol. 258, pp. 5–38, 2015.

[33] F. Min, H. He, Y. Qian, and W. Zhu, "Test-cost-sensitive attribute reduction," *Information Sciences*, vol. 181, no. 22, pp. 4928–4942, 2011.

[34] G. Liu, Z. Hua, and J. Zou, "Local attribute reductions for decision tables," *Information Sciences*, vol. 422, pp. 204–217, 2018.

[35] G. Liu and Z. Hua, "Partial attribute reduction approaches to relation systems and their applications," *Knowledge-Based Systems*, vol. 139, pp. 101–107, 2018.

[36] Y. Jing, T. Li, H. Fujita, Z. Yu, and B. Wang, "An incremental attribute reduction approach based on knowledge granularity with a multi-granulation view," *Information Sciences*, vol. 411, pp. 23–38, 2017.

[37] W. Wei, X. Wu, J. Liang, J. Cui, and Y. Sun, "Discernibility matrix based incremental attribute reduction for dynamic data," *Knowledge-Based Systems*, vol. 140, pp. 142–157, 2018.

[38] Y. Xu, L. Wang, and R. Zhang, "A dynamic attribute reduction algorithm based on 0-1 integer programming," *Knowledge-Based Systems*, vol. 24, no. 8, pp. 1341–1347, 2011.

[39] B. H. Liang, L. Wang, and Y. Liu, "Attribute reduction based on improved information entropy," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 1, pp. 1–10, 2019.

[40] D. Q. Miao and G. M. Lang, "A further investigation to relative reducts of decision information systems," in *Proceedings of the International Conference on Rough Sets and Knowledge Technology*, pp. 26–38, Jeju Island, Korea, November 2015.

[41] A. Skowron, "Boolean reasoning for decision rules generation," in *Proceedings of the International Symposium on Methodologies for Intelligent Systems*, pp. 295–305, Trondheim, Norway, June 1993.

[42] A. Skowron and C. Rauszer, "The discernibility matrices and functions in information systems," *Intelligent decision support*, vol. 11, pp. 331–362, 1992.

[43] S. K. M. Wong and W. Ziarko, "Comparison of the probabilistic approximate classification and the fuzzy set model," *Fuzzy Sets and Systems*, vol. 21, no. 3, pp. 357–362, 1987.

[44] Y. Yao, "Decision-theoretic Rough Set Models," in *Proceedings of the International conference on rough sets and knowledge technology*, pp. 1–12, Springer, Berlin, Heidelberg, 2007.