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Inequalities for a Unified Integral Operator and Associated Results in Fractional Calculus

YOUNG CHEL KWUN[1], GHULAM FARID[2], SALEEM ULLAH[3], WAQAS NAZEER[4], KAHKASHAN MAHREEN[2], AND SHIN MIN KANG[5,6]

[1]Department of Mathematics, Dong-A University, Busan 49315, South Korea
[2]Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock 43600, Pakistan
[3]Department of Mathematics, Air University, Islamabad 44000, Pakistan
[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
[5]Department of Mathematics, Gyeongsang National University, Jinju 52828, South Korea
[6]Center for General Education, China Medical University, Taichung 40402, Taiwan

Corresponding authors: Ghulam Farid (faridphdsms@hotmail.com) and Shin Min Kang (smkang@gnu.ac.kr)

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ABSTRACT Integral operators are useful in real analysis, mathematical analysis, functional analysis and other subjects of mathematical approach. The goal of this paper is to study a unified integral operator via convexity. By using convexity and conditions of unified integral operators, bounds of these operators are obtained. Furthermore consequences of these results are discussed for fractional and conformable integral operators.

INDEX TERMS Convex function, Mittag-Leffler function, integral operator, fractional integral operator, conformable integral operator.

I. INTRODUCTION AND PRELIMINARY RESULTS

A function f satisfying the following inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

where $\lambda \in [0, 1]$, $x, y \in C$ and C is convex set, is called convex function on C. A function satisfying (1) in reverse order is called concave function. For properties and characterizations of convex functions, see [1].

Definition 1 [2]: Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu > 0$ are defined by:

$$ {}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1}g'(t)f(t)dt, \quad x > a \tag{2}$$

and

$$ {}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1}g'(t)f(t)dt, \quad x < b, \tag{3}$$

where $\Gamma(.)$ is the Gamma function.

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A k-fractional analogue of above definition is given as follows:

Definition 2 [3]: Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu, k > 0$ are defined by:

$$ {}_g^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1}g'(t)f(t)dt, \quad x > a \tag{4}$$

and

$$ {}_g^{\mu}I_{b^-}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1}g'(t)f(t)dt, \quad x < b, \tag{5}$$

where $\Gamma_k(.)$ is the k-Gamma function.

A generalized fractional integral with kernel an extended generalized Mittag-Leffler function is defined as follows:

Definition 3 [6]: Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c}f$ and

$\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f$ are defined by:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) = \int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f(t)dt, \quad (6)$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f\right)(x;p) = \int_x^b (t-x)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^{\mu};p)f(t)dt, \quad (7)$$

where

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{t^n}{(l)_{n\delta}} \quad (8)$$

is the extended generalized Mittag-Leffler function.

Recently Farid defined a new unified integral operator from which the fractional as well as conformable integral operators can be derived at once:

Definition 4 [7]: Let $f, g : [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a,b]$, and g be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}, p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then for $x \in [a,b]$ the left and right integral operators are defined by:

$$({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f)(x;p) = \int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)} \times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t)f(t)dt \quad (9)$$

and

$$({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f)(x;p) = \int_x^b \frac{\phi(g(t)-g(x))}{g(t)-g(x)} \times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p)g'(t)f(t)dt. \quad (10)$$

In [7] it is proved that the operators defined in (9) and (10) are bounded, further they are linear hence these are continuous operators.

Theorem 5 [7]: Under the assumptions of Definition 4, the following bounds hold for integral operators (9) and (10):

$$\left|({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f)(x;p)\right| \leq K\|f\|_{[a,b]} \quad (11)$$

and

$$\left|({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f)(x;p)\right| \leq K\|f\|_{[a,b]}. \quad (12)$$

Hence

$$\left|({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f)(x;p)+({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f)(x;p)\right| \leq 2K\|f\|_{[a,b]}, \quad (13)$$

where S is the sum of absolute terms of (8) and $K = S\,|\phi(g(b)-g(a))|$.

Integral operators defined in (9) and (10) are unified in the sense that for specific settings of functions ϕ, g and particular values of involved parameters in Mittag-Leffler function they contain two kinds of general fractional integral operators (4), (5), and (6), (7). These integral operators and their consequences are narrated in the following two remarks.

Remark 6: (i) Let $\phi(x) = \frac{x^{\beta/k}\Gamma(\beta)}{k\Gamma_k(\beta)}$, $k > 0$, $\beta > k$ and $p = \omega = 0$, in unified integral operators (9) and (10). Then generalized Riemann-Liouville fractional integral operators (4) and (5) are obtained.

(ii) For $k = 1$, (4) and (5) fractional integrals coincide with (2) and (3) fractional integrals, which further produce the following fractional and conformable integrals:

(iii) By taking g as identity function, (4) and (5) fractional integrals coincide with k-fractional Riemann-Liouville integrals defined by Mubeen *et al.* in [15].

(iv) For $k = 1$, along with g as identity function, (4) and (5) fractional integrals coincide with Riemann-Liouville fractional integrals [2].

(v) For $k = 1$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$, (4) and (5) produce fractional integrals defined by Chen *et al.* in [10].

(vi) For $k = 1$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, (4) and (5) produce generalized conformable integrals defined by Khan *et al.* in [13].

(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (4) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (5), then conformable (k,s)-fractional integrals will be obtained as defined by Habib *et al.* in [11].

(ix) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then conformable integrals will be obtained as defined by Sarikaya *et al.* in [16].

(x) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (4) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (5) with $k = 1$, then conformable integrals will be obtained as defined by Jarad *et al.* in [12].

Remark 7: Let $\phi(x) = x^{\beta}$ and $g(x) = x$, $\beta > 0$, in unified integral operators (9) and (10). Then fractional integral operators (6) and (7) are obtained, which along with different settings of $p, k, \delta, l, c, \gamma$ in generalized Mittag-Leffler function give the following integral operators:

1. By setting $p = 0$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Salim-Faraj in [5].

2. By setting $l = \delta = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Rahman *et al.* in [18].

3. By setting $p = 0$ and $l = \delta = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Srivastava-Tomovski in [14].

4. By setting $p = 0$ and $l = \delta = k = 1$, fractional integral operators (6) and (7) are reduced to the fractional intagral operators defined by Prabhakar in [19].

5. By setting $p = \omega = 0$, fractional integral operators (6) and (7) are reduced to the left-sided and right-sided Riemann-Liouville fractional integrals.

For detailed study of recent generalized, fractional and conformable integral operators one can consult [3]–[6], [8], [10]–[14], [16], [17] and references therein.

The purpose of this research is the study of all above integral operators via convex functions. We are succeeded to obtain bounds of integral operators defined in (9) and (10). These results provide formulas for bounds of all fractional and conformable integrals comprised in Remark1 and Remark 7. The paper is organized as follows:

In Section II, upper bounds of unified fractional integral operators (9) and (10) are established by using the involved conditions and convex functions. Further by imposing an additional condition of symmetry two sided Hadamard type bounds are obtained. Moreover by using convexity of $|f'|$ and applying integral operator on convolution of two functions some interesting bounds are studied. It is important to note that all these results hold for fractional and conformable integral operators comprised in Remark 6 and Remark 7. Also some fractional differential equations are solved in Section III.

II. MAIN RESULTS

Bounds of integral operators (9), (10) and their sum are obtained in the following theorem.

Theorem 8: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a positive convex function, $0 < a < b$ and $g : [a,b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$ and $\alpha, l, \gamma, c \in \mathbb{C}, p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in [a,b]$ we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \times (\phi(g(x)-g(a)))\,(f(x)+f(a)) \quad (14)$$

and

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p) \times (\phi(g(b)-g(x)))\,(f(x)+f(b)) \quad (15)$$

hence

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)\,(\phi(g(x)-g(a)))\,(f(x)+f(a)) + E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)\,(\phi(g(b)-g(x)))\,(f(x)+f(b)). \quad (16)$$

Proof 9: As g is increasing, therefore for $t \in [a,x]$, $x \in (a,b)$, $g(x)-g(t) \leq g(x)-g(a)$. The function $\frac{\phi}{x}$ is increasing, therefore one can obtain:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)} \leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}. \quad (17)$$

Now by multiplying with $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t)$ the following inequality is yielded:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p) \leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p). \quad (18)$$

Also $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)$ is series of positive terms, therefore $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$ so the following inequality holds:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p) \leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p). \quad (19)$$

Using convexity of f on $[a,x]$ for $x \in (a,b)$ we have

$$f(t) \leq \frac{x-t}{x-a}f(a) + \frac{t-a}{x-a}f(x). \quad (20)$$

Multiplying (19) and (20), then integrating with respect to t over $[a,x]$ we have

$$\int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)f(t) \times E^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega;g}(\omega(g(x)-g(t))^{\mu};p)dt \leq \frac{f(a)}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \times \int_a^x (x-t)g'(t)dt + \frac{f(x)}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \times \int_a^x (t-a)g'(t)dt. \quad (21)$$

By using (9) of Definition 4, and integrating by parts we get

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \times \left(\frac{\phi(g(x)-g(a))}{g(x)-g(a)}\right)\left(\frac{f(a)}{x-a}\left(g(a)(a-x)+\int_a^x g(t)dt\right) + \frac{f(x)}{x-a}\left((x-a)g(x)-\int_a^x g(t)dt\right)\right)$$

which further simplifies as follows:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \times (\phi(g(x)-g(a)))\,(f(x)+f(a)). \quad (22)$$

Now on the other hand for $t \in (x,b]$, $x \in (a,b)$ the following inequality holds true:

$$\frac{\phi(g(t)-g(x))}{g(t)-g(x)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p) \leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p). \quad (23)$$

Also $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p)$ is series of positive terms, therefore $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)$, so the following inequality is valid:

$$\frac{\phi(g(t)-g(x))}{g(t)-g(x)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p) \leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p). \quad (24)$$

The following inequality also holds for convex function f:

$$f(t) \leq \frac{t-x}{b-x}f(b) + \frac{b-t}{b-x}f(x). \quad (25)$$

Multiplying (24) and (25), then integrating with respect to t over $(x, b]$ and adopting the same pattern of simplification as we did for (21), the following inequality is obtained:

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;p)\\ &\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)\\ &\quad\times\left(\frac{\phi(g(b)-g(x))}{g(b)-g(x)}\right)\left(\frac{f(b)}{b-x}\left(g(b)(b-x)-\int_x^b g(t)dt\right)\right.\\ &\quad\left.+\frac{f(x)}{b-x}\left((x-b)g(x)+\int_x^b g(t)dt\right)\right)\end{aligned}$$

which further simplifies as follows:

$$\begin{aligned}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;p) &\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)\\ &\quad\times(\phi(g(b)-g(x)))\,(f(x)+f(b)). \quad (26)\end{aligned}$$

By adding (22) and (26), (16) can be achieved.

Henceforth we give consequences of above theorem for fractional calculus and conformable integral operators defined in [2], [5], [9]–[13], [15], [16].

Proposition 10: Let $\phi(t) = t^{\alpha}$ and $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators (2) and (3) defined in [2], as follows:

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) := {}^{\alpha}_{g}I_{a^+}f(x)$$

and

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) := {}^{\alpha}_{g}I_{b^-}f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \geq 1$, therefore they satisfy the following bound:

$$\begin{aligned}&({}^{\alpha}_{g}I_{a^+}f)(x) + ({}^{\alpha}_{g}I_{b^-}f)(x)\\ &\qquad\leq \frac{1}{\Gamma(\alpha)}\left((g(x)-g(a))^{\alpha}\ (f(x)+f(a))\right.\\ &\qquad\quad\left.+(g(b)-g(x))^{\alpha}(f(x)+f(b))\right).\end{aligned}$$

Proposition 11: Let $g(x) = I(x) = x$ and $p = \omega = 0$. Then (9) and (10) produce integral operators defined in [17] as follows:

$$\begin{aligned}\Gamma(\alpha)\left({}_IF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) &:= ({}_{a^+}I_{\phi}f)(x)\\ &= \int_a^x \frac{\phi(x-t)}{(x-t)}f(t)dt\end{aligned}$$

and

$$\begin{aligned}\Gamma(\alpha)\left({}_IF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) &:= ({}_{b^-}I_{\phi}f)(x)\\ &= \int_x^b \frac{\phi(t-x)}{(t-x)}f(t)dt.\end{aligned}$$

Further they satisfy the following bound:

$$\begin{aligned}&({}_{a^+}I_{\phi}f)(x) + ({}_{b^-}I_{\phi}f)(x)\\ &\quad\leq \phi(x-a)(f(x)+f(a)) + \phi(b-x)(f(x)+f(b)).\end{aligned}$$

Corollary 12: If we take $\phi(t) = \frac{\Gamma(\alpha)t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators (4) and (5) defined in [3] as follows:

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) := {}^{\alpha}_{g}\ I^k_{a^+}f(x)$$

and

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) := {}^{\alpha}_{g}\ I^k_{b^-}f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \geq k$, therefore they satisfy the following bound:

$$\begin{aligned}({}^{\alpha}_{g}I^k_{a^+}f)(x)+({}^{\alpha}_{g}I^k_{b^-}f)(x) &\leq \frac{1}{k\Gamma_k(\alpha)}((g(x)-g(a))^{\frac{\alpha}{k}}(f(x)+f(a))\\ &\quad+(g(b)-g(x))^{\frac{\alpha}{k}}(f(b)+f(x)).\end{aligned}$$

Corollary 13: If we take $\phi(t) = t^{\alpha}$ and $g(x) = I(x) = x$ with $p = \omega = 0$. Then (9) and (10) produce left and right Riemann-Liouville fractional integrals [2] as follows:

$$\left({}_IF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) := {}^{\alpha}I_{a^+}f(x)$$

and

$$\left({}_IF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) := {}^{\alpha}I_{b^-}f(x).$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \geq k$ therefore, they satisfy the following bound:

$$\begin{aligned}&({}^{\alpha}I_{a^+}f)(x) + ({}^{\alpha}I_{b^-}f)(x)\\ &\leq \frac{1}{\Gamma(\alpha)}((x-a)^{\alpha}(f(x)+f(a)) + (b-x)^{\alpha}(f(b)+f(x))).\end{aligned}$$

Corollary 14: If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x) = I(x) = x$, $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators define in [15] as follows:

$$\left({}_IF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) := {}^{\alpha}I^k_{a^+}f(x)$$

and

$$\left({}_IF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) := {}^{\alpha}I^k_{b^-}f(x).$$

Further they satisfy the following bound for $\alpha \geq k$:

$$\begin{aligned}({}^{\alpha}I^k_{b^-}f)(x) + ({}^{\alpha}I^k_{b^-}f)(x) &\leq \frac{1}{k\Gamma_k(\alpha)}((x-a)^{\frac{\alpha}{k}}(f(x)+f(a))\\ &\quad+(b-x)^{\frac{\alpha}{k}}(f(b)+f(x))).\end{aligned}$$

Corollary 15: If we take $\phi(t) = t^{\alpha}, \alpha > 0$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$ with $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [10], as follows:

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^{\rho}I^{\alpha}_{a+}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{\rho}-t^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{\rho}-x^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^{\rho}I^{\alpha}_{a+}f)(x) + ({}^{\rho}I^{\alpha}_{b-}f)(x) \leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}((x^{\rho}-a^{\rho})^{\alpha}(f(x)+f(a)) + (b^{\rho}-x^{\rho})^{\alpha}(f(b)+f(x))).$$

Corollary 16: If we take $\phi(t) = t^{\alpha}, \alpha > 0$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators define as follows:

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^{s}I^{\alpha}_{a+}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{s+1}-t^{s+1})^{\alpha-1}t^{s}f(t)dt$$

and

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = ({}^{s}I^{\alpha}_{b-}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{s+1}-x^{s+1})^{\alpha-1}t^{s}f(t)dt.$$

Further they satisfy the following bound:

$$({}^{s}I^{\alpha}_{a+}f)(x) + ({}^{s}I^{\alpha}_{b-}f)(x) \leq \frac{1}{(s+1)^{\alpha}\Gamma(\alpha)}((x^{s+1}-a^{s+1})^{\alpha}(f(x)+f(a)) + (b^{s+1}-x^{s+1})^{\alpha}(f(b)+f(x))).$$

Corollary 17: If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [16], as follows:

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^{s}_{k}I^{\alpha}_{a+}f)(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}t^{s}f(t)dt$$

and

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = ({}^{s}_{k}I^{\alpha}_{b-}f)(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^{s}f(t)dt.$$

Further $\frac{\phi}{t}$ is increasing for $\alpha \geq k$ therefore, they satisfy the following bound:

$$({}^{s}_{k}I^{\alpha}_{a+}f)(x) + ({}^{s}_{k}I^{\alpha}_{b-}f)(x) \leq \frac{1}{(s+1)^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}((f(x)+f(a))(b^{s+1}-x^{s+1})^{\frac{\alpha}{k}} + (x^{s+1}-a^{s+1})^{\frac{\alpha}{k}}(f(b)+f(x))).$$

Corollary 18: If we take $\phi(t) = t^{\alpha}$ and $g(x) = \frac{x^{\beta+s}}{\beta+s}$, $\beta, s > 0, p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [13] as follows:

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^{s}_{\beta}I^{\alpha}_{a+}f)(x) = \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{\beta+s}-t^{\beta+s})^{\alpha-1}t^{s}f(t)dt$$

and

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = ({}^{s}_{\beta}I^{\alpha}_{b-}f)(x) = \frac{(\beta+s)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{\beta+s}-x^{\beta+s})^{\alpha-1}t^{s}f(t)dt.$$

Further they satisfy the following bound:

$$({}^{s}_{\beta}I^{\alpha}_{a+}f)(x) + ({}^{s}_{\beta}I^{\alpha}_{b-}f)(x) \leq \frac{1}{(\beta+s)^{\alpha}\Gamma(\alpha)}((x^{\beta+s}-a^{\beta+s})^{\alpha}(f(x)+f(a)) + (b^{\beta+s}-x^{\beta+s})^{\alpha}(f(b)+f(x))).$$

Corollary 19: If we take $g(x) = \frac{(x-a)^{\rho}}{\rho}$, $\rho > 0$ in (9) and $g(x) = \frac{-(b-x)^{\rho}}{\rho}$, $\rho > 0$ in (10) with $\phi(t) = t^{\alpha}, \alpha > 0$, $p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [12], as follows:

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x;0) = ({}^{\rho}I^{\alpha}_{a+}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x ((x-a)^{\rho}-(t-a)^{\rho})^{\alpha-1}(t-a)^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x;0) = ({}^{\rho}I^{\alpha}_{b-}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b ((b-x)^{\rho}-(b-t)^{\rho})^{\alpha-1}(b-t)^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^{\rho}I^{\alpha}_{a+}f)(x) + ({}^{\rho}I^{\alpha}_{b-}f)(x) \leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}((x-a)^{\rho\alpha}(f(x)+f(a)) + (b-x)^{\rho\alpha}(f(b)+f(x))).$$

Corollary 20: If we take $g(x) = \frac{(x-a)^{\rho}}{\rho}$, $\rho > 0$ in (9) and $g(x) = \frac{-(b-x)^{\rho}}{\rho}$, $\rho > 0$ in (10) with $\phi(t) = \frac{t^{\frac{\alpha}{k}}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$, $\alpha > k$,

$p = \omega = 0$. Then (9) and (10) produce the fractional integral operators defined in [11], as follows:

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x;0) = ({}^s_kI^\alpha_{a+}f)(x) = ({}^\rho_kI^\alpha_{a+}f)(x) = \frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x ((x-a)^\rho-(t-a)^\rho)^{\frac{\alpha}{k}-1}(t-a)^{\rho-1}f(t)dt$$

and

$$\left({}_gF^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x;0) = ({}^s_kI^\alpha_{a+}f)(x) = ({}^\rho_kI^\alpha_{b-}f)(x) = \frac{\rho^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^b ((b-x)^\rho-(b-t)^\rho)^{\frac{\alpha}{k}-1}(b-t)^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^\rho_kI^\alpha_{a+}f)(x) + ({}^\rho_kI^\alpha_{b-}f)(x) \leq \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}((x-a)^{\frac{\rho\alpha}{k}}(f(x)+f(a)) + (b-x)^{\frac{\rho\alpha}{k}}(f(b)+f(x))).$$

We will use the following lemma to get the next theorem.

Lemma 21 [9]: Let $f : [a,b] \to \mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a,b]. \tag{27}$$

The following theorem provides the Hadamard type estimation of integral operators (9) and (10).

Theorem 22: Along with statement of Theorem 8, if in addition f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b;p)\right) \\ &\leq \left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p)\right) \\ &\leq 2\phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p) \\ &\quad \times(f(a)+f(b)). \end{aligned} \tag{28}$$

Proof 23: For $x \in (a,b)$, under the assumption on g and $\frac{\phi}{x}$ the following inequality holds:

$$\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p) \leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p). \tag{29}$$

Using convexity of f on $[a,b]$ for $x \in (a,b)$ we have

$$f(x) \leq \frac{x-a}{b-a}f(b) + \frac{b-x}{b-a}f(a). \tag{30}$$

Multiplying (29) and (30) and then integrating with respect to x over $[a,b]$, the following inequality is obtained:

$$\begin{aligned} \int_a^b &\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)f(x) \\ &\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)dx \\ &\leq \frac{f(b)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p) \\ &\quad\times\int_a^b (x-a)g'(x)dx \\ &\quad+\frac{f(a)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p) \\ &\quad\times\int_a^b (b-x)g'(x)dx. \end{aligned}$$

By using (9) of Definition 4 and integrating by parts we get

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p) \times(\phi(g(b)-g(a)))(f(a)+f(b)). \tag{31}$$

On the other hand the following inequality holds:

$$\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^\mu;p) \leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p). \tag{32}$$

Multiplying (30) and (32) and then integrating with respect to x over $[a,b]$ and simplifying on the same pattern as we did for (29) and (30), following inequality is obtained:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p) \times (\phi(g(b)-g(a)))(f(a)+f(b)). \tag{33}$$

By adding (31) and (33), we have

$$\begin{aligned} &\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p)\right) \\ &\leq 2\phi(g(b)-g(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^\mu;p)(f(a)+f(b)). \end{aligned} \tag{34}$$

Multiplying both sides of (27) by $\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)$ $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)$, then integrating over $[a,b]$ we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x) \\ &\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)dx \\ &\leq \int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)f(x) \\ &\quad\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^\mu;p)f(x)dx. \end{aligned}$$

By using (10) of Definition 4 we get

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a;p) \leq \left({}_gF^{\phi\gamma\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p). \tag{35}$$

Multiplying both sides of (27) by $\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(x)$ $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)$ and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b;p)\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p) \quad (36)$$

by adding (35) and (36), the following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b;p)\right).$$
$$\leq\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b;p)\right). \quad (37)$$

Combining (34) and (37), inequality (28) can be achieved.

Remark 24: Theorem 22 can be utilized to obtain bounds of Hadamard type for fractional integral operators and conformable integrals like Corollaries $1-9$. We leave them for the readers.

Theorem 25: Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable function. If $|f'|$ is convex, $0<a<b$ and $g:[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function and $\alpha,l,\gamma,c\in\mathbb{C}$, $p,\mu,\delta\geq 0$ and $0<k\leq\delta+\mu$. Then for $x\in(a,b)$ we have

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f*g\right)(x;p)\right|$$
$$\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)\phi(g(x)-g(a))$$
$$(|f'(x)|+|f'(a)|)$$
$$+E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)\phi(g(b)-g(x))$$
$$(|f'(x)|+|f'(b)|). \quad (38)$$

where

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x;p):=\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t)f'(t)dt$$

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f*g\right)(x;p):=\int_x^b\frac{\phi(g(t)-g(x))}{g(t)-g(x)}\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t)f'(t)dt.$$

Proof 26: Using the convexity of $|f'|$ over $[a,b]$ for $t\in[a,x]$ we have

$$|f'(t)|\leq\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|. \quad (39)$$

From which we can write

$$-\left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right)\leq f'(t)$$
$$\leq\left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right) \quad (40)$$

we consider the right hand side inequality of the above inequality i.e.

$$f'(t)\leq\left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right). \quad (41)$$

Further the following inequality holds true:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)$$
$$\leq\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p). \quad (42)$$

Multiplying (41) and (42) and integrating with respect to t over $[a,x]$, the following inequality is obtained:

$$\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)f'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)dt$$
$$\leq\frac{|f'(a)|}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$$
$$\times\int_a^x(x-t)g'(t)dt$$
$$+\frac{|f'(x)|}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$$
$$\times\int_a^x(t-a)g'(t)dt$$

which gives

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x;p)\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$$
$$\times\phi(g(x)-g(a))(|f'(x)|+|f'(a)|). \quad (43)$$

If we consider the left hand side inequality from the inequality (40) and proceed as we did for the right hand side inequality we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x;p)\geq -E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$$
$$\times\phi(g(x)-g(a))(|f'(x)|+|f'(a)|). \quad (44)$$

Combining (43) and (44), the following inequality is obtained:

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x;p)\right|\leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$$
$$\times\phi(g(x)-g(a))(|f'(x)|+|f'(a)|). \quad (45)$$

On the other hand using convexity of $|f'(t)|$ over $[a,b]$ for $t\in(x,b]$ we have

$$|f'(t)|\leq\frac{t-x}{b-x}|f'(b)|+\frac{b-t}{b-x}|f'(x)|. \quad (46)$$

Further the following inequality holds true:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(x)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p) \leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p). \quad (47)$$

By adopting the same treatment as we did for (39) and (42), one can obtain the following inequality from (46) and (47):

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f*g\right)(x;p)\right| \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p) \times\phi(g(b)-g(x))(|f'(x)|+|f'(b)|). \quad (48)$$

Combining (45) and (48), inequality (38) can be achieved.

III. PROPOSED FRACTIONAL DIFFERENTIAL EQUATIONS

Theorem 27: Let $\mu,\alpha,l,\gamma,\nu,c \in \mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$ and $0<k\leq\delta+\Re(\mu)$. Let $g(x)=I(x), f(x)=x^2$ and $\phi(t)=t^{\alpha}$. Then the differential equation

$$(D^{\nu}_{0+}y)(x)=\lambda_1\left({}_IF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,0^+}x^2\right)(x;p)+x^2 \quad (49)$$

with initial condition $(I^{1-\nu}_{0+})=(0+)=C$, has its solution in the $L(0,\infty)$

$$y(x)=C\frac{x^{\nu-1}}{\Gamma(\nu)}+2\lambda_1\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\times\frac{(c)_{nk}}{\Gamma(\mu n+\alpha+\nu+3)}\frac{\omega^n}{(l)_{n\delta}}\times x^{\nu n+\alpha+\nu+2}+2\frac{x^{\nu+2}}{\Gamma(\nu+3)} \quad (50)$$

where C is an arbitrary constant.

Proof 28: For the function $f(x)=x^2$ the generalized fraction integral operator is calculated in [20, Thorem 3.1] as follows:

$$\left({}_IF^{t^{\alpha}\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}x^2\right)(x;p) = (x-a)^{\alpha}\times\Big[a^2E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega(x-a)^{\mu};p) +2a(x-a)E^{\gamma,\delta,k,c}_{\mu,\alpha+2,l}(\omega(x-a)^{\mu};p) +2(x-a)^2\times E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x-a)^{\mu};p)\Big]. \quad (51)$$

Now putting $a=0$ the above equation reduces to

$$\left({}_IF^{t^{\alpha}\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,0^+}x^2\right)(x;p)=2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu};p)). \quad (52)$$

Using (52) in (49) we get

$$(D^{\nu}_{0+}y)(x)=\lambda_1 2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu};p))+x^2. \quad (53)$$

Applying Laplace transform on both sides of (53) we have

$$L[(D^{\nu}_{0+}y)(x);s]=L[\lambda_1 2x^{2+\alpha}(E^{\gamma,\delta,k,c}_{\mu,\alpha+3,l}(\omega(x)^{\mu};p));s] +L[x^2;s]. \quad (54)$$

Laplace transform of Mittag-Leffler function is obtained as follows:

$$L[x^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x)^{\mu};p)] = \int_0^{\infty}x^{\alpha}e^{-sx}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x)^{\mu};p)dx = \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{\omega^n}{(l)_{n\delta}}\times\int_0^{\infty}x^{\alpha+\mu n-1}e^{-sx}dx = \frac{1}{s^{\alpha}}\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{(l)_{n\delta}}\omega^n. \quad (55)$$

And Laplace transform of fractional derivative $D^{\nu}_{0+}f$ is calculated as follows:

$$L[D^{\nu}_{0+}f;s]=s^{\nu}F(s) -\sum_{k=1}^{n}D^{\nu-k}_{0+}f(0+)\ \ (n-1<\nu<n)Re(s)>0. \quad (56)$$

Using (55) and (56) (for $n=1$) in (54) we have

$$y(s)=Cs^{-\nu}+2\lambda_1 s^{-(\mu n+\alpha+\nu+3)}\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\times(c)_{nk}\frac{\omega^n}{(l)_{n\delta}}+2s^{-(\nu+3)}. \quad (57)$$

Now taking the inverse Laplace transformation on both side of (57) and after some simplifications, we achieved the required result (50).

Theorem 29: Let $\mu,\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$ and $0<k\leq\delta+\Re(\mu)$. Let $g(x)=I(x)$, and $\phi(t)=t^{\alpha}$. Then the differential equation

$$(D^{\alpha}_{0+}y)(x)=\lambda_1\left({}_IF^{t^{\alpha},\gamma,\delta,k,c}_{\mu,\alpha,l,0^+}1\right)(x;p)+\lambda_2x^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega x^{\mu};p) \quad (58)$$

with initial condition $(I^{1-\alpha}_{0+})=(0+)=C$, has its solution in the $L(0,\infty)$

$$y(x)=C\frac{x^{\alpha-1}}{\Gamma(\alpha)}+(\lambda_1+\lambda_2)\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\times\frac{(c)_{nk}}{\Gamma(\mu n+2\alpha+1)}\frac{\omega^n}{(l)_{n\delta}}x^{\nu n+2\alpha} \quad (59)$$

where C is an arbitrary constant.

Proof 30: By convenient settings of values of function $g(x) = I(x)$ and $\phi(t) = t^{\alpha}$ in (6), we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^{+}}1\right)(x;p) = \left({}_{I}F^{t^{\alpha}\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^{+}}1\right)(x;p). \qquad (60)$$

By putting $a = 0$ in (60) one can obtained:

$$\begin{aligned}(D^{\alpha}_{0^{+}}y)(x) &= \lambda_1 x^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega x^{\mu};p) + x^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega x^{\mu};p)\\ &= (\lambda_1+\lambda_2)x^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(\omega x^{\mu};p). \qquad (61)\end{aligned}$$

Applying Laplace transform on both sides of (61) and after simplification one can obtained:

$$\begin{aligned}y(s) &= Cs^{-\alpha} + (\lambda_1+\lambda_2)s^{-(\mu n+2\alpha+1)}\\ &\quad\times\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}(c)_{nk}\frac{\omega^n}{(l)_{n\delta}}. \qquad (62)\end{aligned}$$

Applying inverse Laplace transform and after simplification the required result (59) can be achieved.

IV. CONCLUDING REMARKS

The findings of this research provide compact presentation of bounds for fractional integral operators and conformable integrals simultaneously. These bounds can be achieved from the bounds of unified integral operators (9) and (10) which have been established by utilizing convex functions, functions whose derivatives in absolute value are convex, symmetric convex functions, and by applying the conditions involved in definitions of unified operators.

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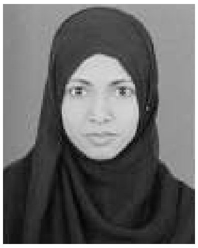
YOUNG CHEL KWUN received the Ph.D. degree in mathematics from Dong-A University, Busan, South Korea, where he is currently a Professor. He is currently a Mathematician from South Korea. He has published more than 100 research articles in different international journals. His research interests include nonlinear analysis, decision theory, and system theory and control.

GHULAM FARID received the Ph.D. degree in mathematics from ASSMS GC University, Lahore, Pakistan, in March 2012. He is currently an Assistant Professor with the Department of Mathematics, COMSATS University Islamabad, Attock Campus. He was a recipient of the Research Productivity Award each year by the university. He received two research grants from COMSATS University Islamabad and one from the Higher Education Commission of Pakistan.

SALEEM ULLAH received the Ph.D. degree in mathematics from COMSATS University Islamabad, in 2013, under the supervision of Dr. M. Aslam Noor in numerical optimization and variational inequalities. He is currently with the Department of Mathematics, Air University, Islamabad, as an Assistant Professor and also teaching M.S. level course and actively involved in research. During his Ph.D., he introduced some new self-adaptive methods for solving the general variational inequalities and related optimization problems. It has been shown that a several unrelated problems, which arise in pure and applied sciences, can be studied in the framework of general variational inequalities. It is known that the general variational inequalities are equivalent to the fixed point problem and the Wiener-Hopf equations. This equivalent formulation is used to study the existence of solution and to develop various iterative methods for solving the general variational inequalities.

WAQAS NAZEER received the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan. He was a recipient of the Outstanding Performance Award in Ph.D. During his studies, he was funded by the Higher Education Commission, Pakistan. He is currently a Mathematician from Pakistan and also an Assistant Professor with the University of Education, Lahore. He has published more than 100 research articles in different international journals. His research interest includes analysis and graph theory.



KAHKASHAN MAHREEN is currently pursuing the M.S. degree with the Department of Mathematics, COMSATS University Islamabad, Attock Campus. She has completed her course work successively. She has also entered in research and working on bounds of integral operators.

SHIN MIN KANG received the Ph.D. degree in mathematics from Dong-A University, Busan, South Korea He is currently a Professor with Gyeongsang National University, South Korea and a Mathematician from South Korea. He has published more than 200 research articles in different international journals. His research interests include fixed point theory, nonlinear analysis, and variational inequality.