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INDIVIDUAL ERGODIC THEOREMS FOR COMMUTING OPERATORS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

RYOTARO SATO

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Introduction. The main purpose of this paper is to prove the following theorem: If $T_1, \cdots, T_d$ are commuting positive contradictions on L_1 of a σ-finite measure space such that each operator T_i satisfies the L_1-mean ergodic theorem, then the multiple ergodic average

$$(1/n)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} T_1^{i_1} \cdots T_d^{i_d} f(x)$$

converges to a finite limit almost everywhere as $n \to \infty$ for all $f \in L_1$.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let $L_p(\mu)$, $1 \leqq p \leqq \infty$, denote the usual Banach spaces of (real or complex) functions on $(X, \mathscr{F}, \mu)$. A linear operator T on $L_p(\mu)$ is called *positive* if $f \geqq 0$ implies $Tf \geqq 0$, and a *contraction* if $\|T\|_p \leqq 1$, $\|T\|_p$ denoting the operator norm of T on $L_p(\mu)$. We shall say that T *satisfies the L_p-mean ergodic theorem* if the average $(1/n) \sum_{i=0}^{n-1} T^i f$ converges in L_p-norm as $n \to \infty$ for all $f \in L_p(\mu)$. Ito [9] proved that if T is a positive contradiction on $L_1(\mu)$ satisfying the L_1-mean ergodic theorem, then the average $(1/n) \sum_{i=0}^{n-1} T^i f(x)$ converges to a finite limit a.e. on X as $n \to \infty$ for all $f \in L_1(\mu)$. In the present paper we intend to extend his result to the case of multiple ergodic averages of d commuting positive contractions on $L_1(\mu)$. To do this, we use Brunel's theory [2] concerning a maximal ergodic inequality for commuting (not necessarily positive) contractions on $L_1(\mu)$. As a corollary to the proof, it follows that if $T_1, \cdots, T_d$ are commuting (not necessarily positive) contractions on $L_1(\mu)$ such that for some $1 < p \leqq \infty$, $\|\tau_i\|_p \leqq 1$ for all $1 \leqq i \leqq d$, τ_i denoting the linear modulus [3] of T_i, then the above multiple average converges to a finite limit a.e. on X as $n \to \infty$ for all $f \in L_1(\mu)$. This is a generalization of McGrath's ergodic theorem [8], who treated the positive operator case. See also Emilion [5].

The continuous versions of these results are obtained by using a standard approximation argument.

2. Ergodic theorems for the discrete case.

THEOREM 1. *Let $T_1, \cdots, T_d$ be positive contractions on $L_1(\mu)$ such that $T_iT_j = T_jT_i$ for all $1 \leqq i, j \leqq d$. Suppose each T_i satisfies the L_1-mean ergodic theorem. Then the limit*

$$\lim_{n\to\infty} (1/n)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} T_1^{i_1} \cdots T_d^{i_d} f(x)$$

exists and is finite a.e. on X for all $f \in L_1(\mu)$.

PROOF. For simplicity we shall consider the case $d = 2$. (The general case follows similarly.) Since T_i satisfies the L_1-mean ergodic theorem, $\{h + (f - T_if): T_ih = h\}$ is a dense subset of $L_1(\mu)$ by a well-known mean ergodic theorem (cf. e.g. [4, VIII, 5.2]). It follows that

$$\{h + (g + f - T_1f) - T_2(g + f - T_1f) : T_2h = h,\ T_1g = g\}$$

is a dense subset of $L_1(\mu)$. Suppose $T_2h = h$. Then Ito's ergodic theorem [9] shows that

$$(1/n)^2 \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} T_1^{i_1}T_2^{i_2}h(x) = (1/n) \sum_{i_1=0}^{n-1} T_1^{i_1}h(x)$$

converges to a finite limit a.e. on X as $n \to \infty$. Next suppose $k = g + f - T_1f$ with $T_1g = g$. Then we get

$$\begin{aligned}(1/n)^2 \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} T_1^{i_1}T_2^{i_2}(k - T_2k) &= (1/n)^2 \sum_{i_1=0}^{n-1} T_1^{i_1}(k - T_2^nk) \\ &= (1/n)^2 \sum_{i_1=0}^{n-1} T_1^{i_1}k - (1/n)^2T_2^n\Big(\sum_{i_1=0}^{n-1} T_1^{i_1}k\Big)\,,\end{aligned}$$

where

$$\lim_{n\to\infty} (1/n)^2 \sum_{i_1=0}^{n-1} T_1^{i_1}k(x) = 0 \quad \text{a.e. on } X$$

by Ito's theorem, and where

$$\begin{aligned}(1/n)^2T_2^n\Big(\sum_{i_1=0}^{n-1} T_1^{i_1}k\Big) &= (1/n)^2T_2^n\Big(\sum_{i_1=0}^{n-1} T_1^{i_1}[g + f - T_1f]\Big) \\ &= (1/n)T_2^ng + (1/n)^2T_2^n(f - T_1^nf)\,.\end{aligned}$$

Ito's theorem shows that $\lim_{n\to\infty} (1/n)T_2^ng(x) = 0$ a.e. on X. On the other hand, since $\sum_{n=1}^{\infty} (1/n)^2 \| T_2^n(f - T_1^nf)\|_1 < \infty$, we must have

$$\lim_{n\to\infty} (1/n)^2T_2^n(f - T_1^nf)(x) = 0 \quad \text{a.e. on } X\,.$$

Thus we have proved that the limit

$$\lim_{n\to\infty} (1/n)^2 \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} T_1^{i_1}T_2^{i_2}f(x)$$

exists and is finite a.e. on X for every f in a dense subset of $L_1(\mu)$. Hence the proof will be completed by Banach's convergence theorem (cf. e.g. [4, Theorem IV. 11.3]), if the following lemma is proved.

LEMMA. *If $T_1, \cdots, T_d$ are commuting positive contractions on $L_1(\mu)$ such that each T_i satisfies the L_1-mean ergodic theorem, then for every $f \in L_1(\mu)$*

$$\sup_{n\geqq 1} (1/n)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} | T_1^{i_1} \cdots T_d^{i_d} f(x) | < \infty \quad a.e. \text{ on } X .$$

To prove this lemma we need the following theorem due to Brunel [2]. (A slightly different form may be seen in [2].)

THEOREM A. *If $T_1, \cdots, T_d$ are commuting (not necessarily positive) contractions on $L_1(\mu)$, then there exists a constant $C_d > 0$ and a positive contraction U on $L_1(\mu)$ of the form*

$$U = \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} a(i_1, \cdots, i_d) \tau_1^{i_1} \cdots \tau_d^{i_d} ,$$

where $a(i_1, \cdots, i_d) \geqq 0$, $\sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} a(i_1, \cdots, i_d) = 1$, and τ_i denotes the linear modulus of T_i, such that for every $f \in L_1(\mu)$

$$\sup_{n\geqq 1} (1/n)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} \tau(i_1, \cdots, i_d) | f | (x) \leqq C_d \cdot \sup_{n\geqq 1} (1/n) \sum_{i=0}^{n-1} U^i | f | (x)$$

a.e. on X, where $\tau(i_1, \cdots, i_d)$ denotes the linear modulus of $T_1^{i_1} \cdots T_d^{i_d}$.

PROOF OF LEMMA. Let U be as in Theorem A. We shall prove that U satisfies the L_1-mean ergodic theorem, which, in turn, implies the lemma by virtue of Ito's theorem. To do this, we first show that for any $0 \leqq h \in L_1(\mu)$, the set $\{T_1^i h : i \geqq 0\}$ is weakly sequentially compact in $L_1(\mu)$. In fact, let C and D denote the conservative and dissipative parts (cf. e.g. [6]) of T_1, respectively. Then, since T_1 satisfies the L_1-mean ergodic theorem, there exists a function $0 \leqq g \in L_1(\mu)$ such that $T_1 g = g$ and $\{g > 0\} = C$ ([9]). Further we have $\lim_{n\to\infty} \int_D (1/n) \sum_{i=0}^{n-1} T_1^i h d\mu = 0$; hence $\lim_{i\to\infty} \int_D T_1^i h d\mu = 0$. Let $E_n \in \mathscr{F}$, $E_{n+1} \subset E_n$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Given an $\varepsilon > 0$, take an $N \geqq 1$ so that $\| (T_1^N h) 1_D \|_1 < \varepsilon$. Write $g_N = (T_1^N h) 1_D$ and $h_N = (T_1^N h) 1_C$. Since $h_N \in L_1(C, \mu)$, an approximation argument implies that $\lim_{n\to\infty} \left(\sup_{i\geqq 0} \int_{E_n} T_1^i h_N d\mu \right) = 0$. Thus

$$\lim_{n\to\infty} \left(\sup_{i\geqq 0} \int_{E_n} T_1^i h d\mu \right) = \lim_{n\to\infty} \left(\sup_{i\geqq 0} \int_{E_n} T_1^i (g_N + h_N) d\mu \right) \leqq \| g_N \|_1 < \varepsilon ;$$

since $\varepsilon >$ was arbitrary, the first expression equals zero. This shows the weak sequential compactness of $\{T_1^i h: i \geqq 0\}$. (See also [7, Theorem 3.2].)

Now, an induction argument implies easily that for any $0 \leqq h \in L_1(\mu)$, the set $\{T_1^{i_1} \cdots T_d^{i_d} h: i_1, \cdots, i_d \geqq 0\}$ is weakly sequentially compact, and thus $\{U^i h: i \geqq 0\}$ is also weakly sequentially compact. By this and a mean ergodic theorem, U satisfies the L_1-mean ergodic theorem. The proof is completed.

The following proposition is needed for the proof of Theorem 3 below. This proposition follows, as in Theorem 1, from an ergodic theorem of Akcoglu and Chacon [1] and a slight modification of McGrath's ergodic theorem ([8, Theorem 3]). Here it should be interesting to note that, when the author was typing the manuscript, he learned from Dr. Emilion that he also proved this proposition by using Brunel's theory [2]. See [5]. Hence we omit the details.

PROPOSITION. *Let $T_1, \cdots, T_d$ be commuting (not necessarily positive) contractions on $L_1(\mu)$ such that for some $1 < p \leqq \infty$, $\|\tau_i\|_p \leqq 1$ for each $1 \leqq i \leqq d$, where τ_i denotes the linear modulus of T_i. Then for any $f \in L_1(\mu)$ the limit*

$$\lim_{n\to\infty} (1/n)^d \sum_{i_1=0}^{n-1} \cdots \sum_{i_d=0}^{n-1} T_1^{i_1} \cdots T_d^{i_d} f(x)$$

exists and is finite a.e. on X.

3. Ergodic theorems for the continuous case. By a strongly continuous semigroup $\{T(t): t > 0\}$ of contractions on $L_p(\mu)$, we mean that $\|T(t)\|_p \leqq 1$, $T(t)T(s) = T(t+s)$ and $\lim_{s\to t} \|T(s)f - T(t)f\|_p = 0$ for all $t, s > 0$ and $f \in L_p(\mu)$. Such a semigroup $\{T(t): t > 0\}$ is said to *satisfy the L_p-mean ergodic theorem* if $(1/a)\int_0^a T(t)f dt$ converges in L_p-norm as $a \to \infty$ for all $f \in L_p(\mu)$.

THEOREM 2. *Let $\{T_i(t): t > 0\}$, $i = 1, \cdots, d$, be strongly continuous semigroups of positive contractions on $L_1(\mu)$ such that $T_i(t)T_j(s) = T_j(s)T_i(t)$ for all $1 \leqq i, j \leqq d$ and $t, s > 0$. Suppose each semigroup $\{T_i(t): t > 0\}$ satisfies the L_1-mean ergodic theorem. Then the limit*

$$\lim_{a\to\infty} (1/a)^d \int_0^a \cdots \int_0^a T_1(t_1) \cdots T_d(t_d) f(x) dt_1 \cdots dt_d$$

exists and is finite a.e. on X for all $f \in L_1(\mu)$.

PROOF. We consider the case $d = 2$. First we prove that each single operator $T_i(1)$ satisfies the L_1-mean ergodic theorem. To do this, take $h \in L_1(\mu)$ such that $h > 0$ a.e. on X, and write $h_0 = \int_0^1 T_i(t)hdt$. Since $\{T_i(t): t > 0\}$ satisfies the L_1-mean ergodic theorem,

$$(1/n)\sum_{j=0}^{n-1} T_i^j(1)h_0 = (1/n)\int_0^n T_i(t)hdt$$

converges in L_1-norm as $n \to \infty$. Therefore the set $\{(1/n)\sum_{j=0}^{n-1} T_i^j(1)h_0: n \geqq 1\}$ is weakly sequentially compact in $L_1(\mu)$.

Now, let $0 \leqq f \in L_1(\mu)$ be given. Then the strong continuity of $\{T_i(t): t > 0\}$ implies that $\{T_i(1)f > 0\} \subset \{T_i(1)h > 0\} \subset \{h_0 > 0\}$, and therefore by an approximation argument, the set $\{(1/n)\sum_{j=0}^{n-1} T_i^j(1)f: n \geqq 1\}$ is also weakly sequentially compact in $L_1(\mu)$. By this and a mean ergodic theorem, $T_i(1)$ satisfies the L_1-mean ergodic theorem.

Next, to finish the proof, write $f_0 = \int_0^1\int_0^1 T_1(t_1)T_2(t_2)fdt_1dt_2$ for $0 \leqq f \in L_1(\mu)$, and $n = [a]$ for $a > 1$, where $[a]$ denotes the integral part of a. Then we obtain

$$\begin{aligned}&\left|(1/n)^2\int_0^a\int_0^a T_1(t_1)T_2(t_2)f(x)dt_1dt_2 - (1/n)^2\int_0^n\int_0^n T_1(t_1)T_2(t_2)f(x)dt_1dt_2\right| \\ &\qquad \leqq (1/n)^2\sum_{i_1=0}^{n}\sum_{i_2=0}^{n} T_1^{i_1}(1)T_2^{i_2}(1)f_0(x) - (1/n)^2\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1} T_1^{i_1}(1)T_2^{i_2}(1)f_0(x)\,,\end{aligned}$$

and the second expression converges the to zero a.e. on X as $n \to \infty$, by Theorem 1. This and Theorem 1 complete the proof.

THEOREM 3. *Let $\{T_i(t): t > 0\}$, $i = 1, \cdots, d$, be commuting strongly continuous semigroups of (not necessarily positive) contractions on $L_1(\mu)$ such that for some $1 < p \leqq \infty$, $\|\tau_i(t)\|_p \leqq 1$ for all $1 \leqq i \leqq d$ and $t > 0$, where $\tau_i(t)$ denotes the linear modulus of $T_i(t)$. Then for any $f \in L_1(\mu)$ the limit*

$$\lim_{a\to\infty}(1/a)^d\int_0^a\cdots\int_0^a T_1(t_1)\cdots T_d(t_d)f(x)dt_1\cdots dt_d$$

exists and is finite a.e. on X.

PROOF. We consider the case $d = 2$. By the Riesz convexity theorem we may assume $p < \infty$. First suppose $f \in L_1(\mu) \cap L_p(\mu)$. Write

$$\tilde{f} = \int_0^1\int_0^1 \tau_1(t_1)\tau_2(t_2)|f|dt_1dt_2 \quad (\in L_1(\mu) \cap L_p(\mu))\,.$$

Here we note that the Bochner integral $\int_0^1\int_0^1 \tau_1(t_1)\tau_2(t_2)|f|dt_1dt_2$ exists, because $\|\tau_1(s)\tau_2(t)|f| - \tau_1(t_1)\tau_2(t_2)|f|\|_1 \to 0$ as $s \to t_1 + 0$ and $t \to t_1 + 0$, independently (cf. Sato [10]). Write $n = [a]$ for $a > 1$. Then we obtain

$$\left|(1/n)^2\int_0^a\int_0^a T_1(t_1)T_2(t_2)f(x)dt_1dt_2 - (1/n)^2\int_0^n\int_0^n T_1(t_1)T_2(t_2)f(x)dt_1dt_2\right|$$
$$\leqq (1/n)^2\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\tau_1(i_1)\tau_2(i_2)\tilde{f}(x) - (1/n)^2\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{n-1}\tau_1(i_1)\tau_2(i_2)\tilde{f}(x)\ ,$$

and the second expression converges to zero a.e. on X as $n\to\infty$, by McGrath's ergodic theorem ([8, Theorem 3]). This and Proposition show that

$$\lim_{a\to\infty}(1/a)^2\int_0^a\int_0^a T_1(t_1)T_2(t_2)f(x)dt_1dt_2$$

exists and is finite a.e. on X.

Next, suppose $f\in L_1(\mu)$. If we denote by $\tau(i_1, i_2)$ the linear modulus of $T_1(i_1)T_2(i_2)$, then

$$(1/a)^2\left|\int_0^a\int_0^a T_1(t_1)T_2(t_2)f(x)dt_1dt_2\right| \leqq (1/n)^2\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\tau(i_1, i_2)\tilde{f}(x)\ .$$

By virtue of Theorem A there exists a constant $C>0$ and a positive contraction U on $L_1(\mu)$ such that

$$\sup(1/n)^2\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\tau(i_1, i_2)\tilde{f}(x)\leqq C\cdot\sup_{n\geqq1}(1/n)\sum_{i=0}^{n}U^i\tilde{f}(x)\quad\text{a.e. on } X\ .$$

Since $\|\tau_1(1)\|_p\leqq 1$ and $\|\tau_2(1)\|_p\leqq 1$, we have $\|U\|_p\leqq 1$, and hence by an ergodic theorem of Akcoglu and Chacon [1], $(1/n)\sum_{i=0}^{n-1}U^i\tilde{f}(x)$ converges to a finite limit a.e. on X as $n\to\infty$. Therefore

$$\sup_{n\geqq1}(1/n)\sum_{i=0}^{n-1}U^i\tilde{f}(x)<\infty\quad\text{a.e. on } X.$$

Thus Banach's convergence theorem completes the proof.

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Department of Mathematics
Faculty of Science
Okayama University
Okayama 700
Japan