Induced Chern-Simons terms

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We examine the claim that the effective action of four-dimensional $\mathrm{SU}(2)_L$ gauge theory at high and low temperature contains a three-dimensional Chern-Simons term which has the chemical potential for baryon number as its coeffcient. The four-dimensional theory has a two-dimensional analogue in which exact calculations can be performed. These calculations demonstrate that the existence of the Chern-Simons term in four dimensions may be rather subtle. [S0556-2821(98)07118-5]

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I. INTRODUCTION

Consider the the four-dimensional Euclidean $\mathrm{SU}(2)_L$ gauge theory at finite temperature $T=1/\beta$, described by

$$S=\int_0^\beta d\tau \int d^3x\left(-\tfrac{1}{2}\ \mathrm{tr}\ F^2+\overline{\psi}_L \not{D}\psi_L\right). \tag{1}$$

There are an even number of massless left-handed fermions to avoid the global SU(2) anomaly [1], and the Dirac operator is $\not{D}=\not{\partial}+ig\not{A}^aT^a+\mu\gamma^0$, where μ is the real chemical potential for the particle-number charge

$$B_L=\int d^3x\,\overline{\psi}_L\gamma^0\psi_L. \tag{2}$$

It has been suggested by Redlich and Wijewardhana [2], Tsokos [3], and Rutherford [4], that — at both high and low temperature — the effective action obtained by integrating out the fermions contains a term reminiscent of the three-dimensional Chern-Simons term with the coefficient μ:

$$S_{\mathrm{eff}}=\mu\int_0^\beta d\tau\int d^3x\,\epsilon_{ijk}\ \mathrm{tr}\left(A_i\partial_jA_k-\tfrac{2}{3}gA_iA_jA_k\right)+\cdots. \tag{3}$$

This model has been used [5,6] to describe baryogenesis by weak interactions at temperatures around the weak scale in the early universe. The authors note that because of the U(1) anomaly, B_L is only quasi-conserved. Then, when the gauge configurations tunnel from one vacuum sector to another, baryons will be created or destroyed. Because μ is real, the ''Chern-Simons'' term in Eq. (3) is not gauge invariant, and so breaks the degeneracy of the topological vacua. Thus the system would be biased to ''fall'' in one particular direction resulting in more baryons being created than antibaryons.

Let us now present a calculation that produces no Chern-Simons term at low temperature. We use Pauli-Villars regularization which is manifestly gauge invariant. Since μ is real we are only interested in the real part of the effective action, log det $\not{D}\not{D}^\dagger$. The standard way [2,4,5] to obtain this is to ''vectorize'' the model by adding $\overline{\psi}_R\not{D}^\dagger\psi_R$ which yields a theory of Dirac fermions with an axial quasi-conserved charge

$$S=\int\overline{\psi}(\not{\partial}+ig\not{A}^aT^a+\mu\gamma^0\gamma^5)\psi. \tag{4}$$

The coefficient of $\mu A^a_\lambda A^a_\delta$ in the Chern-Simons term is

$$\Gamma^{\lambda\delta0}(p,M,T)=\int_k \mathrm{tr}\ \gamma^\lambda\Delta(k,M)\gamma^0\gamma^5\Delta(k,M)\gamma^\delta \times\Delta(k+p,M). \tag{5}$$

Here $\Delta(k,M)$ is the propagator of a Dirac fermion with mass M and the integral over momentum space is $\int_k=\beta^{-1}\Sigma_n d^3\boldsymbol{k}$ for nonzero temperature. Following Refs. [2,4] we add a mass m for the fermions at low temperature. Expanding the denominator in powers of $(2k\cdot p+p^2)(k^2+M^2)^{-1}$ yields

$$\Gamma^{\lambda\delta0}(p,M,T)=C\epsilon^{0\lambda\delta\alpha}p^\alpha+O(p^2/M). \tag{6}$$

Since C is mass independent, Pauli-Villars regularization will yield, in apparent contradiction to [2–4],

$$\begin{aligned}\Gamma^{\lambda\delta0}_{\mathrm{PV}}&(p,m,T\sim0)\\ &\equiv\lim_{M\to\infty}[\Gamma^{\lambda\delta0}(p,m,T\sim0)-\Gamma^{\lambda\delta0}(p,M,T\sim0)]\\ &=0+O(m^{-1}).\end{aligned} \tag{7}$$

It is tempting to invoke gauge invariance in order to rule out the appearance of the Chern-Simons term. However, this is too naive, because—although the term is not gauge invariant by itself—it is still possible that the entire effective action may be invariant [4,7,8]. In later sections we shall present simple examples of this phenomena.

In light of the apparent contradiction of Pauli-Villars regularization with the results of Refs. [2–4], and the subtlety of gauge invariance, we feel that the problem needs more study. Fortunately, there is a related model in two dimensions in which further calculations can be made more simply. We believe there is nothing in the following calculations that suggests our results are particularly specific to

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two dimensions. Indeed, in the conclusion we reproduce the result of Ref. [2] by performing an exact calculation in the 2D model.

II. THE TOY MODEL

We work in a flat two-dimensional (2D) Euclidean space $\mathcal{M}$ with coordinates (τ,x) where $0 \leqslant \tau \leqslant \beta$. Our gamma matrices are Hermitian and satisfy

$$[\gamma^\mu,\gamma^\nu]_+ = 2\delta^{\mu\nu} \quad \text{and} \quad \gamma_5 = -i\gamma_0\gamma_1. \tag{8}$$

The 2D equivalent of the vectorized theory of Eq. (4) is

$$Z[A,\mu,\bar{\eta},\eta] = \int [d\bar{\psi} d\psi] e^{-S-\int \bar{\eta}\psi - \bar{\psi}\eta}, \tag{9}$$

with

$$S = \int_{\mathcal{M}} \bar{\psi} \not{D} \psi \quad \text{and} \quad \not{D} = \not{\partial} + m + \mu\gamma^0\gamma^5 + ie\not{A}. \tag{10}$$

A mass term has been included for generality at this point. We shall see later on that it infrared (IR) regulates the theory at zero temperature. The chemical potential μ for the Hermitian axial charge $Q_5 = \int \bar{\psi}\gamma^0\gamma^5\psi$ is real. One can check this through a derivation of the path-integral representation of the partition function.[1]

The U(1) gauge transformations are

$$A_\mu \to A_\mu - ie^{-1} e^{i\theta} \partial_\mu e^{-i\theta},$$

$$\psi \to e^{i\theta}\psi. \tag{12}$$

A gauge transformation is called ''small'' when θ is well defined on $\mathcal{M}$, while if only $e^{i\theta}$ is well defined (but not θ itself) the transformation is called ''large.'' An example of a large gauge transformation is

$$\theta(x,\tau) = 2\pi \tilde{N}\tau/\beta, \quad \text{for } \tilde{N} \in \mathbb{Z}. \tag{13}$$

This shifts A_0 by a constant

$$A_0 \to A_0 - 2\pi\tilde{N}/e\beta. \tag{14}$$

The Chern-Simons term in this context is

$$\mu \int_{\mathcal{M}} A^1. \tag{15}$$

Let us first present some perturbative calculations that suggest that this term does not appear in the effective action. Then we will study the effective action nonperturbatively.

III. PERTURBATIVE RESULTS

Since μ is constant, it is efficient to put it into the propagator

$$\Delta(k) = \frac{1}{i\not{k} + m + \mu\gamma^0\gamma^5} = \frac{1}{i\not{k} + m - i\mu\gamma^1}. \tag{16}$$

The second equality holds in two dimensions because of the identity $\gamma^\lambda\gamma^5 = -i\epsilon^{\lambda\delta}\gamma^\delta$ and shows that a constant μ simply shifts the momentum in the loop. Expanding the path integral in powers of A we find the coefficient of the linear term is the superficially linearly divergent one-point function

$$\Gamma^\lambda(m,T,\mu) = \int_k \operatorname{tr} e\gamma^\lambda \frac{m - i\tilde{\not{k}}}{\tilde{k}^2 + m^2}, \tag{17}$$

where $\tilde{k}_1 \equiv k_1 - \mu$.

To regulate this expression we will use Pauli-Villars regularization in which a massive spinor χ is added into the path integral[2]

$$Z = \lim_{M\to\infty} \int [d\bar{\psi} d\psi d\bar{\chi} d\chi] e^{-S(\bar{\psi},\psi,A,m) + S(\bar{\chi},\chi,A,M)}. \tag{18}$$

This is manifestly gauge invariant and, in the usual fashion, gives

$$\Gamma^\lambda_{\rm PV}(m) \equiv \lim_{M\to\infty} [\Gamma^\lambda(m) - \Gamma^\lambda(M)]. \tag{19}$$

Since the momentum integral is now finite we can shift away all dependence on μ. It is possible to go further and explicitly calculate each separate term on the right-hand side (RHS) of Eq. (19). The mass term in the numerator of Eq. (17) gets killed by $\operatorname{tr}\gamma^\mu = 0$. When $\lambda = 0$ symmetric summation (or integration) gives $\Gamma^0(m,T) = 0$. For $\lambda = 1$ the answer obtained depends on the order of integration. Performing the k^1 integral first gives

[1]In the derivation of the path-integral representation of the partition function Tr $\exp -\beta(H+\mu Q_5)$, we insert a complete basis at each time slice and then express the action thus derived in terms of relativistic fields in Euclidean space. This last part is relatively nontrivial, but it is found that with the choice $\bar{\psi} = \psi^\dagger\gamma^5$, the path integral of Eq. (9) correctly calculates the partition function. This careful calculation thereby confirms the recent work of Waldron *et al.* [9] who studied the continuous rotation of spinors from Minkowsky to Euclidean space. It was found that with the definitions (subscripts M and E refer to Minkowsky and Euclidean, respectively)

$$\psi_M \equiv e^{-i\pi\gamma^0_M\gamma^5_M/4}\psi_E,$$

and

$$\psi^\dagger_M \equiv \psi^\dagger_E e^{-i\pi\gamma^0_M\gamma^5_M/4}, \tag{11}$$

with $\gamma^0_M \equiv i\gamma^5_E$, $\gamma^i_M = \gamma^i_E$, and $\gamma^5_M = \gamma^0_E$, the SO(4) invariant Euclidean action was given by Eq. (10). Parity, for example, acts on the Euclidean space spinors as $\psi_E \to \eta_P\gamma^0_E\psi_E$ and $\bar{\psi}_E \to \eta^*_P\bar{\psi}_E\gamma^0_E$, so that the μQ_5 term breaks parity invariance as required.

[2]In principle two spinors are needed, however, this is an unnecessary notational complication.

$$\Gamma^1(m,T)=e\int_{k_0}\int_{-\Lambda-\mu}^{\Lambda-\mu}d\tilde{k}_1\frac{\tilde{k}_1}{\tilde{k}_1^2+m^2+k_0^2}\overset{\Lambda\to\infty}{\to}\int_{k_0}0=0. \tag{20}$$

However, performing the k_0 summation first yields

$$\Gamma^1(m,T)=\beta^2 e\int dk_1\frac{\tilde{k}_1}{\beta\sqrt{\tilde{k}_1^2+m^2}}\pi\tanh(\pi\beta\sqrt{\tilde{k}_1^2+m^2})$$
$$=2e\pi\mu. \tag{21}$$

The same result is obtained at zero temperature. However, all answers are mass independent, so Pauli-Villars regularization yields

$$\Gamma^\lambda_{\rm PV}(m,T)=0\quad\forall m,T. \tag{22}$$

An alternative treatment is not to put μ into the propagator, but to expand the path integral in powers of both μ and A. The correlation function of interest is the logarithmically divergent two-point function

$$\Gamma^{\lambda 0}(m,T)=\int_k \mathrm{tr}\,\frac{m-i\not{k}}{k^2+m^2}\gamma^0\gamma^5\frac{m-i\not{k}}{k^2+m^2}ie\gamma^\lambda. \tag{23}$$

This method has the advantage that we can easily make μ nonconstant. The momentum p, flowing into the associated Feynman diagram will then be nonzero, and only after calculating will we set $p=0$. With nonzero p, Adler's regularization-independent method [10] can be applied. At zero temperature, the most general expression with the correct Lorentz structure and parity is

$$\Gamma^{\lambda\delta}(p,m,T=0)=Y(p^2,m^2)\epsilon^{\lambda\delta}+Z(p^2,m^2)p_\sigma\epsilon^{\sigma(\lambda}p^{\delta)}. \tag{24}$$

The parentheses indicate symmetrization. Gauge invariance implies

$$p_\lambda\Gamma^{\lambda\delta}=0\Rightarrow p_1\Gamma^{10}=p_0\Gamma^{00}\Rightarrow Y=-\tfrac{1}{2}p^2Z. \tag{25}$$

However, Z is finite so we can calculate it. For the massive case we find $Z\propto m^{-2}+O(p^2)$. Then setting $p^2=0$ gives

$$Y=0\Rightarrow\Gamma^{\lambda\delta}(m\neq 0,T=0)=0. \tag{26}$$

However, for $m=0$ we obtain

$$\Gamma^{10}(p,m=0,T=0)=\frac{2e\pi p_0^2}{p_0^2+p_1^2}. \tag{27}$$

Interestingly, this is ambiguous in the zero-momentum limit

$$\Gamma^{10}(m=0,T=0)\to\begin{cases}0 & p_0\to 0\ \text{ then }\ p_1\to 0,\\ 2e\pi & p_1\to 0\ \text{ then }\ p_0\to 0.\end{cases} \tag{28}$$

We attribute this to the IR divergence contained in the two-point function of Eq. (23) for $M=0$ and $T=0$. We find a similar problem when naively applying Pauli-Villars regularization at zero temperature. Namely, after taking the trace over gamma matrices,

$$\Gamma^{\lambda\delta}(M\neq 0,T=0)=ieM^2\mathrm{tr}\ \gamma^\delta\gamma^5\gamma^\lambda\int_k(k^2+M^2)^{-2}$$
$$=-2e\pi\epsilon^{\lambda\delta}, \tag{29}$$

while

$$\Gamma^{\lambda\delta}(M=0,T=0)=0. \tag{30}$$

This implies, in contradiction to the null result obtained using the one-point function,

$$\Gamma^{10}_{\rm PV}(m,T=0)=\begin{cases}0 & m\neq 0,\\ 2e\pi & m=0.\end{cases} \tag{31}$$

However, this occurs only because the IR divergence has made the result somewhat arbitrary. In this situation a natural prescription is to define the massless theory as the limit of the massive one:

$$\Gamma^{10}_{\rm PV}(m,T=0)=0\quad\forall m. \tag{32}$$

At nonzero temperature there is no IR problem because k_0 is never zero. Pauli-Villars regularization gives zero in agreement with the one-point function. The Adler argument is more complicated because the heat bath breaks Lorentz invariance and so $\Gamma^{\lambda\delta}$ can depend on the normal vector in the p_0 direction. It turns out [11], that Γ^{10} has the same form as Eq. (27). However, this time p_0 is quantized, which means it cannot be taken to zero smoothly. We argue that this implies that p_0 must be set to zero from the very start, and so the top limit in Eq. (28) is the correct one.

IV. NONPERTURBATIVE RESULTS

The partition function can also be calculated directly to all orders in μ by functional methods.[3] To make the eigenvalue problem well defined, $\mathcal{M}$ is chosen to be the torus with $0\leqslant\tau\leqslant\beta$ and $0\leqslant x\leqslant R$. Here we can make the Hodge decomposition on the background gauge field

$$A_\mu=\frac{1}{e}\partial_\mu\sigma+\frac{1}{e}\epsilon_{\mu\nu}\partial_\nu\rho+h_\mu. \tag{33}$$

The fields σ and ρ are well defined on $\mathcal{M}$ and h_μ is constant. Our case differs from the Schwinger model [12] on the torus only by the μ term. However, using the identity $\gamma^0\gamma^5$

[3]We are interested in the trivial sector of the model. The effective action when the gauge field is in a nontrivial winding sector is also well known [15,16]. Nontrivial sectors may be of interest when studying baryogenesis in the early universe. A nonzero chemical potential for the conserved *electric* charge has also been considered [17]. In this case the Dirac operator is no longer Hermitian and the phase in the partition function leads to interesting results.

$=-i\gamma^1$ we can shift the μ into h_1. The form of the generating functional is well known [13]

$$Z[A,\bar{\eta},\eta]=\exp\left(\int \bar{\eta}e^{-i\sigma-\gamma^5\rho}\Delta_0 e^{i\sigma-\gamma^5\rho}\eta + \frac{1}{2\pi}\int \rho\Box\rho\right)\ \det \not{D}_0. \tag{34}$$

Here $\not{D}_0=\not{\partial}+ie\not{h}-i\mu\gamma^1$ and has associated propagator Δ_0. The determinant of this operator can be calculated using zeta-function regularization. The result can be written in terms of a theta function and Dedekind's eta function [14,16]

$$\begin{aligned}\det \not{D}_0&=\left|\frac{1}{\eta(iR/\beta)}\Theta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0,iR/\beta)\right|^2\\ &=\left|q^{1/24}\prod_{m=1}^{\infty}(1-q^m)\sum_{n\in\mathbb{Z}}q^{(n+\theta)^2/2}e^{2\pi i(n+\theta)\phi}\right|^2.\end{aligned} \tag{35}$$

In this formula $\theta=-\beta eh^0/2\pi$ and $\phi=\frac{1}{2}+R(eh^1-\mu)/2\pi$ and the parameter $q=e^{-2\pi R/\beta}$.

The partition function is clearly invariant under small gauge transformations since $e^{i\sigma}\eta$ and its conjugate are invariant. It is also invariant under large gauge transformations in the x and τ directions

$$x\ \text{ direction: }\ \delta h^1=\frac{2\pi\tilde{N}}{eR}\quad\text{and}\quad \bar{\eta}\rightarrow\bar{\eta}e^{2\pi i\tilde{N}x/R},$$

$$\tau\ \text{ direction: }\ \delta h^0=\frac{2\pi\tilde{N}}{e\beta}\quad\text{and}\quad \bar{\eta}\rightarrow\bar{\eta}e^{2\pi i\tilde{N}\tau/\beta}. \tag{36}$$

The first transformation changes the summand in Eq. (35) by a phase which is then canceled by the mod squared. The second transformation can be soaked up by relabeling the index of summation.

Let us study the partition function as we take the cylindrical limit. The determinant (35) of $\not{D}_0$ obtained by zeta-function regularization is nonlocal in the gauge field. Also, each term in the expansion of the effective action $S_{\text{eff}}=\log\det\not{D}_0$ in powers of $h^\lambda=(1/R\beta)\int A^\lambda$ is not gauge invariant. For example, at large R (the limit to the cylinder) or small β (high temperature), the parameter q is small. Then we can expand, for $\theta=0$,

$$S_{\text{eff}}=8\sqrt{q}\frac{R}{\beta}e\mu\int A^1+\cdots, \tag{37}$$

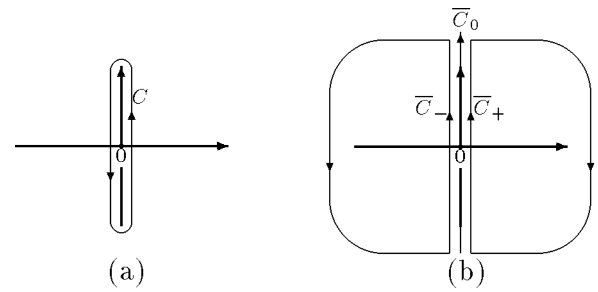

FIG. 1. Contours of integration in the z-plane. (a) The contour C encircles the imaginary axis, and (b) contour $\overline{C}_0$ passes up the imaginary axis and $\overline{C}_+$ ($\overline{C}_-$) encircles the RHS (LHS) of the plane.

where, in the last equality, the Chern-Simons term has been extracted. The term by itself is not gauge invariant. In the Appendix we study the one dimensional analogue, det $\not{D}$ on the circle. Once again zeta-function regularization results in a nonlocal but gauge-invariant result. Each term in the expansion in powers of the gauge field is not gauge invariant. We also study the limit to the line. One would not expect the limit to depend upon whether the boundary conditions on the circle were initially periodic or antiperiodic. The only subtlety is that one has to be careful with IR divergences (zeromodes). In the 2D model there are no IR problems because the fermions are antiperiodic along the time direction. Thus, by setting $q=0$ in Eq. (37), we see that there is no induced Chern-Simons term on the cylinder according to zeta-function regularization.

V. CONCLUSIONS

The effective action of the 2D toy model of baryogenesis has been calculated in various ways. Because the chemical potential is real, the Chern-Simons-type term that has been proposed to appear in the effective action is not gauge invariant. As we have seen in one and two dimensions, this does not rule out its appearance in the effective action. However, all our gauge-invariant calculations at nonzero temperature gave no Chern-Simons term. It was only for the massless theory at zero temperature that there was any chance of getting a term. This was attributed to an ambiguity brought about through an IR divergence.

How then, did other authors [2] obtain a nonzero result? The regularization scheme was to subtract off the zero-temperature, zero-μ result. Let us perform the same calculation in 2D. The one-point function of Eq. (17) can be written in the form

$$\Gamma^1(m,T,\mu)=\int dk_1\oint_C\frac{dz}{2\pi i}\left(\frac{k_1-\mu}{-z^2+(k_1-\mu)^2+m^2}\right)\tanh\tfrac{1}{2}\beta z, \tag{38}$$

where the contour of integration is shown in Fig. 1(a). Using partial fractions and expressing tanh in terms of exponentials leads to

$$\Gamma^1(m,T,\mu)=\int dk_1\frac{k_1-\mu}{\omega}\Bigg[-\oint_{\bar{C}_+}\frac{dz}{2\pi i}\times\left(\frac{1}{z+w}-\frac{1}{z-w}\right)\frac{1}{1+e^{\beta z}}-\oint_{\bar{C}_-}\frac{dz}{2\pi i}\left(\frac{1}{z+w}-\frac{1}{z-w}\right)\frac{1}{1+e^{-\beta z}}+\int_{\bar{C}_0}\frac{dz}{2\pi i}\left(\frac{1}{z+w}-\frac{1}{z-w}\right)\Bigg],\tag{39}$$

where $\omega=\sqrt{(k_1-\mu)^2+m^2}$ and the various contours are shown in Fig. 1(b). Evaluating these integrals leads to

$$\Gamma^1(m,T,\mu)=2e\pi\mu+\Gamma^1(m,0,0).\tag{40}$$

Thus, if we follow Ref. [2] and regulate by subtracting off the zero-temperature, zero-μ result, we will obtain a Chern-Simons term. This is in contrast to Pauli-Villars regularization which gave no Chern-Simons term.

One might try to justify this procedure by casting it into a Pauli-Villars-like form

$$Z=\lim_{M\to\infty}\int[d\bar{\psi}d\psi d\bar{\chi}d\chi]\exp[-S(\bar{\psi},\psi,A,m,T,\mu)+S(\bar{\chi},\chi,A,M,T=0,\mu=0)].\tag{41}$$

In the second action the spinor fields χ are defined over the plane. The gauge field must be the same in both actions. Presumably it is extended periodically to the plane in the second action. The second action also has no axial charge. A standard argument shows that there are no new divergences introduced by insertions of the charge of a conserved current. In the present case, Q_5 is the charge of an anomalous current, so this argument must be reexamined. Clearly it is somewhat uncertain as to whether this scheme can be implemented as a gauge-invariant regularization to all orders in perturbation theory. In contrast, the regularization schemes used in this paper are gauge invariant and implementable to all orders. If the unusual regularization scheme in Eq. (41) can be implemented then it amounts to a definition of the theory, and it would be interesting to reexamine the cosmological models using it to see whether the Chern-Simons term arises in their effective description. Using zeta function regularization, the effective action for gauge fields in nontrivial winding sectors has also been calculated [15,16]. It would be of interest to calculate matrix elements corresponding to baryogenesis in the early universe with this action.

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APPENDIX A: DETERMINANT ON A ONE-DIMENSIONAL MANIFOLD

A nonperturbative result for the partition function on the torus has been presented. The effective action was nonlocal and the expansion in small A naively looked gauge variant. The one-dimensional theory has these properties too. It also provides us with a testing ground to check for nontrivialities in the torus $\rightarrow$ cylinder limit. Start with the operator

$$D=i\partial+eA(t)+iM,$$

where $-\pi R\leq t\leq\pi R$. We have included a mass term iM for generality, and it will serve to IR regulate the theory. On the circle the eigenvectors are

$$\psi_\lambda=\exp\left[i\left(\lambda t-e\int^t A\right)-Mt\right].$$

The boundary conditions then imply $\lambda_n=\mathcal{A}+(n/R)$ where

$$\mathcal{A}\equiv\begin{cases}\dfrac{e}{2\pi R}\displaystyle\int A-iM & \text{periodic,}\\[2ex] \dfrac{1}{2R}+\dfrac{e}{2\pi R}\displaystyle\int A-iM & \text{antiperiodic.}\end{cases}$$

If $M\neq 0$ there are no zero modes, however, if $M=0$ there is a possibility of one zero mode depending on the value of $\int A$. The product of eigenvalues needs regularization. A non-gauge-invariant way to proceed is to calculate $\det D(i\partial+iM)^{-1}$. This leads to a sine in the periodic case and a cosine for antiperiodic boundary conditions. Alternately, zeta-function regularization is gauge invariant, and results in (for values of the Riemann zeta function see Ref. [18], Sec. 9.53)

$$\det D=\exp\left[-\frac{d}{ds}\sum_n\left(\frac{n}{R}+\mathcal{A}\right)^{-s}\Bigg|_{s=0}\right],$$
$$=1-e^{-2\pi i\mathcal{A}R}.$$

Consider the antiperiodic massless theory. Expanding the effective action in powers of A gives

$$S_{\text{eff}}=\log 2-\tfrac{1}{2}ei\int A+O(A^2).$$

Although the whole effective action is gauge invariant, this term is only invariant under $A\rightarrow A-2\pi\tilde{N}/eR$ for even $\tilde{N}$. It is clear that the effective action for the periodic massless case does not have an expansion in small A. This is because there is a zero mode which must be removed

$$\det{}'_{\text{periodic}}D=\frac{1-e^{-ie\int A}}{ie\displaystyle\int A}.$$

The same problem crops up in perturbation theory, where we get IR divergent terms such as $\Sigma_n(1/n)$.

The limit to the line of the above result is (mod$2\pi i$):

$$\log\det D \to -\pi R(M-|M|) - i\theta(-M)e \times \int A + \begin{cases} \pi i & \text{periodic,} \\ 0 & \text{antiperiodic,} \end{cases}$$

for $M \neq 0$, while for $M=0$ the antiperiodic case gives

$$\log\det D \to \log\left(1+e^{-ie\int A}\right).$$

The M-dependent normalization is physically unimportant. If we had taken the limit of the massless periodic case without removing the zero mode, the effective action would not have had an expansion in small A. It is only when the compact theory is properly IR regulated that the noncompact effective action can be properly defined. In our 2D example, the antiperiodicity over the time direction at nonzero temperature will provide the necessary IR regulator.

Let us compare this with the expression obtained from $\det D(i\partial+iM)^{-1}$. The Green's function for $i\partial+iM$ with $M\neq 0$ is

$$G(x-y) = \int \frac{dk}{2\pi}\frac{e^{ik(x-y)}}{-k+iM} = \begin{cases} ie^{-M(x-y)}[\theta(M)\theta(x-y)-\theta(-M)\theta(y-x)] & \text{for } x-y\neq 0, \\ -\frac{1}{2}i \operatorname{sgn} M & \text{for } x-y=0, \end{cases}$$

where θ is a step function. Expanding the effective action in powers of A, the step functions destroy all terms but the linear one, resulting in

$$\det D(i\partial+iM)^{-1} = \exp\left(\tfrac{1}{2}i \operatorname{sgn} M \int_{-\infty}^{\infty} dx A(x)\right).$$

Because there are no large gauge transformations on the line this is gauge invariant. It it differs from the zeta function result $-i\theta(-M)\int A$. It is well known that the imaginary part of the effective action can be defined in many ways (see Ref. [19] for a review).

As in the 2D case, zeta function regularization has resulted in a nonlocal expression for the effective action. It is of interest to see if the derivative expansion, which is local, feels these nonlocalities in any way. To calculate the derivative expansion we use the heat-kernel method. This has the disadvantage that only the real part of the effective action, $\log\det DD^\dagger$, can be calculated, because the heat kernel is then quadratic in derivatives. However, it has the advantage that at finite R we can apply the well-known result that the heat kernel is not temperature (R) dependent (see, for example, Ref. [20]). Then

$$\begin{aligned}\log\det DD^\dagger &= \int_0^\infty \frac{d\epsilon}{\epsilon}\operatorname{Tr} e^{-\epsilon DD^\dagger} \\ &= \int_0^\infty \frac{d\epsilon}{\epsilon} e^{-\epsilon M^2}\int \frac{dk}{2\pi} e^{ikx} e^{-\epsilon[-\partial^2+2iA\partial+(i\partial A+A^2)]} e^{ikx} \\ &= \int_0^\infty \frac{d\epsilon}{\epsilon}\frac{1}{\sqrt{\epsilon}} e^{-\epsilon M^2}\int \frac{dk}{2\pi} e^{-k^2} e^{-2\sqrt{\epsilon}kD_0-\epsilon D_0 D_0} \\ &= \int_0^\infty \frac{d\epsilon}{\epsilon}\frac{1}{\sqrt{4\pi\epsilon}} e^{-\epsilon M^2},\end{aligned}$$

where $D_0 = i\partial + A$. The last line follows by expanding the exponential in powers of ϵ. Thus, the real part of the effective action does not depend on the gauge field A. This does not agree with the nonlocal zeta-function result. It is, however, the same as $\det D(i\partial+iM)^{-1}$ on the line.

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