*Inequalities for a Classical Eigenvalue Problem**

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§1. Introduction. (*a*) Let $\mathfrak{D}$ be a simply-connected domain in the xy-plane bounded by the simple closed curve $\mathfrak{C}$, assumed to be analytic. We consider the problem of determining those twice-continuously differentiable functions $\phi = \phi(x, y)$ which satisfy

$$\nabla^2 \phi = 0 \quad \text{in} \quad \mathfrak{D}, \tag{1.1}$$

$$\frac{\partial \phi}{\partial n} = h\phi \quad \text{on} \quad \mathfrak{C}, \tag{1.2}$$

where $\partial/\partial n$ indicates differentiation with respect to the *exterior* normal to $\mathfrak{C}$, ∇^2 is the two-dimensional laplacian operator, and h is a constant. We call this the "problem of Stekloff" [5].

It can be shown [1, 2] that only for a discrete infinite set of non-negative real values of h (eigenvalues) do there exist functions ϕ (eigenfunctions) that satisfy (1.1) and (1.2). We denote the totality of eigenvalues by $h_0, h_1, h_2, \cdots$ (with $h_k \leqq h_{k+1}$ for all k), and the corresponding eigenfunctions by $\phi_0, \phi_1, \phi_2, \cdots$. We suppose the latter to be normalized according to the rule

$$\int_{\mathfrak{C}} \phi_k^2 \, ds = 1 \qquad (k = 0, 1, 2, \cdots). \tag{1.3}$$

(It is clear from (1.1) and (1.2) that $h_0 = 0$, $\phi_0 =$ constant.) The main interest of the present paper lies in determining an upper bound for the first non-trivial

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eigenvalue h_1. In fact, adaptation of a method devised by Szegö [6] leads to the result (§2, below)

$$h_1 \leqq 2\pi/L, \tag{1.4}$$

where L is the perimeter of $\mathfrak{C}$. Since the equality in (1.4) holds if and only if $\mathfrak{C}$ is a circle (§3, below), we prove that

of all simply-connected domains with analytic boundary of assigned perimeter L, the circle yields the largest value of h_1.

A weaker result is achieved if we substitute "area" for "perimeter." This follows from (1.4) with the aid of the classical isoperimetric inequality $L^2 \geqq 4\pi A$:

$$h_1 \leqq \sqrt{\frac{\pi}{A}}, \tag{1.5}$$

where A is the area of $\mathfrak{D}$. Equality in (1.5) holds only if $\mathfrak{C}$ is a circle.

(b) In order to achieve an elementary derivation of (1.4) valid when $\mathfrak{D}$ is convex, the following purely geometric inequality [7] is employed in §5(a) below:

$$LA \leqq \pi J, \tag{1.6}$$

where L is the length of $\mathfrak{C}$, A is the area of $\mathfrak{D}$ (assumed convex), and J is the polar moment of inertia of $\mathfrak{C}$ with respect to the centroid of $\mathfrak{C}$.

(c) If $\mathfrak{C}$ is the circle $x^2 + y^2 = a^2$, the Stekloff problem (1.1), (1.2) is completely soluble in explicit form. We have, in fact,

$$h_{2k-1} = h_{2k} = \frac{k}{a}, \qquad \begin{Bmatrix} \phi_{2k-1} = b_k\, r^k \cos k\theta \\ \phi_{2k} \quad = b_k\, r^k \sin k\theta \end{Bmatrix} \qquad (k = 1, 2, 3, \cdots), \tag{1.7}$$

where (r, θ) are plane polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$) and the b_k are numerical factors chosen so as to effect the normalization (1.3).

In particular, we have $h_1 = h_2 = 1/a$, so that, since $L = 2\pi a$ and $A = \pi a^2$, (1.4) and (1.5) reduce to equalities. Also

$$\phi_1 = b_1\, r \cos \theta = b_1\, x, \qquad \phi_2 = b_1\, r \sin \theta = b_1\, y \tag{1.8}$$

are eigenfunctions corresponding to the lowest non-trivial eigenvalue $h_1 = h_2$.

§2. Upper bound for h_1. (a) A definition of the lowest non-trivial Stekloff eigenvalue equivalent to the one given in §1(a) is the following [1, 2]:

$$h_1 = \min_{u \in \Omega} \frac{\iint_D (u_x^2 + u_y^2)\, dx\, dy}{\int_C u^2\, ds}, \tag{2.1}$$

where $u_x \equiv \partial u/\partial x$, $u_y \equiv \partial u/\partial y$, and Ω is the class of functions u continuously differentiable in $\mathfrak{D}$, continuous on $\mathfrak{C}$, and such that

$$\int_{\mathfrak{C}} u \, ds = 0. \tag{2.2}$$

Thus for $u \, \varepsilon \, \Omega$ we have the inequality

$$h_1 \leqq \frac{\iint_{\mathfrak{D}} (u_x^2 + u_y^2) \, dx \, dy}{\int_{\mathfrak{C}} u^2 \, ds}, \tag{2.3}$$

with equality achieved if and only if $u = \phi$, where ϕ is an eigenfunction satisfying

$$\nabla^2 \phi = 0 \quad \text{in} \quad \mathfrak{D}, \qquad \frac{\partial \phi}{\partial n} = h_1 \phi_1 \quad \text{on} \quad \mathfrak{C}. \tag{2.4}$$

To select a pair of functions from the class Ω for substitution into (2.3) we write $z \equiv x + iy$ and $w \equiv \xi + i\eta$ and consider the family of analytic functions $z = f(w)$ which map $\mathfrak{D}$ univalently onto the circle $| w | < 1$ in the $\xi\eta$–plane. (Since $\mathfrak{C}$ is analytic, each such function is analytic in $| w | \leqq 1$.) We denote the (unique) mapping inverse to $z = f(w)$ by

$$\xi + i\eta = w = g(z) = U(x, y) + iV(x, y), \tag{2.5}$$

where U and V are real functions harmonic in $\mathfrak{D}$. (We note that because of the analyticity of $\mathfrak{C}$

$$g'(z) = 1/f'(w) \tag{2.6}$$

holds for $z \, \varepsilon \, \mathfrak{C}$ as well as for $z \, \varepsilon \, \mathfrak{D}$.)

In (*d*) below it is shown that there exists at least one $f(w)$ such that both $U \, \varepsilon \, \Omega$ and $V \, \varepsilon \, \Omega$. In fact by (2.2)

$$\int_{\mathfrak{C}} (U + iV) \, ds = \int_{\mathfrak{C}} g(z) \, ds = 0. \tag{2.7}$$

(*b*) In what follows we need the following elementary arithmetic result: When m, n, m', and n' are all positive

$$\left. \begin{aligned} h &\leqq \frac{m}{n} \\ h &\leqq \frac{m'}{n'} \end{aligned} \right\} \quad \text{implies} \quad h \leqq \frac{m + m'}{n + n'}. \tag{2.8}$$

(*c*) Assuming the validity of (2.7) until its proof in (*d*) below, we employ (2.3) to obtain the pair of inequalities

$$h_1 \leqq \frac{\iint_{\mathfrak{D}} (W_x^2 + W_y^2)\, dx\, dy}{\int_{\mathfrak{C}} W^2\, ds} \qquad (W = U, V). \tag{2.9}$$

On introduction of the change of variables $z = f(w)$, whose inverse is provided by (2.5), the inequalities (2.9) read

$$h_1 \leqq \frac{\iint_{|w|<1} (W_\xi^2 + W_\eta^2)\, d\xi\, d\eta}{\int_{|w|=1} W^2\, |f'(w)\, dw|} \qquad (W = \xi, \eta), \tag{2.10}$$

where we have used the well-known invariance of form of the Dirichlet integral under conformal transformation. Substitution of $W = \xi$, $W = \eta$ then gives

$$h_1 \leqq \frac{\iint_{|w|<1} d\xi\, d\eta}{\int_{|w|=1} \xi^2\, |f'(w)\, dw|}, \qquad h_1 \leqq \frac{\iint_{|w|<1} d\xi\, d\eta}{\int_{|w|=1} \eta^2\, |f'(w)\, dw|}, \tag{2.11}$$

from which follows directly, according to (2.8),

$$h_1 \leqq \frac{2\iint_{|w|<1} d\xi\, d\eta}{\int_{|w|=1} (\xi^2 + \eta^2)\, |f'(w)\, dw|} = \frac{2\pi}{\int_{|w|=1} |f'(w)\, dw|} = \frac{2\pi}{L}, \tag{2.12}$$

where

$$L = \int_{\mathfrak{C}} ds = \int_{|w|=1} |f'(w)\, dw| \tag{2.13}$$

is the perimeter of $\mathfrak{C}$. This is the result announced in (1.4).

(*d*) We fix our attention upon $z = f_0(w)$, a particular univalent conformal mapping of $\mathfrak{D}$ onto $|w| < 1$, and note that another such mapping is [3]

$$z = f(w) = f_0\left(\frac{w - \alpha}{1 - \bar{\alpha}w}\right), \qquad \alpha = \rho e^{i\gamma}, \qquad 0 \leqq \rho < 1, \tag{2.14}$$

for arbitrary real γ. Writing $w = g(z)$ for the inverse of (2.14), we may express the line integral (2.7) as

$$\begin{aligned} \int_{\mathfrak{C}} g(z)\, ds &= \int_{|w|=1} w\, |f'(w)\, dw| = \int_{|w|=1} w\, \frac{1 - |\alpha|^2}{|1 - \bar{\alpha}w|^2} \left| f_0'\left(\frac{w - \alpha}{1 - \bar{\alpha}w}\right) dw \right| \\ &= \int_{-\delta}^{2\pi-\delta} \frac{e^{i\theta}(1 - \rho^2)}{|1 - \rho e^{i(\theta-\gamma)}|^2} \left| f_0'\left(\frac{e^{i\theta} - \rho e^{i\gamma}}{1 - \rho e^{i(\theta-\gamma)}}\right) \right| d\theta \equiv I(\alpha), \end{aligned} \tag{2.15}$$

where δ is an arbitrary real number. We note that $I(\alpha)$, defined by (2.15), is a continuous function of α for $|\alpha| < 1$.

Since the integrand has the period 2π with respect to θ, we may rewrite (2.15) as

$$I(\alpha) = e^{i\gamma}H(\alpha), \tag{2.16}$$

where

$$H(\alpha) = \int_{-\delta}^{2\pi-\delta} \frac{e^{i\theta}(1-\rho^2)}{|1-\rho e^{i\theta}|^2}\left|f_0'\left(\frac{e^{i\gamma}(e^{i\theta}-\rho)}{1-\rho e^{i\theta}}\right)\right| d\theta. \tag{2.17}$$

The steps involved in the transformation of (2.15) similarly lead from (2.13) to

$$L = \int_{-\delta}^{2\pi-\delta} \frac{1-\rho^2}{|1-\rho e^{i\theta}|^2}\, |f_0'|\, d\theta, \tag{2.18}$$

in which the argument of f_0' is the same as that in (2.17).

In order to justify the use of (2.7) in (*c*) above, we employ the method of Szegö [6] to demonstrate the existence of at least one α, with $|\alpha| < 1$, for which $I(\alpha) = 0$. This method employs the fact that the mapping $\beta = I(\alpha)$ is continuous for $|\alpha| < 1$, as follows from (2.15); and, moreover, that the index (with respect to $\beta = 0$) of the β-plane image of $\alpha = \rho e^{i\gamma}$, as γ runs from 0 to 2π, is unity if ρ (<1) is a constant sufficiently close to one. To prove the latter fact, by (2.16) it is adequate to show that

$$|H(\alpha) - L| < \epsilon \tag{2.19}$$

for any preassigned positive ϵ, provided positive $(1-\rho)$ is sufficiently close to zero. The proof of (2.19) follows:

We choose δ so that

$$|1 - e^{i\theta}| < \epsilon/2L \quad \text{for} \quad -\delta < \theta < \delta \quad (0 < \delta < \pi). \tag{2.20}$$

With the aid of (2.17) and (2.18) a short calculation then gives

$$\begin{aligned} |H(\alpha) - L| &\leqq \int_{-\delta}^{2\pi-\delta} \frac{|1-e^{i\theta}|(1-\rho^2)}{|1-\rho e^{i\theta}|^2}\,|f_0'|\,d\theta \\ &< \frac{\epsilon}{2L}\int_{-\delta}^{\delta}\frac{(1-\rho^2)|f_0'|}{|1-\rho e^{i\theta}|^2}\,d\theta + \int_{\delta}^{2\pi-\delta}\frac{(2\sin\frac{1}{2}\theta)(1-\rho^2)|f_0'|}{(1-\rho)^2+4\rho\sin^2\frac{1}{2}\theta}\,d\theta \\ &< \frac{\epsilon}{2L}\int_{-\delta}^{2\pi-\delta}\frac{(1-\rho^2)|f_0'|}{|1-\rho e^{i\theta}|^2}\,d\theta + \frac{1-\rho^2}{2\rho}\int_{\delta}^{2\pi-\delta}\frac{|f_0'|}{\sin\frac{1}{2}\theta}\,d\theta \\ &< \frac{\epsilon}{2} + \frac{2\pi M}{\rho\sin\frac{1}{2}\delta}(1-\rho), \end{aligned} \tag{2.21}$$

where $|f_0'(w)| < M$ on $|w| = 1$. (That M is finite follows from the analyticity of $\mathcal{C}$.) With δ already fixed by (2.20), we choose ρ so close to one that the final term of (2.21) is less than $\frac{1}{2}\epsilon$. This completes the proof of (2.19).

§3. The case of equality. (*a*) Although it is clear (§1 (*c*)) that (2.12) becomes an equality if $\mathfrak{C}$ is a circle, the mode of derivation fails to indicate whether there exists any other form of $\mathfrak{C}$ for which $h_1 = 2\pi/L$. To answer this question (in the negative), we first note that equality holds in (2.3) if and only if $u = \phi$, where ϕ satisfies (2.4); we next proceed to show that the functions $U(x, y)$ and $V(x, y)$, introduced in (2.5), satisfy

$$(3.1) \qquad \frac{\partial U}{\partial n} = h_1 U, \qquad \frac{\partial V}{\partial n} = h_1 V \qquad \text{on } \mathfrak{C}$$

only if $\mathfrak{C}$ is a circle.

Since the mapping $z = f(w)$ of $\mathfrak{D}$ onto $| w | < 1$ is conformal, we have

$$(3.2) \qquad \frac{\partial U}{\partial n} = \frac{1}{|f'(e^{i\theta})|}\left[\frac{\partial \xi}{\partial r}\right]_{r=1} \quad \text{on} \quad \mathfrak{C} \qquad (| w | = 1),$$

because $U = \xi = r \cos \theta$. Since the partial derivative on the right is thus equal to $\cos \theta = \xi$ (for $r = 1$) $= U$, from (3.2) and (2.6) follows

$$(3.3) \qquad \frac{\partial U}{\partial n} = | g'(z) | U \quad \text{on} \quad \mathfrak{C}.$$

Thus the first relation of (3.1) requires $| g'(z) | = h_1$, a positive constant, on $\mathfrak{C}$. (Clearly the same requirement follows also from the second relation of (3.1).) Since the univalence of $w = g(z)$ requires $g'(z) \neq 0$ in $\mathfrak{D}$, this implies that $g'(z) =$ constant throughout $\mathfrak{D}$. Hence (3.1)—and therefore equality in (2.12)—holds only if $g(z) = b_1 z + b_0$, so that $\mathfrak{C}$ is necessarily a circle.

(*b*) It is clear that the results of §2 and (*a*) above remain valid if $\mathfrak{C}$ is merely piecewise analytic, provided that $| f'(w) |$ and $| f'(w) |^{-1}$ remain bounded as $| w | \to 1$. We do not investigate the geometric implications of this requirement.

§4. A generalization. (*a*) A modified Stekloff problem involves determination of functions ψ that satisfy

$$(4.1) \qquad \nabla^2 \psi = 0 \quad \text{in} \quad \mathfrak{D}, \qquad \frac{\partial \psi}{\partial n} = j p(s) \psi \quad \text{on} \quad \mathfrak{C},$$

where j is a constant and $p(s)$ is a given positive continuous function of position on $\mathfrak{C}$. As in the ordinary Stekloff problem ($p \equiv 1$) there exists [1, 2] a discrete infinite set of non-negative real eigenvalues of j and corresponding eigenfunctions. If j_1 is the lowest nonzero eigenvalue, then [1, 2]

$$(4.2) \qquad j_1 \leqq \frac{\iint_{\mathfrak{D}} (v_x^2 + v_y^2)\, dx\, dy}{\int_{\mathfrak{C}} v^2 p(s)\, ds},$$

where $v = v(x, y)$ satisfies

$$\int_{\mathfrak{C}} vp(s)\, ds = 0. \tag{4.3}$$

Step-by-step use of the technique employed in §2 provides the inequality

$$j_1 \leqq \frac{2\pi}{\int_{\mathfrak{C}} p(s)\, ds}, \tag{4.4}$$

of which (2.12) is a special case ($p \equiv 1$). If $w = g(z)$ maps $\mathfrak{D}$ univalently onto $|w| < 1$ with the normalization which (4.3) indicates to be the proper extension of (2.7), namely,

$$\int_{\mathfrak{C}} p(s) g(z)\, ds = 0, \tag{4.5}$$

equality holds in (4.4) if and only if

$$|g'(z)| = j_1 p(s) \quad \text{on} \quad \mathfrak{C}. \tag{4.6}$$

(This follows, in applying the technique of §3(a) to the modified Stekloff problem, on comparison of (3.3) with the second relation of (4.1).)

(b) An interesting special case of the modified Stekloff problem is that in which $\mathfrak{D}$ is convex and $p(s) \equiv K(s)$, the (positive) curvature of $\mathfrak{C}$. Since

$$\int_{\mathfrak{C}} K(s)\, ds = 2\pi, \tag{4.7}$$

(4.4) reads

$$j_1 \leqq 1 \qquad [p \equiv K(s)]. \tag{4.8}$$

If $z = f(w)$ is the mapping inverse to $w = g(z)$ mentioned in (a) above, the curvature of $\mathfrak{C}$ is given by the formula [4]

$$K(s) = \frac{1 + \operatorname{Re}\left\{ w \dfrac{f''(w)}{f'(w)} \right\}}{|f'(w)|} \qquad (|w| = 1). \tag{4.9}$$

It is obvious that if $\mathfrak{C}$ is a circle the equality in (4.8) holds; conversely, by (4.6), (2.6), and (4.9) it follows that equality in (4.8) leads to the relation

$$\operatorname{Re}\left\{ w \frac{f''(w)}{f'(w)} \right\} = 0 \qquad (|w| = 1). \tag{4.10}$$

Since $f'(w) \neq 0$ for $|w| \leqq 1$, the left-hand member of (4.10) is a regular harmonic function; therefore it is identically zero, in $|w| \leqq 1$. It then follows that Im $\{wf''(w)/f'(w)\}$ is constant; since this quantity vanishes at $w = 0$, it also is zero. That is, we must have

$$w\frac{f''(w)}{f'(w)} = 0 \qquad (|w| \leqq 1), \tag{4.11}$$

whence $f''(w) = 0$ identically; equality in (4.8) thus holds only if $f(w) = b_0 + b_1 w$, so that $\mathfrak{C}$ is a circle.

§5. A second upper bound for h_1. (*a*) We suppose the coordinate system in the xy-plane so situated that its origin is the centroid of the boundary curve $\mathfrak{C}$ of $\mathfrak{D}$; that is,

$$\int_{\mathfrak{C}} x\,ds = \int_{\mathfrak{C}} y\,ds = 0. \tag{5.1}$$

Thus, according to (2.2) and (2.3), we have

$$h_1 \leqq \frac{\iint_{\mathfrak{D}} dx\,dy}{\int_{\mathfrak{C}} x^2\,ds}, \qquad h_1 \leqq \frac{\iint_{\mathfrak{D}} dx\,dy}{\int_{\mathfrak{C}} y^2\,ds}, \tag{5.2}$$

whence it follows from (2.8) that

$$h_1 \leqq 2A/J, \tag{5.3}$$

where A is the area of $\mathfrak{D}$ and

$$J = \int_{\mathfrak{C}} (x^2 + y^2)\,ds \tag{5.4}$$

is the polar moment of inertia of $\mathfrak{C}$ with respect to its centroid.

(*b*) For a circle of radius a, $A = \pi a^2$ and $J = 2\pi a^3$, so that, according to §1(*c*), equality holds in (5.3); that such is the case only if $\mathfrak{C}$ is a circle is seen as follows.

Equality in (5.3) implies equality in both of the relations (5.2); by (2.4), these latter require that

$$\frac{\partial x}{\partial n} = h_1 x, \qquad \frac{\partial y}{\partial n} = h_1 y \quad \text{on} \quad \mathfrak{C}, \tag{5.5}$$

whence

$$h_1^2(x^2 + y^2) = \left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2 = 1 \quad \text{on} \quad \mathfrak{C}. \tag{5.6}$$

(*c*) It is clear that at least for some domains the upper bound presented by (5.3) is less than that given by (2.12). For a square of side b, for example, we easily compute

$$2A/J = 3/(2b) < \pi/(2b) = 2\pi/L; \tag{5.7}$$

clearly the same inequality holds for any long, slender domain whose area A is close to zero. When $\mathfrak{D}$ is convex and $\mathfrak{C}$ has piecewise continuous curvature the inequality $2A/J \leqq 2\pi/L$ always holds [7]; thus, for such domains the inequality (5.3) implies (1.4). That this is not the case generally for non-convex domains is exhibited by the counter example furnished by the case in which $\mathfrak{C}$ is the cardioid

$$r = 1 - \cos\theta, \tag{5.8}$$

in plane polar coordinates: We readily compute

$$2\pi/L = 2\pi/8 < 3\pi\tfrac{75}{896} = 2A/J. \tag{5.9}$$

(*d*) Demonstration of the validity of (5.3) is effected under a far weaker assumption as to $\mathfrak{C}$ than that which is required for the proof of (2.12) carried out in §2. It is sufficient in (*a*) and (*b*) above to assume merely piecewise smoothness of $\mathfrak{C}$. In (*c*), as stated, in deriving from (5.3) and (1.6) the main inequality (1.4) for convex domains we make the additional assumption of piecewise continuous curvature—still an assumption less severe than the restricted piecewise analyticity called for in §3(*b*).

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