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Inequalities connecting the eigenvalues of a hermitian matrix with the eigenvalues of complementary principal submatrices

Robert C. Thompson and S. Therianos

Let $C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a hermitian matrix in partitioned form. Let the eigenvalues of A, B, C be $\alpha_1 \geq \ldots \geq \alpha_a$, $\beta_1 \geq \ldots \geq \beta_b$, $\gamma_1 \geq \ldots \geq \gamma_n$, respectively. In this paper four classes of inequalities are proved comparing the α_i and β_j with the γ_k. The simplest of these is:

$$\sum_{s=1}^{m} \gamma_{i_s+j_s-s} + \sum_{s=1}^{m} \gamma_{n-m+s} \leq \sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_{j_s}$$

if the subscripts i_s, j_s satisfy $1 \leq i_1 < \ldots < i_m \leq a$, $1 \leq j_1 < \ldots < j_m \leq b$.

1. Introduction

Let $A, B, C = A + B$ be hermitian matrices with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \ldots \geq \beta_n$, $\gamma_1 \geq \ldots \geq \gamma_n$, respectively. The inequality

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j , \quad 1 \leq i, j \leq n, \; i + j - 1 \leq n , \tag{1}$$

is due to Weyl [15]. The inequality

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(2) $$\sum_{s=1}^{m} \gamma_{i_s} \le \sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_s , \quad 1 \le i_1 < \dots < i_m \le n ,$$

is due to Lidskiĭ [8] and Wielandt [16]. An inequality containing both (1) and (2) as special cases was found by Amir-Moéz [1]. His somewhat complicated result goes as follows. If we are given integers $i_1, \dots, i_m$ satisfying $1 \le i_1 \le \dots \le i_m \le n$ and $i_s \le n - m + s$ for $s = 1, \dots, m$, define $i_1'', \dots, i_m''$ by $i_1'' = i_1$, $i_s'' = \max(i_s, i_{s-1}''+1)$, $2 \le s \le m$. If $1 \le i_1 \le \dots \le i_m \le n$, $1 \le j_1 \le \dots \le j_m \le n$, and if $i_s + j_s - 1 \le n - m + s$ for $s = 1, \dots, m$, Amir-Moéz's inequality for the eigenvalues of A, B, $C = A + B$ then takes the form

(3) $$\sum_{s=1}^{m} \gamma_{(i_s+j_s-1)''} \le \sum_{s=1}^{m} \alpha_{i_s''} + \sum_{s=1}^{m} \beta_{j_s''} .$$

Recently it has been shown [11] that a simpler and sharper generalization of (1) and (2) may be found: if

(4) $$1 \le i_1 < \dots < i_m \le n , \quad 1 \le j_1 < \dots < j_m \le n , \quad i_m + j_m - m \le n ,$$

then

(5) $$\sum_{s=1}^{m} \gamma_{i_s+j_s-s} \le \sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_{j_s} .$$

It was shown in [11] that (5) implies (3). (It is also shown in [9] that (5) is equal in strength to a more complicated inequality given by Hersch and Zwahlen [7, 19].)

Now let

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

be a partitioned hermitian matrix, where A, B are square (not necessarily of the same size), and where the eigenvalues of A, B, C are

$$\alpha_1 \ge \dots \ge \alpha_a , \quad \beta_1 \ge \dots \ge \beta_b , \quad \gamma_1 \ge \dots \ge \gamma_n ,$$

respectively. Then an inequality of Aronszajn [3, 6] states that

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(6) $$\gamma_{i+j-1} + \gamma_n \leq \alpha_i + \beta_j \ , \quad 1 \leq i \leq a, \ 1 \leq j \leq b \ .$$

In [10] a generalization of (6) in the spirit of (3) was found, namely

(7) $$\sum_{s=1}^{m} \gamma_{(i_s+j_s-1)''} + \sum_{s=1}^{m} \gamma_{n-m+s} \leq \sum_{s=1}^{m} \alpha_{i''_s} + \sum_{s=1}^{m} \beta_{j''_s}$$

if

(7.1) $$1 \leq i_1 \leq \ldots \leq i_m \leq a \ , \quad 1 \leq j_1 \leq \ldots \leq j_m \leq b \ ,$$

(7.2) $$i_s \leq a - m + s \ , \quad j_s \leq b - m + s \ , \quad s = 1, \ldots, m \ .$$

The proof of (7) given in [10] has recently been simplified by Amir-Moéz and Perry [2].

Since (5) is sharper then (3), it is natural to ask whether an improvement and simplification of (7) along the lines of (5) is possible. That such a simplification will exist is suggested by the fact that some of the subscripts in the first part of the left-hand side of (7) may coincide with some of the subscripts in the second part of the left-hand side. The proposed generalization of (6) along the lines suggested by (5) should take the following form:

(8) $$\sum_{s=1}^{m} \gamma_{i_s+j_s-s} + \sum_{s=1}^{m} \gamma_{n-m+s} \leq \sum_{s=1}^{m} \alpha_{i_s} + \sum_{s=1}^{m} \beta_{j_s}$$
$$\text{if} \quad 1 \leq i_1 < \ldots < i_m \leq a, \ 1 \leq j_1 < \ldots < j_m \leq b \ .$$

It is not difficult to show that (8) is free of the defect that blemishes (7), that is, the subscripts in the left-hand side of (8) are distinct. Moreover, were (8) true, it would be sharper than (and simpler than) (7), in the same way that (5) is sharper and simpler than (3).

After this preamble, we announce one of the main results of this paper: the inequality (8) is valid. We shall in fact prove four classes of inequalities comparing the eigenvalues of a partitioned hermitian matrix

$$C = (A_{st})_{1 \leq s,t \leq k}$$

with those of its main diagonal blocks A_{tt} , $t = 1, \ldots, k$. One of these classes will contain (8) as a special case. Two proofs will be given. The first will use a device of Wielandt [18, p. 120] to derive (8) from (5), and the second will derive (8) directly by invoking the properties of a subspace constructed in [13].

2. The basic result

THEOREM 1. *Let*

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

be a hermitian n-square matrix partitioned as indicated, where A *is* $a \times a$ *and* B *is* $b \times b$ *, and where*

$$(9) \qquad \alpha_1 \geq \ldots \geq \alpha_a , \quad \beta_1 \geq \ldots \geq \beta_b , \quad \gamma_1 \geq \ldots \geq \gamma_n$$

denote the eigenvalues of A, B, C *respectively. Let* $0 \leq \mu \leq a$ *and* $0 \leq \nu \leq b$. *Let integers* $i_1, \ldots, i_\mu$, $j_1, \ldots, j_\nu$ *satisfy*

$$1 \leq i_1 < \ldots < i_\mu \leq a , \quad 1 \leq j_1 < \ldots < j_\nu \leq b .$$

Define $i_s = a - \mu + s$ *for* $s > \mu$ *and* $j_s = b - \nu + s$ *for* $s > \nu$. *Then*

$$(10) \qquad \sum_{s=1}^{\mu+\nu} \gamma_{i_s+j_s-s} \leq \sum_{s=1}^{\mu} \alpha_{i_s} + \sum_{s=1}^{\nu} \beta_{j_s} .$$

REMARK. If one sets $\mu = \nu = m$ then (10) reduces to (8).

First proof. (Compare [18], p. 120.) The inequality (10) is invariant under translation of A, B, C by scalar matrices. We may therefore assume C is positive definite. Let $C = X^*X$. Partition $X = (X_1, X_2)$ where X_1 is $n \times a$ and X_2 is $n \times b$. Then $A = X_1^*X_1$ and $B = X_2^*X_2$. Also $XX^* = X_1X_1^* + X_2X_2^*$. The eigenvalues of $X_1X_1^*$ and $X_2X_2^*$ coincide, except for zeros, with the eigenvalues of $X_1^*X_1 = A$ and $X_2^*X_2 = B$, and the eigenvalues of XX^* are those of C . Thus if we apply (5) to $XX^* = X_1X_1^* + X_2X_2^*$, we obtain

$$\sum_{s=1}^{\mu+\nu} \gamma_{i_s+j_s-s} \leq \sum_{s=1}^{\mu} \alpha_{i_s} + 0 + \sum_{s=1}^{\nu} \beta_{j_s} + 0 ,$$

completing the proof.

Second proof. Let $g_1, \ldots, g_n$ be an orthonormal system of column

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n-tuple eigenvectors of C associated with the eigenvalues $\gamma_1, \ldots, \gamma_n$. Let $e_1, \ldots, e_a$ be an orthonormal system of column a-tuple eigenvectors of A associated respectively with $\alpha_1, \ldots, \alpha_a$ and let $f_1, \ldots, f_b$ be an orthonormal system of column b-tuple eigenvectors of B associated respectively with $\beta_1, \ldots, \beta_b$. Define column n-tuples E_s, F_s by

$$E_s = \begin{bmatrix} e_s \\ 0 \end{bmatrix}, \quad s = 1, \ldots, a, \quad F_s = \begin{bmatrix} 0 \\ f_s \end{bmatrix}, \quad s = 1, \ldots, b.$$

It is known [13] that a ρ-dimensional space $L_\rho = \langle X_1, \ldots, X_\rho \rangle = \langle Y_1, \ldots, Y_\rho \rangle$ exists (the symbol $\langle \ \rangle$ denotes the linear span of the enclosed vectors) such that

$$X_s \in \left\langle E_{i_s}, \ldots, E_a \right\rangle, \quad s = 1, \ldots, \mu,$$

$$\text{(11)} \qquad X_{s+a} \in \left\langle F_{j_s}, \ldots, F_b \right\rangle, \quad s = 1, \ldots, \nu,$$

$$Y_s \in \left\langle g_1, \ldots, g_{i_s+j_s-s} \right\rangle, \quad s = 1, \ldots, \mu+\nu = \rho.$$

Here $X_1, \ldots, X_{\mu+\nu}$ are orthonormal, as are $Y_1, \ldots, Y_{\mu+\nu}$. Set

$$X_s = \begin{bmatrix} x_s \\ 0 \end{bmatrix}, \quad s = 1, \ldots, \mu,$$

$$X_{s+\mu} = \begin{bmatrix} 0 \\ x_{s+\mu} \end{bmatrix}, \quad s = 1, \ldots, \nu.$$

Taking the trace of the restriction of C to L_ρ, we get

$$\text{(12)} \qquad \sum_{s=1}^{\mu+\nu} Y_s^* C Y_s = \sum_{s=1}^{\mu} X_s^* C X_s + \sum_{s=\mu+1}^{\mu+\nu} X_s^* C X_s$$
$$= \sum_{s=1}^{\mu} x_s^* A x_s + \sum_{s=\mu+1}^{\mu+\nu} x_s^* B x_s.$$

Since

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$$Y_s^* C Y_s \ge \gamma_{i_s+j_s-s} \;, \quad s = 1, \ldots, \mu+\nu \;, \text{ by } (11),$$

$$x_s^* A x_s \le \alpha_{i_s} \;, \quad s = 1, \ldots, \mu \;, \text{ because } x_s \in \left\langle e_{i_s}, \ldots, e_a \right\rangle ,$$

$$x_{\mu+s}^* B x_{\mu+s} \le \beta_{j_s} \;, \quad s = 1, \ldots, \nu \;, \text{ because } x_{s+\mu} \in \left\langle f_{j_s}, \ldots, f_b \right\rangle ,$$

we immediately obtain (8) from (12).

3. The four principal classes of inequalities

Throughout this section we let

$$C = \left(A_{st}\right)_{s,t=1,\ldots,k}$$

be a partitioned hermitian matrix, in which diagonal block A_{tt} is n_t-square, $t = 1, \ldots, k$. Let

(13) $$\alpha_{t1} \ge \ldots \ge \alpha_{tn_t}$$

be the eigenvalues of A_{tt}, $t = 1, \ldots, k$, and let $\gamma_1 \ge \ldots \ge \gamma_n$ be the eigenvalues of C. By induction on k it is relatively simple to establish the following generalization of Theorem 1.

THEOREM 2. *Let* $C = \left(A_{st}\right)$ *be as described above. Let integers* m_t, j_{ts} *satisfy*

(14) $$0 \le m_t \le n_t \;, \quad 1 \le j_{t1} < \ldots < j_{t,m_t} \le n_t \;,$$

and define

(15) $$j_{ts} = n_t - m_t + s \quad \text{for all} \quad s > m_t \;.$$

Let $m = m_1 + \ldots + m_k$. *Then the eigenvalues* γ_i *of* C *and the eigenvalues* α_{ts} *of its main diagonal blocks* A_{tt} *satisfy*

(16) $$\sum_{s=1}^{m} \gamma_{j_{1s}+j_{2s}+\ldots+j_{ks}-(k-1)s} \le \sum_{t=1}^{k} \left(\sum_{s=1}^{m_t} \alpha_{t,j_{ts}} \right) .$$

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REMARK. If we set $k = n$ and each $n_t = 1$, then specifying $m_1 = \dots = m_r = 1$, $m_{r+1} = \dots = m_n = 0$ reduces the inequality (16) to

$$\sum_{s=1}^{r} \gamma_{s+n-r} \leq \sum_{t=1}^{r} c_{tt} , \tag{17}$$

where $C = \left(c_{st}\right)_{s,t=1,\dots,n}$. The inequality (17) is a classical result of Fan [4] asserting that the sum of r diagonal elements of a hermitian matrix C dominates the sum of the r lowest eigenvalues of C. Thus Fan's result is included in (16) as a special case.

In the following we let $\delta_x(y)$ be a jump function: $\delta_x(y) = 0$ if $y \leq x$, $\delta_x(y) = 1$ if $y > x$.

THEOREM 3. *Let integers* $p_1, \dots, p_k$ *satisfy* $0 \leq p_1 \leq n_1, \dots, 0 \leq p_k \leq n_k$. *Suppose that integers* z_{ts} *satisfy*

$$0 \leq z_{t1} \leq \dots \leq z_{t,n_t-p_t} \leq p_t , \quad t = 1, \dots, k . \tag{18}$$

Define

$$z_{ts} = p_t \quad \text{for} \quad s > n_t - p_t . \tag{19}$$

Set $p = p_1 + \dots + p_k$. *Let*

$$\zeta_t = \sum_{\rho=1}^{k} z_{\rho,t} , \quad t = 1, \dots, n-p . \tag{20}$$

Define subscripts i_{ts} *and* k_t *by*

$$i_{ts} = s + \delta_{z_{t1}}(s) + \dots + \delta_{z_{t,n_t-p_t}}(s) , \qquad s = 1, \dots, p_t, \ t = 1, \dots, k , \tag{21}$$

$$k_s = s + \delta_{\zeta_1}(s) + \dots + \delta_{\zeta_{n-p}}(s) , \quad s = 1, \dots, p . \tag{22}$$

Then the eigenvalues γ_i *of* $C = \left(A_{st}\right)$ *and the eigenvalues* α_{ti} *of* A_{tt}, *its main diagonal blocks, satisfy*

(23) $$\sum_{s=1}^{p} \gamma_{k_s} \geq \sum_{t=1}^{k} \left(\sum_{s=1}^{p_t} \alpha_{t,i_{ts}} \right) .$$

Proof. Define $j_{t\rho} = z_{t\rho} + \rho$ for all $\rho \geq 1$ and $m_t = n_t - p_t$. Then the conditions of Theorem 2 are satisfied. We now use the following fact proved in the Lemma of [9]: if integers $a_1, \ldots, a_{n-p}$ satisfy $1 \leq a_1 < \ldots < a_{n-p} \leq n$ then the integers $a'_1, \ldots, a'_p$ satisfying $1 \leq a'_1 < \ldots < a'_p \leq n$ and distinct from $a_1, \ldots, a_{n-p}$ are given by the formula

$$a'_s = s + \sum_{\rho=1}^{n-p} \delta_{a_\rho - \rho}(s) , \quad s = 1, \ldots, p .$$

By this fact the integers in $1, \ldots, n_t$ complementary to the j_{ts}, $s = 1, \ldots, m_t$, are the i_{ts} defined above, and the integers in $1, \ldots, n$ complementary to the integers $j_{1s} + \ldots + j_{ks} - (k-1)s$, $s = 1, \ldots, m$, are the k_s, $s = 1, \ldots, p$, given above. Since $\operatorname{trace} C = \operatorname{trace} A_{11} + \ldots + \operatorname{trace} A_{kk}$, it is clear that the inequality of Theorem 2 induces an inequality in the opposite sense involving these complementary subscripts.

THEOREM 4. *Let* $C = (A_{st})$ *be as described above. For each fixed* t, $1 \leq t \leq k$, *let integers* p_t, Z_{ts}, $s = 1, 2, \ldots$, *satisfy*

$$0 \leq p_t \leq n_t ,$$

$$p_t \geq Z_{t1} \geq Z_{t2} \geq \ldots \geq Z_{t,n_t-p_t} \geq 0 ,$$

$$Z_{t\rho} = 0 \quad \text{for} \quad \rho > n_t - p_t .$$

Define subscripts I_{ts} *and* K_s *by*

$$I_{ts} = s + \delta_{Z_{t1}}(s) + \ldots + \delta_{Z_{t,n_t-p_t}}(s) , \quad s = 1, \ldots, p_t, \ t = 1, \ldots, k ,$$

$$K_s = s + \delta_{\xi_1}(s) + \ldots + \delta_{\xi_{n-p}}(s) \qquad , \quad s = 1, \ldots, p ,$$

where $p = p_1 + \ldots + p_k$, *and* $\xi_\rho = Z_{1\rho} + \ldots + Z_{k\rho}$, $\rho = 1, \ldots, n-p$. *Then the eigenvalues* γ_i *of* C *and the eigenvalues* α_{ti} *of* A_{tt} , *its main diagonal blocks, satisfy*

$$\sum_{s=1}^{p} \gamma_{K_s} \le \sum_{t=1}^{k} \left(\sum_{s=1}^{p_t} \alpha_{t,I_{ts}} \right) .$$

Proof. Apply Theorem 3 to $-C = (-A_{st})$, setting $z_{ts} = p_t - Z_{ts}$ and using the fact that

$$1 - \delta_z(u) = \delta_{q-z}(q+1-u) .$$

THEOREM 5. *Let* $C = (A_{st})$ *be as described above. For each fixed* t , $1 \le t \le k$, *let* m_t, J_{ts} *satisfy* $0 \le m_t \le n_t$,

(24) $$n_t \ge J_{t1} > \ldots > J_{t,m_t} \ge 1 ,$$

(25) $$J_{ts} = m_t + 1 - s \quad for \quad s > m_t .$$

Let $m = m_1 + \ldots + m_k$. *Then the eigenvalues* γ_i *of* C *and* α_{ti} *of* A_{tt} , *its main diagonal blocks, satisfy*

(26) $$\sum_{s=1}^{m} \gamma_{J_{1s}+\ldots+J_{ks}+(k-1)(s-1)} \ge \sum_{t=1}^{k} \left(\sum_{s=1}^{m_t} \alpha_{t,J_{ts}} \right) .$$

Proof. Apply Theorem 3 to $-C = (-A_{st})$, taking $j_{ts} = n_t + 1 - J_{ts}$.

REMARK. The γ subscripts on the left-hand side of (26) decrease as t increases.

4. Comparison with previously known inequalities

The previously known inequalities are those in [*10*]. We compare the inequalities in [*10*, Theorem 2] with the inequalities in Theorem 1 above. Thus we shall compare the subscripts in (7) and (8).

Given a set of integers i_s, j_s satisfying (7.1) and (7.2) let

(27) $$I_s = i''_s , \quad J_s = j''_s , \quad 1 \le s \le m .$$

Then

$$1 \le I_1 < \dots < I_m \le a \ , \quad 1 \le J_1 < \dots < J_m \le b \ .$$

We may sharpen the inequality (7) if the integers i_s, j_s are decreased in such a fashion that the I_s, J_s remain unaltered and such that (7.1) continues to hold. Assuming that all possible such decreases in the i_s, j_s have been made, we say that the resulting set of i_s, j_s are fully reduced. For a fully reduced set of i_s, j_s, let $K_s = (i_s+j_s-1)''$, $s = 1, \dots, m$. The (7) becomes

$$\text{(28)} \qquad \sum_{s=1}^{m} \gamma_{K_s} + \sum_{s=1}^{m} \gamma_{n-m+s} \le \sum_{s=1}^{m} \alpha_{I_s} + \sum_{s=1}^{m} \beta_{J_s} \ .$$

For each fixed s, $1 \le s \le m$, the proof of Theorem 2 of [*11*] gives

$$\begin{aligned} K_s &= I_s + J_s - 1 - \max\left(I_s-i_s,\ J_s-j_s\right) \\ &= I_s + J_s - s + \left\{(s-1)-\max\left(I_s-i_s,\ J_s-j_s\right)\right\} \ . \end{aligned}$$

Thus

$$\text{(29)} \qquad K_s \ge I_s + J_s - s \ , \quad s = 1, \dots, m \ .$$

In (8) take the α and β subscripts to be I_s, J_s, respectively. Then (8) becomes

$$\text{(30)} \qquad \sum_{s=1}^{m} \gamma_{I_s+J_s-s} + \sum_{s=1}^{m} \gamma_{n-m+s} \le \sum_{s=1}^{m} \alpha_{I_s} + \sum_{s=1}^{m} \beta_{J_s} \ .$$

By virtue of (29), it is clear that (30) is a sharper assertion than (28).

Thus the inequalities in this paper are stronger than the inequalities in [*10*].

It is also clear from the first proof of Theorem 1 that the inequalities of [*10*] could have been derived from [*1*]. This was not realized until some time after Theorem 1 was proved (by the method of the second proof).

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5. Singular value inequalities

Throughout this section we let $C = (A_{st})_{1\le s,t\le k}$ be a not necessarily hermitian matrix, in partitioned form, with A_{tt} having dimensions $n_t \times n_t$, $t = 1, \ldots, k$. We let (13) be the singular values of A_{tt}, for $t = 1, \ldots, k$, and we let $\gamma_1 \ge \ldots \ge \gamma_n$ be the singular values of C. Thus $\gamma_1 \ge \ldots \ge \gamma_n \ge -\gamma_n \ge \ldots \ge -\gamma_1$ are the eigenvalues of the $(2n)$-square hermitian matrix

$$\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} .$$

On this matrix perform the unitary similarity in which we rearrange the block rows and block columns in the same way, by taking them in the order $1, k+1, 2, k+2, 3, k+3, \ldots, k, 2k$. Let K be the resulting matrix. Its eigenvalues are still $\gamma_1 \ge \ldots \ge \gamma_n \ge -\gamma_n \ge \ldots \ge -\gamma_1$, but now down the block diagonal we see the matrices

$$A_s = \begin{bmatrix} 0 & A_{ss} \\ A_{ss}^* & 0 \end{bmatrix} ,$$

which have eigenvalues $\alpha_{s1} \ge \ldots \ge \alpha_{s,n_s} \ge -\alpha_{s,n_s} \ge \ldots \ge -\alpha_{s,1}$; $s = 1, \ldots, k$.

THEOREM 6. *Let the not necessarily hermitian matrix* $C = (A_{st})$ *be as described above. Let* $0 \le p_s \le n_s$, $s = 1, \ldots, k$, *and let integers* z_{st} *satisfy* (18) *and* (19). *Define subscripts* i_{st}, k_t *by* (20), (21), *and* (22). *Then the singular values* γ_i *of* C *and the singular values* α_{ti} *of* A_{tt}, *its main diagonal blocks, satisfy* (23).

Proof. Apply Theorem 3 to the $2n$-square matrix K in which the main diagonal blocks are the $2n_t$-square matrices A_{tt}. Note that

$$\delta_{z_{t\rho}}(s) = 0 \quad \text{for} \quad s \le p_t \quad \text{and} \quad \rho > n_t - p_t ,$$

since $z_{t\rho} = p_t$ for $\rho > n_t - p_t$. Also note that

$$\delta_{z_{1\rho}+\dots+z_{k\rho}}(s) = 0 \quad \text{for} \quad s \le p \quad \text{and} \quad \rho > n - p ,$$

since if $\rho > n - p = (n_1-p_1) + \dots + (n_k-p_k) \ge n_i - p_i$, we have $z_{i\rho} = p_i$, hence $z_{1\rho} + \dots + z_{k\rho} = p_1 + \dots + p_k = p$. Using these facts, Theorem 3 applied to K yields (23).

THEOREM 7. *Let the not necessarily hermitian matrix* C *be as described above. Let* $0 \le m_t \le n_t$, $t = 1, \dots, k$ *and let integers* J_{ts} *satisfy* (24) *and* (25). *Then the singular values* γ_i *of* C *and the singular values* α_{ti} *of* A_{tt} *, its main diagonal block, satisfy* (26).

Proof. Apply Theorem 5 to K . One may verify that $J_{1s} + \dots + J_{ks} + (k-1)(s-1) \le n$ for $1 \le s \le m$ and so none of the negative eigenvalues of K enter when we apply Theorem 5 to K .

REMARK 1. In Theorem 6 set each $z_{s\rho} = p_s$. Then the inequality (23) becomes

$$\sum_{s=1}^{p} \gamma_s \ge \sum_{t=1}^{k} \left(\sum_{s=1}^{p_t} \alpha_{ts} \right) . \tag{31}$$

In Theorem 7 set $J_{ts} = m_t + 1 - s$ for all s, t . Then the inequality (26) reduces to (31). The inequality (31) is known; it is due to Gohberg and Kreĭn and appears as (5.4) on page 53 of [5]. Thus both Theorems 5 and 7 generalize the inequality of Gohberg and Kreĭn.

REMARK 2. By considering a nonsingular matrix with zero blocks on its main diagonal it is easy to see that Theorem 2 and 4 cannot be valid for singular values.

6. Applications

1. Let

$$L = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

be hermitian. Let $\alpha_1 \ge \alpha_2 \ge \dots$, $\lambda_1 \ge \lambda_2 \ge \dots$ be the eigenvalues of A and L respectively. Let

$$D = \begin{bmatrix} 0 & B \\ B^* & C \end{bmatrix}$$

and let $\delta_1^2 \geq \delta_2^2 \geq \dots$ be the eigenvalues of D^2. In [16] it was shown that if $\alpha_p \geq 0$ then

(32) $$\lambda_p^2 - \alpha_p^2 \leq 2\delta_1^2 .$$

The proof involved a combination of the Aronszajn inequality $\gamma_{i+j-1} + \gamma_n \leq \alpha_i + \beta_j$ with the Weyl inequality $\gamma_{i+j-1} \leq \alpha_i + \beta_j$ for the eigenvalues of a sum $C = A + B$. By using the generalization (8) of Aronszajn's inequality and the Lidskiĭ inequality (see [8] or [17]) for the eigenvalues of a sum, and slightly sharpening the argument in [16], the following generalization of (32) may be established: If $i_1 < i_2 < \dots < i_p$ and $\alpha_{i_p} \geq 0$, then

$$\sum_{t=1}^{p} \left(\lambda_{i_t}^2 - \alpha_{i_t}^2\right) \leq \sum_{t=1}^{2p} \delta_t^2 .$$

Here $\delta_t = 0$ if t exceeds the number of rows in D.

2. Let $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where all blocks A, X, Y, B are k-square. Let $\alpha_1 \geq \dots$, $x_1 \geq \dots$, $\beta_1 \geq \dots$, $y_1 \geq \dots$ be the singular values of A, X, B, Y, respectively. Let $\gamma_1 \geq \dots$ be the singular values of C. If $1 \leq i_1 < \dots < i_m \leq k$, $1 \leq j_1 < \dots < j_m \leq k$, then

$$\sum_{s=1}^{m} \gamma_{i_s+j_s-s}^2 + \sum_{s=1}^{m} \gamma_{k-m+s}^2 \leq \sum_{s=1}^{m} \alpha_{i_s}^2 + \sum_{s=1}^{m} \beta_{j_s}^2 + \sum_{s=1}^{m} x_s^2 + \sum_{s=1}^{m} y_s^2 .$$

If, instead, we have $k \geq i_1 > \dots > i_m \geq 1$, $k \geq j_1 > \dots > j_m \geq 1$, then

$$\sum_{s=1}^{m} \gamma_{i_s+j_s+s-1}^2 + \sum_{s=1}^{m} \gamma_s^2 \geq \sum_{s=1}^{m} \alpha_{i_s}^2 + \sum_{s=1}^{m} \beta_{j_s}^2 + \sum_{s=1}^{m} x_{k-m+s}^2 + \sum_{s=1}^{m} y_{k-m+s}^2 .$$

These inequalities may be obtained by applying Theorems 3 and 5 to CC^* in which $AA^* + XX^*$, $YY^* + BB^*$ are the main diagonal blocks, and using Lidskiĭ's inequalities.

Many other inequalities of this nature may be proved by combining Theorems 3-6 with the inequalities in [9, 11] for the eigenvalues of the sum of hermitian matrices.

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University of California,
Santa Barbara,
California, USA.