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Research Article

Inequalities between Arithmetic-Geometric, Gini, and Toader Means

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We find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a,b) < M(a,b) < S_{q_1}(a,b)$ and $S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals. Here $M(a,b)$, $T(a,b)$, and $S_p(a,b)$ are the arithmetic-geometric, Toader, and pth Gini means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the pth Gini mean $S_p(a,b)$ and power mean $M_p(a,b)$ of two positive real numbers a and b are defined by

$$S_p(a,b) = \begin{cases} \left(\dfrac{a^{p-1}+b^{p-1}}{a+b}\right)^{1/(p-2)}, & p \neq 2, \\ \left(a^a b^b\right)^{1/(a+b)}, & p = 2, \end{cases} \tag{1.1}$$

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.2}$$

respectively.

It is well known that $S_p(a,b)$ and $M_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many means are special case of these means, for example,

$$\begin{gathered} S_1(a,b) = M_1(a,b) = \frac{a+b}{2} = A(a,b) \text{ is the arithmetic mean}, \\ S_0(a,b) = M_0(a,b) = \sqrt{ab} = G(a,b) \text{ is the geometric mean}, \\ M_{-1}(a,b) = \frac{2ab}{a+b} = H(a,b) \text{ is the harmonic mean}. \end{gathered} \tag{1.3}$$

Recently, the Gini and power means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–7].

In [8], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta \\ &= \begin{cases} \dfrac{2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)}{\pi}, & a > b, \\ \dfrac{2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)}{\pi}, & a < b, \\ a, & a = b, \end{cases} \end{aligned} \tag{1.4}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1 - r^2\sin^2 t)^{1/2}dt$, $r \in [0,1]$, is the complete elliptic integrals of the second kind.

The classical arithmetic-geometric mean $M(a,b)$ of two positive number a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{gathered} a_0 = a, \qquad b_0 = b, \\ a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \qquad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n). \end{gathered} \tag{1.5}$$

The Gauss identity [9] shows that

$$M(1,r)\mathcal{K}\left(\sqrt{1-r^2}\right) = \frac{\pi}{2} \tag{1.6}$$

for $r \in (0,1)$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2\sin^2 t)^{-1/2}dt$, $r \in [0,1)$, is the complete elliptic integrals of the first kind.

Vuorinen [10] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.7}$$

for all $a,b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen in [11] and Barnard et al. in [12], respectively.

In [13], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.8}$$

for all $a,b > 0$ with $a \neq b$.

In [14–17], the authors proved that

$$M_0(a,b) = G(a,b) < M(a,b) < M_{1/2}(a,b), \tag{1.9}$$

$$L(a,b) < M(a,b) < \frac{\pi}{2}L(a,b) \tag{1.10}$$

for all $a,b > 0$ with $a \neq b$, where

$$L(a,b) = \begin{cases} \dfrac{a-b}{\log a - \log b}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.11}$$

denotes the classical logarithmic mean of two positive numbers a and b.

Very recently, Chu and Wang [18] and Guo and Qi [19] proved that

$$L_0(a,b) < T(a,b) < L_{1/4}(a,b) \tag{1.12}$$

for all $a,b > 0$ with $a \neq b$, and $L_0(a,b)$ and $L_{1/4}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the Toader mean $T(a,b)$, respectively. Here, the pth Lehmer mean $L_p(a,b)$ of two positive numbers a and b is defined by $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$.

The main purpose of this paper is to find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a,b) < M(a,b) < S_{q_1}(a,b)$ and $S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals.

2. Preliminary Knowledge

Throughout this paper, we denote $r' = \sqrt{1-r^2}$ for $r \in [0,1]$.

For $0 < r < 1$, the following derivative formulas were presented in [9, Appendix E, pages 474–475]:

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, \qquad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r},$$
$$\frac{d\left[\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]}{dr} = r\mathcal{K}(r), \qquad \frac{d[\mathcal{K}(r) - \mathcal{E}(r)]}{dr} = \frac{r\mathcal{E}(r)}{r'^2}. \tag{2.1}$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \tag{2.2}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}(r) - r'^2\mathcal{K}(r)}{1+r}. \tag{2.3}$$

Lemma 2.1 can be found in [9, Theorem 3.21(7), (8), and (10), and Exercise 3.43(13) and (46)].

Lemma 2.1. *(1) $r'^c\mathcal{K}(r)$ is strictly decreasing from $[0,1)$ onto $(0,\pi/2]$ for $c \in [1/2,\infty)$;*

(2) $r'^c\mathcal{E}(r)$ is strictly increasing on $(0,1)$ if and only if $c \le -1/2$ and strictly decreasing if and only if $c > 0$;

(3) $\mathcal{K}(r)/\log(4/r')$ is strictly decreasing from $(0,1)$ onto $(1,\pi/\log 16)$;

(4) $2\mathcal{E}(r) - r'^2\mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$;

(5) $[\mathcal{E}(r) - r'^2\mathcal{K}(r)]/[r^2\mathcal{K}(r)]$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$.

3. Main Results

Theorem 3.1. *Inequality $S_{1/2}(a,b) < M(a,b) < S_1(a,b)$ holds for all $a,b > 0$ with $a \ne b$, and $S_{1/2}(a,b)$ and $S_1(a,b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a,b)$.*

Proof. From (1.1) and (1.5) we clearly see that both $S_p(a,b)$ and $M(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1-t)/(1+t)$. Then from (1.1) and (1.6) together with (2.2) we clearly see that

$$\begin{aligned}
M(a,b) - S_{1/2}(a,b) &= \frac{\pi}{2\mathcal{K}\left(\sqrt{1-t^2}\right)} - \left[\frac{(1+t)\sqrt{t}}{1+\sqrt{t}}\right]^{2/3} \\
&= \frac{\pi}{2(1+r)\mathcal{K}(r)} - \left[\frac{2\sqrt{1-r}}{(1+r)\left(\sqrt{1+r}+\sqrt{1-r}\right)}\right]^{2/3} \\
&= \frac{1}{1+r}\left[\frac{\pi}{2\mathcal{K}(r)} - \left(\frac{2r'}{\sqrt{1+r}+\sqrt{1-r}}\right)^{2/3}\right].
\end{aligned} \tag{3.1}$$

Let

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)}\right]^3 - \left(\frac{2r'}{\sqrt{1+r}+\sqrt{1-r}}\right)^2. \tag{3.2}$$

Then $F(r)$ can be rewritten as

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)}\right]^3 - \frac{2r'^2}{1+r'} = \frac{2r'^2}{1+r'}F_1(r), \tag{3.3}$$

where

$$F_1(r) = \left(\frac{\pi}{2}\right)^3 \frac{1+r'}{2r'^2\mathcal{K}(r)^3} - 1. \tag{3.4}$$

It is well known that the function $r \rightarrow \sqrt{r} + 1/\sqrt{r}$ is positive and strictly decreasing in $(0,1)$. Then (3.4) and Lemma 2.1(1) lead to the conclusion that $F_1(r)$ is strictly increasing in $(0,1)$, so that $F_1(r) > F_1(0) = 0$ for $r \in (0,1)$.

Therefore, $M(a,b) > S_{1/2}(a,b)$ follows from (3.1)–(3.3).

On the other hand, $M(a,b) < S_1(a,b) = A(a,b)$ follows directly from (1.9).

Next, we prove that $S_{1/2}(a,b)$ and $S_1(a,b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a,b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1), (1.6), and Lemma 2.1(3) we have

$$\begin{aligned} [M(1,1-x)]^{3-2\varepsilon} - [S_{1/2+\varepsilon}(1,1-x)]^{3-2\varepsilon} &= \left[\frac{\pi}{2\int_0^{\pi/2}\left[1-(2x-x^2)\sin^2 t\right]^{-1/2}dt}\right]^{3-2\varepsilon} \\ &\quad - \left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1+(1-x)^{1/2-\varepsilon}}\right]^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \lim_{x\to 0}\frac{M(1,x)}{S_{1-\varepsilon}(1,x)} &= \lim_{x\to 0}\left[\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\mathcal{K}\left(\sqrt{1-x^2}\right)}\left(\frac{1+x^{\varepsilon}}{1+x}\right)^{1/(1+\varepsilon)}\right] \\ &= \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\mathcal{K}\left(\sqrt{1-x^2}\right)} = \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\log(4/x)}\frac{\log(4/x)}{\mathcal{K}\left(\sqrt{1-x^2}\right)} \\ &= \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\log(4/x)} = +\infty. \end{aligned} \tag{3.6}$$

Letting $x \to 0$ and making use of the Taylor expansion, one has

$$\begin{aligned}
&\left[\frac{\pi}{2\int_0^{\pi/2}\left[1-(2x-x^2)\sin^2 t\right]^{-1/2}dt}\right]^{3-2\varepsilon}-\left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1+(1-x)^{1/2-\varepsilon}}\right]^2\\
&\quad=1+\left(-\frac{3}{2}+\varepsilon\right)x+\frac{(2\varepsilon-3)(4\varepsilon-3)}{16}x^2+o\left(x^2\right)\\
&\qquad-\left[1+\left(-\frac{3}{2}+\varepsilon\right)x+\frac{(2\varepsilon-3)^2}{16}x^2+o\left(x^2\right)\right]\\
&\quad=-\frac{\varepsilon(3-2\varepsilon)}{8}x^2+o\left(x^2\right).
\end{aligned}\tag{3.7}$$

Equations (3.5)–(3.7) imply that for any $1 < \varepsilon < 1/2$ there exist $\delta_1 = \delta_1(\varepsilon) \in (0,1)$ and $\delta_2 = \delta_2(\varepsilon) \in (0,1)$, such that $M(1,1-x) < S_{1/2+\varepsilon}(1,1-x)$ for $x \in (0,\delta_1)$ and $M(1,x) > S_{1-\varepsilon}(1,x)$ for $x \in (0,\delta_2)$. □

Theorem 3.2. *Inequality $S_1(a,b) < T(a,b) < S_{3/2}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, and $S_1(a,b)$ and $S_{3/2}(a,b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a,b)$.*

Proof. From (1.1) and (1.4) we clearly see that both $S_p(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1-t)/(1+t)$. Then from (1.1), (1.4), and (2.3) we have

$$\begin{aligned}
\frac{T(a,b)}{S_{3/2}(a,b)}&=\frac{2}{\pi}\mathcal{E}\left(\sqrt{1-t^2}\right)\cdot\left(\frac{1+\sqrt{t}}{1+t}\right)^2\\
&=\frac{2}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)\cdot(1+r)\cdot\left(\frac{\sqrt{1+r}+\sqrt{1-r}}{2}\right)^2\\
&=\frac{2}{\pi}\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right]\cdot\left(\frac{\sqrt{1+r}+\sqrt{1-r}}{2}\right)^2\\
&=\frac{1}{\pi}\left(1+r'\right)\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right].
\end{aligned}\tag{3.8}$$

Let

$$G(r)=\frac{1}{\pi}\left(1+r'\right)\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right].\tag{3.9}$$

Then simple computations lead to

$$G(0) = 1, \tag{3.10}$$

$$\begin{aligned} G'(r) &= \frac{1}{\pi}\left[\left(-\frac{r}{r'}\right)\left(2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right) + (1+r')\left(\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r}\right)\right] \\ &= \frac{r'(1+r')\left[\mathcal{E}(r) - r'^2\mathcal{K}(r)\right] - r^2\left[2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]}{\pi r r'} \\ &= \frac{r}{\pi r'}G_1(r), \end{aligned} \tag{3.11}$$

where

$$G_1(r) = (1+r')r'\mathcal{K}(r)\left[\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r^2\mathcal{K}(r)}\right] - \left[2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]. \tag{3.12}$$

It follows from (3.12) and Lemma 2.1(1), (4), and (5) that $G_1(r)$ is strictly decreasing from $(0,1)$ onto $(-2,0)$. Then (3.11) leads to the conclusion that $G'(r) < 0$ for $r \in (0,1)$. Hence $G(r)$ is strictly decreasing in $(0,1)$.

Therefore, $T(a,b) < S_{3/2}(a,b)$ follows from (3.8)–(3.10) together with the monotonicity of $G(r)$.

On the other hand, $T(a,b) > S_1(a,b) = A(a,b)$ follows directly from (1.7).

Next, we prove that $S_1(a,b)$ and $S_{3/2}(a,b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a,b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1) and (1.4) one has

$$\begin{aligned} [T(1,1-x)]^{1+2\varepsilon} - [S_{3/2-\varepsilon}(1,1-x)]^{1+2\varepsilon} &= \left[\frac{2}{\pi}\int_0^{\pi/2}\left[1-\left(2x-x^2\right)\sin^2 t\right]^{1/2}dt\right]^{1+2\varepsilon} \\ &\quad - \left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}}\right]^2, \end{aligned} \tag{3.13}$$

$$\lim_{x\to 0}\frac{T(1,x)}{S_{1+\varepsilon}(1,x)} = \lim_{x\to 0}\left[\frac{2}{\pi}\mathcal{E}\left(\sqrt{1-x^2}\right)\left(\frac{1+x^\varepsilon}{1+x}\right)^{1/(1-\varepsilon)}\right] = \frac{2}{\pi} < 1. \tag{3.14}$$

Letting $x \to 0$ and making use of the Taylor expansion, we get

$$\begin{aligned}
&\left[\frac{2}{\pi}\int_0^{\pi/2}\left[1-\left(2x-x^2\right)\sin^2 t\right]^{1/2}dt\right]^{1+2\varepsilon}-\left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}}\right]^2\\
&\quad=1-\left(\frac{1}{2}+\varepsilon\right)x+\frac{(2\varepsilon+1)(4\varepsilon+1)}{16}x^2+o\left(x^2\right)\\
&\qquad-\left[1-\left(\frac{1}{2}+\varepsilon\right)x+\frac{(2\varepsilon+1)^2}{16}x^2+o\left(x^2\right)\right]\\
&\quad=\frac{\varepsilon(2\varepsilon+1)}{8}x^2+o\left(x^2\right).
\end{aligned}\tag{3.15}$$

Equations (3.13)–(3.15) imply that for any $0 < \varepsilon < 1/2$ there exist $\delta_3 = \delta_3(\varepsilon) \in (0,1)$ and $\delta_4 = \delta_4(\varepsilon) \in (0,1)$, such that $T(1,1-x) > S_{3/2-\varepsilon}(1,1-x)$ for $x \in (0,\delta_3)$ and $T(1,x) < S_{1+\varepsilon}(1,x)$ for $x \in (0,\delta_4)$. □

4. Remarks and Corollaries

Remark 4.1. From (3.9) and Lemma 2.1(4) we clearly see that $G(1^-) = 2/\pi$. Then (3.8) and (3.9) together with the monotonicity of $G(r)$ lead to the conclusion that

$$\frac{2}{\pi}S_{3/2}(a,b) < T(a,b)\tag{4.1}$$

for all $a, b > 0$ with $a \neq b$.

Remark 4.2. We find that the lower bound $L(a,b)$ in (1.10) and the best possible lower Gini mean bound $S_{1/2}(a,b)$ in Theorem 3.1 are not comparable. In fact, from (1.1) and (1.11) we have

$$\lim_{x\to+\infty}\frac{S_{1/2}(1,x)}{L(1,x)}=\lim_{x\to+\infty}\left[\frac{1+x^{-1}}{1+x^{-1/2}}\right]^{2/3}\frac{x^{2/3}\log x}{x-1}=\lim_{x\to+\infty}\frac{\log x}{x^{1/3}-x^{-2/3}}=0,$$

$$\begin{aligned}
S_{1/2}(1,1+x)-L(1,1+x)&=1+\frac{1}{2}x-\frac{1}{16}x^2+o\left(x^2\right)-\left[1+\frac{1}{2}x-\frac{1}{12}x^2+o\left(x^2\right)\right]\\
&=\frac{1}{48}x^2+o\left(x^2\right)\quad(x\longrightarrow 0).
\end{aligned}\tag{4.2}$$

Table 1: Comparison of $\mathcal{K}(r)$ with $H(r)$ for some $r \in (0,1)$.

r	$\mathcal{K}(r)$	$H(r)$
0.1	1.574745561517···	1.574745561518···
0.2	1.586867847···	1.586867848···
0.3	1.608048620···	1.608048634···
0.4	1.639999866···	1.640000021···
0.5	1.685750355···	1.685751528···
0.6	1.750753803···	1.750760840···
0.7	1.845693998···	1.845732233···
0.8	1.995302778···	1.995519211···

Remark 4.3. The following two equations show that the best possible upper power mean bound $M_{\log 2/\log(\pi/2)}(a,b)$ in (1.8) and the best possible upper Gini mean bound $S_{3/2}(a,b)$ in Theorem 3.2 are not comparable:

$$\lim_{x\to+\infty}\frac{S_{3/2}(1,x)}{M_{\log 2/\log(\pi/2)}(1,x)} = 2^{\log(\pi/2)/\log 2} = \frac{\pi}{2},$$

$$\begin{aligned} M_{\log 2/\log(\pi/2)}(1,1+x) - S_{3/2}(1,1+x) &= 1+\frac{1}{2}x+\frac{1}{8}\left[\frac{\log 2}{\log(\pi/2)}-1\right]x^2 \\ &\quad + o\left(x^2\right) - \left[1+\frac{1}{2}x+\frac{1}{16}x^2+o\left(x^2\right)\right] \\ &= \frac{1}{16}\left[\frac{2\log 2}{\log(\pi/2)}-3\right]x^2+o\left(x^2\right) \\ &= 0.00436\cdots\times x^2+o\left(x^2\right) \quad (x\longrightarrow 0). \end{aligned} \tag{4.3}$$

From Theorem 3.1 we get an upper bound for the complete elliptic integrals of the first kind $\mathcal{K}(r)$ as follows.

Corollary 4.4. *Inequality*

$$\mathcal{K}(r) < \frac{\pi}{2}\left[\frac{1+\left(1-r^2\right)^{1/4}}{\left(1+\sqrt{1-r^2}\right)\left(1-r^2\right)^{1/4}}\right]^{2/3} \tag{4.4}$$

holds for all $r \in (0,1)$.

Remark 4.5. Computational and numerical experiments show that the upper bound in (4.4) for $\mathcal{K}(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $H(r) = \pi[1+(1-r^2)^{1/4}]^{2/3}/\{2[(1+\sqrt{1-r^2})(1-r^2)^{1/4}]^{2/3}\}$, then we have Table 1 via elementary computation.

Table 2: Comparison of $\mathcal{E}(r)$ with $J(r)$ for some $r \in (0,1)$.

r	$\mathcal{E}(r)$	$J(r)$
0.1	$1.566861942021\cdots$	$1.566861942028\cdots$
0.2	$1.554968546\cdots$	$1.554968548\cdots$
0.3	$1.534833465\cdots$	$1.534833516\cdots$
0.4	$1.505941612\cdots$	$1.505942206\cdots$
0.5	$1.467462209\cdots$	$1.467466484\cdots$
0.6	$1.418083394\cdots$	$1.418107161\cdots$
0.7	$1.355661136\cdots$	$1.355777213\cdots$
0.8	$1.276349943\cdots$	$1.276910677\cdots$

The following bounds for the complete elliptic integrals of the second kind $\mathcal{E}(r)$ follow from Theorem 3.2 and Remark 4.1.

Corollary 4.6. *Inequality*

$$\left[\frac{1+\sqrt{1-r^2}}{1+(1-r^2)^{1/4}}\right]^2 < E(r) < \frac{\pi}{2}\left[\frac{1+\sqrt{1-r^2}}{1+(1-r^2)^{1/4}}\right]^2 \tag{4.5}$$

holds for all $r \in (0,1)$.

Remark 4.7. Computational and numerical experiments show that the upper bound in (4.5) for $\mathcal{E}(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $J(r) = \pi[1+\sqrt{1-r^2}]^2/\{2[1+(1-r^2)^{1/4}]^2\}$, then we have Table 2 via elementary computation.

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