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*Research Article*

# Inequalities for a Unified Integral Operator via $(\alpha, m)$-Convex Functions

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Recently, a unified integral operator has been introduced by Farid, 2020, which produces several kinds of known fractional and conformable integral operators defined in recent decades (Kwun, 2019, Remarks 6 and 7). The aim of this paper is to establish bounds of this unified integral operator by means of $(\alpha, m)$-convex functions. The resulting inequalities provide the bounds of all associated fractional and conformable integral operators in a compact form. Also, the results of this paper hold for different kinds of convex functions connected with $(\alpha, m)$-convex functions.

## 1. Introduction

To prove the mathematical inequalities, fractional integral operators play an important role in the field of different branches of mathematics and engineering. Many mathematicians have used fractional integrals and conformable fractional integrals to develop integral inequalities [1–14]. We start from definitions of fractional integral operators which are direct consequences of unified integral operators given in (8) and (9).

*Definition 1* (see [15]). Let $\eta_1\colon [a, b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let $\eta_2$ be an increasing and positive function on $(a, b]$, having a continuous derivative $\eta_2'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $\eta_1$ with respect to another function $\eta_2$ on $[a, b]$ of order $\mu$, where $\Re(\mu) > 0$, are defined by

$$
\begin{aligned}
{}^{\mu}_{\eta_2}I_{a^+}\eta_1(x) &= \frac{1}{\Gamma(\mu)}\int_a^x \left(\eta_2(x)-\eta_2(t)\right)^{\mu-1}\eta_2'(t)\eta_1(t)\mathrm{d}t, \quad x>a,\\
{}^{\mu}_{\eta_2}I_{b^-}\eta_1(x) &= \frac{1}{\Gamma(\mu)}\int_x^b \left(\eta_2(t)-\eta_2(x)\right)^{\mu-1}\eta_2'(t)\eta_1(t)\mathrm{d}t, \quad x<b,
\end{aligned}
\tag{1}
$$

where $\Gamma(\cdot)$ is the gamma function.

A $k$-analogue of the above definition is defined as follows.

*Definition 2* (see [16]). Let $\eta_1\colon [a, b] \longrightarrow \mathbb{R}$ be an integrable function. Also, let $\eta_2$ be an increasing and positive function on $(a, b]$, having a continuous derivative $\eta_2'$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $\eta_1$ with respect to another function $\eta_2$ on $[a, b]$ of order $\mu$; $\Re(\mu)$, $k > 0$ are defined by

$$
{}^{\mu}_{\eta_2}I^k_{a^+}\eta_1(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x \left(\eta_2(x)-\eta_2(t)\right)^{(\mu/k)-1}\eta_2'(t)\eta_1(t)\mathrm{d}t, \quad x>a, \tag{2}
$$

$$
{}^{\mu}_{\eta_2}I^k_{b^-}\eta_1(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b \left(\eta_2(t)-\eta_2(x)\right)^{(\mu/k)-1}\eta_2'(t)\eta_1(t)\mathrm{d}t, \quad x<b, \tag{3}
$$

where $\Gamma_k(\cdot)$ is defined by [17]

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-(t^k/k)} \mathrm{d}t, \quad \Re(x) > 0. \tag{4}$$

A well-known function named Mittag-Leffler function is defined by [18]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{5}$$

where $\alpha, z \in \mathbb{C}$ and $\Re(\alpha) > 0$.

One can see [19–22] to study the Mittag-Leffler function and its generalizations. Also, (2) and (3) produce many types of fractional integral operators (see [10], Remark 6).

A generalized fractional integral operator containing an extended generalized Mittag-Leffler function is defined as follows.

*Definition 3* (see [1]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $\eta_1 \in L_1[a,b]$ and $x \in [a,b]$. Then, the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\eta_1$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\eta_1$ are defined by

$$\begin{aligned}
\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}\eta_1\right)(x;p) &= \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(x-t)^\mu; p\right)\eta_1(t)\mathrm{d}t,\\
\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}\eta_1\right)(x;p) &= \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\left(\omega(t-x)^\mu; p\right)\eta_1(t)\mathrm{d}t,
\end{aligned} \tag{6}$$

where

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{7}$$

is the extended generalized Mittag-Leffler function.

Recently, Farid defined a unified integral operator which unifies several kinds of fractional and conformable integrals in a compact formula which is defined as follows.

*Definition 4* (see [23]). Let $\eta_1, \eta_2\colon [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$, be the functions such that $\eta_1$ be positive and $\eta_1 \in L_1[a,b]$ and $\eta_2$ be differentiable and strictly increasing. Also, let $\phi/x$ be an increasing function on $[a,\infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$. Then, for $x \in [a,b]$, the left and right integral operators are defined by

$$\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\eta_1\right)(x,\omega;p) = \int_a^x K_x^y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, \eta_2; \phi\right)\eta_1(y)\mathrm{d}(\eta_2(y)), \tag{8}$$

$$\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\eta_1\right)(x,\omega;p) = \int_x^b K_y^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, \eta_2; \phi\right)\eta_1(y)d(\eta_2(y)), \tag{9}$$

where $K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, \eta_2; \phi) = (\phi(\eta_2(x) - \eta_2(y))/\eta_2(x) - \eta_2(y))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\eta_2(x) - \eta_2(y))^\mu; p)$.

For suitable settings of function $\phi$, $\eta_2$, and certain values of parameters included in Mittag-Leffler function (7), very interesting consequences are obtained which are comprised in Remarks 6 and 7 of [10].

The objective of this paper is to obtain bounds of unified integral operators explicitly which are directly linked with various fractional and conformable integrals. The $(\alpha, m)$-convexity has been used for establishing these bounds. The notion of $(\alpha, m)$-convexity is defined by Mihesan in [24].

*Definition 5.* A function $\eta_1\colon [0,b] \longrightarrow \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)$-convex, where $(\alpha, m) \in [0,1]^2$, if

$$\eta_1(tx + m(1-t)y) \leq t^\alpha \eta_1(x) + m(1-t^\alpha)\eta_1(y), \tag{10}$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$.

*Remark 1*

(i) If we put $(\alpha, m) = (1, m)$, then (10) gives the definition of $m$-convex function

(ii) If we put $(\alpha, m) = (1, 1)$, then (10) gives the definition of convex function

(iii) If we put $(\alpha, m) = (1, 0)$, then (10) gives the definition of star-shaped function

For some recent citations and utilizations of $(\alpha, m)$-convex functions, one can see [9, 25–28] and references therein. In the upcoming section, bounds of unified integral operators are established by using $(\alpha, m)$-convexity. These bounds provide general formulas to obtain bounds of fractional and conformable integral operators described in Remarks 6 and 7 of [10]. Among the well-known inequalities which are related to the integral mean of a convex function, the Hadamard inequality is of great importance. Many mathematicians worked on new types of Hadamard inequalities using convex functions, see [8, 29–31]. We also established the general Hadamard-type inequality by applying Lemma 1 which further produces various inequalities of Hadamard type for fractional and conformable integrals. At the end, by using $(\alpha, m)$-convexity of $|\eta_1'|$, a modulus inequality is obtained.

## 2. Main Results

Bounds of unified integral operators (8) and (9) using $(\alpha, m)$-convexity are studied in the following result:

**Theorem 1.** *Let $\eta_1\colon [a,b] \longrightarrow \mathbb{R}$ be a positive integrable $(\alpha, m)$-convex function with $m \in (0,1]$. Let $\eta_2\colon [a,b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, and also, let $\phi/x$ be an increasing function on $[a,b]$. If $\alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$, and $0 < k \leq \delta + \mu$, then for $x \in (a,b)$, we have*

$$\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\eta_1\right)(x,\omega;p) \leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\left(\left(m\eta_1\left(\frac{x}{m}\right)\eta_2(x)-\eta_1(a)\eta_2(a)\right)-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\eta_1\left(\frac{x}{m}\right)-\eta_1(a)\right)^\alpha I_{a^+}\eta_2(x)\right),$$

$$\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\eta_1\right)(x,\omega;p) \leq K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\left(\left(\eta_1(b)\eta_2(b)-m\eta_1\left(\frac{x}{m}\right)\eta_2(x)\right)\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(\eta_1(b)-m\eta_1\left(\frac{x}{m}\right)\right)^\alpha I_{b^-}\eta_2(x)\right), \tag{11}$$

*and hence,*

$$\begin{aligned}&\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\eta_1\right)(x,\omega;p)+\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}\eta_1\right)(x,\omega;p)\\ &\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\left(\left(m\eta_1\left(\frac{x}{m}\right)\eta_2(x)-\eta_1(a)\eta_2(a)\right)\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\eta_1\left(\frac{x}{m}\right)-\eta_1(a)\right)^\alpha I_{a^+}\eta_2(x)\right)\\ &+K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\left(\left(\eta_1(b)\eta_2(b)-m\eta_1\left(\frac{x}{m}\right)\eta_2(x)\right)\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(\eta_1(b)-m\eta_1\left(\frac{x}{m}\right)\right)^\alpha I_{b^-}\eta_2(x)\right).\end{aligned} \tag{12}$$

*Proof.* Under the assumptions of $\phi$ and $\eta_2$, one can write the following inequality:

$$\frac{\phi(\eta_2(x)-\eta_2(t))}{\eta_2(x)-\eta_2(t)} \leq \frac{\phi(\eta_2(x)-\eta_2(a))}{\eta_2(x)-\eta_2(a)}; \quad t\in[a,x], x\in(a,b). \tag{13}$$

Multiplying with $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\eta_2(x)-\eta_2(t))^\mu;p)\eta_2'(t)$, we can obtain

$$\begin{aligned}&K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\eta_2'(t)\\ &\leq \frac{\phi(\eta_2(x)-\eta_2(a))}{\eta_2(x)-\eta_2(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\eta_2(x)-\eta_2(t))^\mu;p)\eta_2'(t).\end{aligned} \tag{14}$$

By using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(\eta_2(x)-\eta_2(t))^\mu;p) \leq E_{\mu,}\ \alpha, l^{\gamma,\delta,k,c}(\omega(\eta_2(x)-\eta_2(a))^\mu;p)$, the following inequality is obtained:

$$K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\eta_2'(t) \leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\eta_2'(t). \tag{15}$$

Using the definition of $(\alpha,m)$-convexity for $\eta_1$, the following inequality is valid:

$$\eta_1(t) \leq \left(\frac{x-t}{x-a}\right)^\alpha \eta_1(a)+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\eta_1\left(\frac{x}{m}\right). \tag{16}$$

Multiplying (15) with (16) and integrating over $[a,x]$, one can obtain

$$\begin{aligned}\int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\eta_1(t)d(\eta_2(t)) &\leq \eta_1(a)K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\int_a^x\left(\frac{x-t}{x-a}\right)^\alpha d(\eta_2(t))\\ &+m\eta_1\left(\frac{x}{m}\right)K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)d(\eta_2(t)).\end{aligned} \tag{17}$$

By using (8) of Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\eta_1\right)(x,\omega;p) \leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)&\left((x-a)^\alpha\left(m\eta_1\left(\frac{x}{m}\right)\eta_2(x)-\eta_1(a)\eta_2(a)\right)\right.\\ &\left.\cdot\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\eta_1\left(\frac{x}{m}\right)-\eta_1(a)\right)^\alpha I_{a^+}\eta_2(x)\right).\end{aligned} \tag{18}$$

Now, on the other side, for $t \in (x, b]$ and $x \in (a, b)$, the following inequality holds true:

$$K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\eta_2'(t) \leq K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\eta_2'(t). \tag{19}$$

Using $(\alpha, m)$-convexity of $\eta_1$, we have

$$\eta_1(t) \leq \left(\frac{t-x}{b-x}\right)^{\alpha}\eta_1(b) + m\left(1 - \left(\frac{t-x}{b-x}\right)^{\alpha}\right)\eta_1\left(\frac{x}{m}\right). \tag{20}$$

Adopting the same procedure as we did for (15) and (16), the following inequality from (19) and (20) can be obtained:

$$\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(x,\omega;p) \leq K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\left(\left(\eta_1(b)\eta_2(b) - m\eta_1\left(\frac{x}{m}\right)\eta_2(x)\right) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left(\eta_1(b) - m\eta_1\left(\frac{x}{m}\right)\right)^{\alpha} I_{b^-}\eta_2(x)\right). \tag{21}$$

By adding (18) and (21), (12) can be obtained. □

*Remark 2*

(i) If we consider $(\alpha, m) = (1,1)$ in (12), Theorem 8 in [10] is obtained

(ii) If we consider $\phi(t) = (\Gamma(\mu)t^{\mu/k}/k\Gamma_k(\mu))$ for the left-hand integral and $\phi(t) = (\Gamma(\nu)t^{\nu/k}/k\Gamma_k(\nu))$ for the right-hand integral and $p = \omega = 0$ in (12), then Theorem 1 in [9] can be obtained

(iii) If we consider $\mu = \nu$ in the result of (ii), then Corollary 1 in [9] can be obtained

(iv) If we consider $\phi(t) = \Gamma(\mu)t^{\mu}$, $p = \omega = 0$, and $(\alpha, m) = (1,1)$ in (12), Theorem 1 in [6] is obtained

(v) If we consider $\mu = \nu$ in the result of (iv), Corollary 1 in [6] is obtained

(vi) If we consider $\phi(t) = (\Gamma(\mu)t^{\mu/k}/k\Gamma_k(\mu))$ for the left-hand integral and $\phi(t) = (\Gamma(\nu)t^{\nu/k}/k\Gamma_k(\nu))$for the right-hand integral, $(\alpha, m)$ =(1,1), $\eta_2(x) = x$, and $p = \omega = 0$, then Theorem 1 in [4] can be obtained

(vii) If we consider $\mu = \nu$ in the result of (vi), then Corollary 1 in [4] can be obtained

(viii) If we consider $\phi(t) = \Gamma(\mu)t^{\mu}$ for the left-hand integral and $\phi(t) = \Gamma(\nu)t^{\nu}$ for the right-hand integral, $\eta_2(x) = x$, and $p = \omega = 0$ and $(\alpha, m)$ =(1,1) in (12), then Theorem 1 in [5] is obtained

(ix) By setting $\mu = \nu$ in the result of (viii), Corollary 1 in [5] can be obtained

(x) By setting $\mu = \nu = 1$ and $x = a$ or $x = b$ in the result of (ix), Corollary 2 in [5] can be obtained

(xi) By setting $\mu = \nu = 1$ and $x = (a + b/2)$ in the result of (ix), Corollary 3 in [5] can be obtained

To prove the next result, we need the following lemma [9].

**Lemma 1.** *Let $\eta_1$: $[0, \infty] \longrightarrow \mathbb{R}$ be an $(\alpha, m)$-convex function with $m \in (0, 1]$. If $\eta_1(x) = \eta_1(a + b - x/m)$, $0 < a < b$, then the following inequality holds:*

$$\eta_1\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha}}\left(1 + m\left(2^{\alpha} - 1\right)\right)\eta_1(x), \tag{22}$$

*for all $x \in [a, b]$ and $m \in (0, 1]$.*

The following result provides upper and lower bounds of the sum of operators (8) and (9) in the form of a Hadamard inequality.

**Theorem 2.** *With the assumptions of Theorem 1 in addition, if $\eta_1(x) = \eta_1(a + b - x/m)$, then we have*

$$\begin{aligned}
&\frac{2^{\alpha}\eta_1(a+b/2)}{(1+m(2^{\alpha}-1))}\left(\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a,\omega;p) + \left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b,\omega;p)\right) \\
&\leq \left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(a,\omega;p) + \left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(b,\omega;p) \\
&\leq 2K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\left(\left(\eta_1(b)\eta_2(b) - m\eta_1\left(\frac{a}{m}\right)\eta_2(a)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\eta_1(b) - m\eta_1\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-}\eta_2(a)\right).
\end{aligned} \tag{23}$$

*Proof.* Under the assumptions of $\phi$ and $\eta_2$, we have

$$\frac{\phi(\eta_2(x) - \eta_2(a))}{\eta_2(x) - \eta_2(a)} \leq \frac{\phi(\eta_2(b) - \eta_2(a))}{\eta_2(b) - \eta_2(a)}. \tag{24}$$

Multiplying with $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\eta_2(x) - \eta_2(a))^{\mu}; p)\eta_2'(x)$, we can obtain from (24) the following inequality:

$$\begin{aligned}
&K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\eta_2'(x) \\
&\leq \frac{\phi(\eta_2(b) - \eta_2(a))}{\eta_2(b) - \eta_2(a)} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\eta_2(x) - \eta_2(a))^{\mu}; p)\eta_2'(x).
\end{aligned} \tag{25}$$

By using $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(\eta_2(x) - \eta_2(a))^{\mu}; p) \leq E_{\mu,\alpha,\ l}{}^{\gamma,\delta,k,c}(\omega(\eta_2(b) - \eta_2(a))^{\mu}; p)$, the following inequality is obtained:

$$K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\eta_2'(x) \leq K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \eta_2; \phi\right)\eta_2'(x). \tag{26}$$

Using $(\alpha, m)$-convexity of $\eta_1$ for $x \in (a, b)$, we have

$$\eta_1(x) \le \left(\frac{x-a}{b-a}\right)^{\alpha}\eta_1(b) + m\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)\eta_1\left(\frac{a}{m}\right). \quad (27)$$

Multiplying (26) and (27) and integrating the resulting inequality over $[a,b]$, one can obtain

$$\begin{aligned}&\int_a^b K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_1(x)d(\eta_2(x))\\ &\le m\eta_1\left(\frac{a}{m}\right)K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\int_a^b\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)d(\eta_2(x))\\ &+\eta_1(b)\frac{\phi(\eta_2(b)-\eta_2(a))}{\eta_2(b)-\eta_2(a)}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega(\eta_2(b)-\eta_2(a))^{\mu};p\right)\int_a^b\left(\frac{x-a}{b-a}\right)^{\alpha}d(\eta_2(x)).\end{aligned} \quad (28)$$

By using Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(b,\omega;p) &\le K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\Bigg(\left(\eta_1(b)\eta_2(b)-m\eta_1\left(\frac{a}{m}\right)\eta_2(a)\right)\\ &-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\eta_1(b)-m\eta_1\left(\frac{a}{m}\right)\right)^{\alpha}I_{b^-}\eta_2(a)\Bigg).\end{aligned} \quad (29)$$

On the other hand, for $x \in (a,b)$, the following inequality holds true:

$$K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_2'(x) \le K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_2'(x). \quad (30)$$

Adopting the same pattern of simplification as we did for (26) and (27), the following inequality can be observed from (27) and (30):

$$\begin{aligned}\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(a;p) &\le K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\Bigg(\left(\eta_1(b)\eta_2(b)-m\eta_1\left(\frac{a}{m}\right)\eta_2(a)\right)\\ &-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\eta_1(b)-m\eta_1\left(\frac{a}{m}\right)\right)^{\alpha}I_{b^-}\eta_2(a)\Bigg).\end{aligned} \quad (31)$$

By adding (29) and (31), the following inequality can be obtained:

$$\begin{aligned}&\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(a,\omega;p)+\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(b,\omega;p)\\ &\le K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\Bigg(\left(\eta_1(b)\eta_2(b)-m\eta_1\left(\frac{a}{m}\right)\eta_2(a)\right)\\ &-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\eta_1(b)-m\eta_1\left(\frac{a}{m}\right)\right)^{\alpha}I_{b^-}\eta_2(a)\Bigg).\end{aligned} \quad (32)$$

Multiplying both sides of (22) by $K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi)d(\eta_2(x))$ and integrating over $[a,b]$, we have

$$\begin{aligned}&\eta_1\left(\frac{a+b}{2}\right)\int_a^b K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)d(\eta_2(x))\\ &\le\left(\frac{1}{2^{\alpha}}\right)\left(1+tmn\left(2^{\alpha}-1\right)\right)\\ &\times\int_a^b K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_1(x)d(\eta_2(x)).\end{aligned} \quad (33)$$

From Definition 4, the following inequality is obtained:

$$\begin{aligned}&\eta_1\left(\frac{a+b}{2}\right)\frac{2^{\alpha}}{\left(1+m\left(2^{\alpha}-1\right)\right)}\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a;p)\\ &\le\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(a;p).\end{aligned} \quad (34)$$

Similarly, multiplying both sides of (22) by $K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi)d(\eta_2(x))$ and integrating over $[a,b]$, we have

$$\eta_1\left(\frac{a+b}{2}\right)\frac{2^\alpha}{(1+m(2^\alpha-1))}\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b;p) \leq \left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(b;p). \tag{35}$$

By adding (34) and (35), the following inequality is obtained:

$$\begin{aligned}&\eta_1\left(\frac{a+b}{2}\right)\frac{2^\alpha}{(1+m(2^\alpha-1))}\\&\cdot\left(\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a,\omega;p)+\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b,\omega;p)\right)\\&\leq\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(a,\omega;p)+\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(b,\omega;p).\end{aligned} \tag{36}$$

Using (32) and (36), inequality (23) can be achieved. □

*Remark 3*

(i) If we consider $(\alpha,m)=(1,1)$ in (23), Theorem 22 in [10] is obtained

(ii) If we consider $\phi(t)=\Gamma(\mu)t^{(\mu/k)+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{(\nu/k)+1}$ and $p=\omega=0$ in (23), then Theorem 3 in [9] can be obtained

(iii) If we consider $\mu=\nu$ in the result of (ii), then Corollary 3 in [9] can be obtained

(iv) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$for the right-hand integral in (23), $p=\omega=0$, and $(\alpha,m)=(1,1)$ in (23), Theorem 3 in [6] is obtained

(v) If we consider $\mu=\nu$ in the result of (iv), Corollary 3 in [6] is obtained

(vi) If we consider $\phi(t)=\Gamma(\mu)t^{(\mu/k)+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{(\nu/k)+1}$for the right-hand integral, $(\alpha,m)=(1,1)$, $\eta_2(x)=x$, and $p=\omega=0$ in (23), then Theorem 3 in [4] can be obtained

(vii) If we consider $\mu=\nu$ in the result of (vi), then Corollary 6 in [4] can be obtained

(viii) By setting $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for the right-hand integral, $p=\omega=0$, $(\alpha,m)=1$, and $g(t)=t$ in (23), Theorem 3 in [5] can be obtained

(ix) By setting $\mu=\nu$ in the result of (viii), Corollary 6 in [5] can be obtained

**Theorem 3.** *Let $\eta_1\colon [a,b]\longrightarrow\mathbb{R}$ be a differentiable function. $|\eta_1'|$ is $(\alpha,m)$-convex with $m\in(0,1]$, and let $\eta_2\colon[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function; also, let $\phi/x$ be an increasing function on $[a,b]$. If $\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$, $p,\mu,\delta\geq 0$, and $0<k\leq\delta+\mu$, then for $x\in(a,b)$, we have*

$$\begin{aligned}&\left|\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1\right)(x,\omega;p)+\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1\right)(x,\omega;p)\right|\\&\leq K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\left(\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)-\left|\eta_1'(a)\right|\eta_2(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|-\left|\eta_1'(a)\right|\right)^\alpha I_{a^+}\eta_2(x)\right)\\&+K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\left(\left(\left|\eta_1'(b)\right|\eta_2(b)-m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(\left|\eta_1'(b)\right|-m\left|\eta_1'\left(\frac{x}{m}\right)\right|\right)^\alpha I_{b^-}\eta_2(x)\right),\end{aligned} \tag{37}$$

*where*

$$\begin{aligned}\left({}_{\eta_2}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\eta_1*\eta_2\right)(x,\omega;p)&\coloneqq\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_1'(t)d(\eta_2(t)),\\\left({}_{\eta_2}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\eta_1*\eta_2\right)(x,\omega;p)&\coloneqq\int_x^b K_t^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\eta_1'(t)d(\eta_2(t)).\end{aligned} \tag{38}$$

*Proof.* Let $x\in(a,b)$ and $t\in[a,x]$. Then, using $(\alpha,m)$-convexity of $|\eta_1'|$, we have

$$\left|\eta_1'(t)\right|\leq\left(\frac{x-t}{x-a}\right)^\alpha\left|\eta_1'(a)\right|+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\eta_1'\left(\frac{x}{m}\right)\right|. \tag{39}$$

Inequality (39) can be written as follows:

$$\begin{aligned}&-\left(\left(\frac{x-t}{x-a}\right)^\alpha\left|\eta_1'(a)\right|+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\eta_1'\left(\frac{x}{m}\right)\right|\right)\\&\leq\eta_1'(t)\leq\left(\frac{x-t}{x-a}\right)^\alpha\left|\eta_1'(a)\right|+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\eta_1'\left(\frac{x}{m}\right)\right|.\end{aligned} \tag{40}$$

Let us consider the second inequality of (40):

$$\eta_1'(t)\leq\left(\frac{x-t}{x-a}\right)^\alpha\left|\eta_1'(a)\right|+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\eta_1'\left(\frac{x}{m}\right)\right|. \tag{41}$$

Multiplying (15) and (41) and integrating over $[a,x]$, we can obtain

$$\begin{aligned}&\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)d(\eta_2(t))\\&\leq\left|\eta_1(a)\right|K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\int_a^x\left(\frac{x-t}{x-a}\right)^\alpha d(\eta_2(t))\\&+m\left|\eta_1\left(\frac{x}{m}\right)\right|K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},\eta_2;\phi\right)\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)d(\eta_2(t)).\end{aligned} \tag{42}$$

By using (8) of Definition 4 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}&\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}\eta_1\right)(x,\omega;p)\\&\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\\&\times\left(\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)-\left|\eta_1'(a)\right|\eta_2(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|-\left|\eta_1'(a)\right|\right)^\alpha I_{a^+}\eta_2(x)\right).\end{aligned}\tag{43}$$

If we consider the left-hand side from inequality (40) and adopt the same pattern as we did for the right-hand side inequality, then

$$\begin{aligned}&\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(\eta_1*\eta_2)\right)(x,\omega;p)\\&\geq -K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\\&\times\left(\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)-\left|\eta_1'(a)\right|\eta_2(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|-\left|\eta_1'(a)\right|\right)^\alpha I_{a^+}\eta_2(x)\right).\end{aligned}\tag{44}$$

From (43) and (44), the following inequality is observed:

$$\begin{aligned}&\left|\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(\eta_1*\eta_2)\right)(x,\omega;p)\right|\\&\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\times\left(\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)-\left|\eta_1'(a)\right|\eta_2(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|\eta_1'\left(\frac{x}{m}\right)\right|-\left|\eta_1'(a)\right|\right)^\alpha I_{a^+}\eta_2(x)\right).\end{aligned}\tag{45}$$

Now, using $(\alpha,m)$-convexity of $|\eta_1'|$ on $(x,b]$ for $x\in(a,b)$, we have

$$\left|\eta_1'(t)\right|\leq\left(\frac{t-x}{b-x}\right)^\alpha\left|\eta_1'(b)\right|+m\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\left|\eta_1'\left(\frac{x}{m}\right)\right|.\tag{46}$$

On the same procedure as we did for (15) and (39), one can obtain the following inequality from (19) and (46):

$$\begin{aligned}&\left|\left({}_{\eta_2}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(\eta_1*\eta_2)\right)(x,\omega;p)\right|\\&\leq K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\eta_2;\phi\right)\\&\times\left(\left(\left|\eta_1'(b)\right|\eta_2(b)-m\left|\eta_1'\left(\frac{x}{m}\right)\right|\eta_2(x)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(\left|\eta_1'(b)\right|-m\left|\eta_1'\left(\frac{x}{m}\right)\right|\right)^\alpha I_{b^-}\eta_2(x)\right).\end{aligned}\tag{47}$$

By adding (45) and (47), inequality (37) can be achieved. □

*Remark 4*

(i) If we consider $(\alpha,m)=(1,1)$ in (37), then Theorem 25 in [10] is obtained

(ii) If we consider $\phi(t)=\Gamma(\mu)t^{(\mu/k)+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{(\nu/k)+1}$ for the right-hand integral and $p=\omega=0$ in (37), then Theorem 2 in [9] can be obtained

(iii) If we consider $\mu=\nu$ in the result of (ii), then Corollary 2 in [9] can be obtained

(iv) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for the right-hand integral, $p=\omega=0$, and $(\alpha,m)=(1,1)$ in (37), then Theorem 2 in [6] is obtained

(v) If we consider $\mu=\nu$ in the result of (iv), then Corollary 2 in [6] is obtained

(vi) If we consider $\phi(t)=\Gamma(\mu)t^{(\mu/k)+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{(\nu/k)+1}$ for the right-hand integral, $(\alpha,m)=(1,1)$, $\eta_2(x)=x$, and $p=\omega=0$ in (37), then Theorem 2 in [4] can be obtained

(vii) If we consider $\mu=\nu$ in the result of (vi), then Corollary 4 in [4] can be obtained

(viii) If we consider $\mu=\nu=k=1$ and $x=(a+b/2)$ in the result of (vii), then Corollary 5 in [4] can be obtained

(ix) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for the left-hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for the right-hand integral, $\eta_2(x)=x$, $p=\omega=0$, and $(\alpha,m)=(1,1)$ in (37), then Theorem 2 in [5] is obtained

(x) By setting $\mu=\nu$ in the result of (ix), then Corollary 5 in [5] can be obtained

## 3. Concluding Remarks

This research paper explores fractional and conformable fractional integral inequalities in a unified form, which provide the bounds of conformable fractional integral operators and fractional integral operators containing Mittag-Leffler functions in their kernels. The results of this paper hold for fractional and conformable integral operators and convex, $m$-convex, and star-shaped functions (see Remarks 6 and 7 of [10] and Remark 1) simultaneously.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors have declared that no conflicts of interest exist.

## Authors' Contributions

All authors have equal contribution to the formation of this manuscript.

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## References

[1] M. Andrić, G. Farid, and J. Pečarić, "A further extension of Mittag-Leffler function," *Fractional Calculus and Applied Analysis*, vol. 21, no. 5, pp. 1377–1395, 2018.

[2] H. Chen and U. N. Katugampola, "Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[3] S. S. Dragomir, "Inequalities of Jensens type for generalized $k$-$g$-fractional integrals of functions for which the composite $f^\circ g^{-1}$is convex," *RGMIA Research Report Collection*, vol. 20, Article ID 133, 24 pages, 2017.

[4] G. Farid, "Estimations of Riemann-Liouville k-fractional integrals via convex functions," *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol. 23, no. 1, pp. 71–78, 2019.

[5] G. Farid, "Some Riemann-Liouville fractional integral for inequalities for convex functions," *The Journal of Analysis*, vol. 27, no. 4, pp. 1095–1102, 2019.

[6] G. Farid, W. Nazeer, M. Saleem, S. Mehmood, and S. Kang, "Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications," *Mathematics*, vol. 6, no. 11, p. 248, 2018.

[7] S. Habib, S. Mubeen, and M. N. Naeem, "Chebyshev type integral inequalities for generalized $k$-fractional conformable integrals," *Journal of Inequalities and Special Functions*, vol. 9, no. 4, pp. 53–65, 2018.

[8] C. J. Huang, G. Rahman, K. S. Nisar, A. Ghafar, and F. Qi, "Some inequalities of the Hermite-Hadamard type for $k$-fractional conformable integrals," *AJMAA*, vol. 16, no. 1, p. 9, 2019.

[9] S. M. Kang, G. Farid, M. Waseem, S. Ullah, W. Nazeer, and S. Mehmood, "Generalized $k$-fractional integral inequalities associated with $(\alpha, m)$-convex functions," *Journal of Inequalities and Applications*, vol. 2019, p. 255, 2019.

[10] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.

[11] S. Mehmood, G. Farid, K. A. Khan, and M. Yussouf, "New fractional Hadamard and Fejr-Hadamard inequalities associated with exponentially $(h, m)$-convex functions," *Engineering and Applied Science Letters*, vol. 3, no. 2, pp. 9–18, 2020.

[12] S. Mehmood, G. Farid, K. A. Khan et al., "New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m-convex function," *Engineering and Applied Science Letters*, vol. 3, no. 1, pp. 45–55, 2020.

[13] K. S. Nisar, S. Tassaddiq, G. Rehman, and A. Khan, "Some inequalities via fractional conformable integral operators," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 217, 2019.

[14] G. Rahman, A. Khan, T. Abdeljwad, and K. S. Nisar, "The Minkowski inequalities via generalized proportional fractional integral operators," *Advances in Difference Equations*, vol. 2019, p. 287, 2019.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory andApplications of Fractional Differential Equations, North-olland Mathematics Studies*, vol. 204, Elsevier, Amsterdam, Netherlands.

[16] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized riemann-liouville $k$ -fractional integrals associated with ostrowski type inequalities and error bounds of hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.

[17] A. R. Diaz and E. Parigunan, "On hypergeometric functions and $k$-Pochhammar symbol," *Divulgaciones Matematicas*, vol. 15, no. 2, pp. 179–192, 2007.

[18] G. Mittag-Leffler, "Sur la nouvelle fonction $E_\alpha(x)$." *Comptes Randus l'Academie des Sciences Paris*, vol. 137, pp. 554–558, 1903.

[19] M. Arshad, J. ChoI, S. Mubeen, K. S. Nisar, and G. Rahman, "A new extension of MittagLeffler function," *Communications of the Korean Mathematical Society*, vol. 33, no. 2, pp. 549–560, 2018.

[20] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.

[21] T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.

[22] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen, and M. Arshad, "The extended Mittag-Leffler function via fractional calculus," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 4244–4253, 2013.

[23] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[24] V. G. Mihesan, *A Generalization of the Convexity, Seminar on Functional Equations, approximation and Convex*, Science and Education, Cluj-Napoca, Romania, 1993.

[25] M. K. Bakula, M. E. Ozdemir, and J. Pečarić, "Hadamard-type inequalities for $m$-convex and $(\alpha, m)$-convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, Article ID 96, 2007.

[26] I. Iscan, H. Kadakal, and M. Kadakal, "Some new integral inequalities for functions whose nth derivatives in absolute value are $(\alpha, m)$-convex functions," *New Trends in Mathematical Science*, vol. 5, no. 2, pp. 180–185, 2017.

[27] E. Set, M. Sardari, M. E. Ozdemir, and J. Rooin, "On generalizations of the Hadamard inequality for $(\alpha, m)$-convex functions," *RGMIA Research Report Collection*, vol. 12, no. 4, Article ID 4, 2009.

[28] W. Sun and Q. Liu, "New Hermite-Hadamard type inequalities for $(\alpha, m)$-convex functions and applications to special means," *Journal of Mathematical Inequalities*, vol. 11, no. 2, pp. 383–397, 2017.

[29] M. E. Ozdemir, M. Avcı, and H. Kavurmacı , "Hermite-Hadamard-type inequalities via $(\alpha, m)$-convexity," *Computers & Mathematics with Applications*, vol. 61, pp. 2614–2620, 2011.

[30] M. Z. Sarikaya, E. Set, and M. E. Ozdemir, "Some new Hadamard's type inequalities for co-ordinated $m$-convex and $(\alpha, m)$-convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, pp. 219–229, 2011.

[31] S.-H. Wang, B.-Y. Xi, and F. Qi, "Some new inequalities of Hermite-Hadamard type forn-time differentiable functions which arem-convex," *Analysis*, vol. 32, no. 3, pp. 247–262, 2012.

[32] G. Farid, "Existence of an integral operator and its consequences in fractional and conformable integrals," *Open Journal of Mathematical Sciences*, vol. 3, no. 3, pp. 210–216, 2019.

[33] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Advances in Difference Equations*, vol. 2017, no. 1, 2017.

[34] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 378–389, 2019.

[35] A. A. Kilbas, O. I. Marichev, and S. G. Samko, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach, Chur, Switzerland, 1993.

[36] S. Mubeen and G. M. Habibullah, ".$k$-fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.

[37] T. O. Salim and A. W. Faraj, "A generalization of Mittag-Leffler function and integral operator associated with integral calculus," *Journal of Fractional Calculus and Applications*, vol. 3, no. 5, pp. 1–13, 2012.

[38] M. Z. Sarikaya, M. Dahmani, M. E. Kiris, and F. Ahmad, ".$(k, s)$-Riemann-Liouville fractional integral and applications," *Hacettepe Journal of Mathematics and Statistics*, vol. 45, no. 1, pp. 77–89, 2016.

[39] H. M. Srivastava and Ž. Tomovski, "Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel," *Applied Mathematics and Computation*, vol. 211, no. 1, pp. 198–210, 2009.

[40] T. Tunc, H. Budak, F. Usta, and M. Z. Sarikaya, "On New Generalized Fractional Integral Operators and Related Fractional Inequalities," 2017, https://www.researchgate.net/publication/313650587.