*Inequalities for Certain Eigenvalues of a Membrane of Given Area**

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§1. INTRODUCTION

1.1. This paper deals with relations between certain eigenvalues of a vibrating membrane and the surface area of the membrane in its position of equilibrium. We represent the membrane in this position as a domain $\mathfrak{D}$ of the complex $z = z + iy$–plane bounded by a single analytic curve $\mathfrak{C}$. We are concerned with the differential equation

$$\nabla^2 u + h^2 u = 0, \tag{1}$$

where h is a constant and the function $u = u(x, y)$ is defined in the domain $\mathfrak{D}$. We deal with the following two, basically different, boundary conditions:

$$u = 0 \quad \text{on} \quad \mathfrak{C}; \tag{2}$$

$$\partial u / \partial n = 0 \quad \text{on} \quad \mathfrak{C}. \tag{3}$$

In both cases infinitely many eigenvalues h exist, $h \geqq 0$. We denote them in the case (2) by

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots, \tag{4}$$

and in the case (3) by

$$\mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots. \tag{5}$$

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It is well-known that $0 < \lambda_1 < \lambda_2$. Also $\mu_1 = 0$ since $u = 1$ satisfies the equation $\nabla^2 u = 0$ and condition (3).

1.2. In the special case when $\mathfrak{D}$ is a circle of radius 1, we have* ([**5**], p. 3) $\lambda_1 = j$ where $j = 2.4048$ is the least positive zero of the Bessel function $J_0(r)$. This eigenvalue λ_1 is simple and the corresponding eigenfunction is $J_0(jr)$.

In the same special case we have $\mu_2 = \mu_3 = p$ where $p = 1.8412$ is the least positive zero of the Bessel function $J_1'(r)$. This is a double eigenvalue and the corresponding eigenfunctions are

$$J_1(pr) \cos \phi, \qquad J_1(pr) \sin \phi. \tag{6}$$

In these cases r and ϕ denote polar coordinates with the center of the circle as pole. In case of a circle of radius a we have

$$\lambda_1 = j/a, \qquad \mu_2 = \mu_3 = p/a. \tag{7}$$

1.3. Returning again to the case of an arbitrary domain $\mathfrak{D}$ we assume, for the sake of simplicity, that its boundary $\mathfrak{C}$ is an analytic curve. Let $A = A(\mathfrak{D})$ be the area of $\mathfrak{D}$ and $\lambda_1 = \lambda_1(\mathfrak{D})$ the first (principal) eigenvalue of $\mathfrak{D}$ corresponding to the boundary condition (2). An interesting relation of λ_1 to A, conjectured by Lord Rayleigh ([**6**], p. 339), can be stated in terms of the following extremum property:

For all domains $\mathfrak{D}$ of given area A the circle yields the minimum value of λ_1.

Since the radius of the circle of area A is $(A/\pi)^{\frac{1}{2}}$, this property can also be expressed in the form of the following inequality:

$$\lambda_1 \geqq j(A/\pi)^{-\frac{1}{2}} \tag{8}$$

with the sign $=$ for any circle.

Lord Rayleigh supported this assertion by the following facts:

(a) Inequality (8) can be confirmed in a number of special cases ([**6**], p. 345);

(b) for a "nearly circular" domain (see §3) both quantities λ_1 and A can be evaluated up to infinitesimals of the second order, inclusive, and the difference of the quantities appearing in (8) can be ascertained to be of constant sign.

A formal proof of (8) was given by G. Faber [**1**] and E. Krahn [**3**] in 1923. A refinement of (8) was proved by G. Pólya & G. Szegö ([**5**], p. 6) asserting that $\lambda_1(\mathfrak{D})$ is diminished (not increased) if the domain $\mathfrak{D}$ is subjected to a so-called Steiner symmetrization.

1.4. A counterpart of Rayleigh's assertion was formulated recently by E. T. Kornhauser & I. Stakgold [**2**]. Let $\mathfrak{D}$ have the same meaning as before and let $\mu_2 = \mu_2(\mathfrak{D})$ be the second (first non-trivial) eigenvalue of $\mathfrak{D}$ corresponding to the boundary condition (3). Then:

* Numbers in square brackets refer to the bibliography at the end of the text.

For all domains $\mathfrak{D}$ of given area A, the circle yields the maximum value of μ_2. This can be expressed in the form of the following inequality:

$$\mu_2 \leqq p(A/\pi)^{-\frac{1}{2}} \tag{9}$$

with the sign = for any circle.

The supporting facts in this case are as follows:

(a) This inequality can be confirmed in various special cases ([**2**], p. 47);

(b) no infinitesimal, area-preserving deformation of a circle can increase μ_2;

(c) if $\mathfrak{D}$ is not a circle, there exist infinitesimal, area-preserving deformations of $\mathfrak{D}$ which do increase μ_2.

As to (b) and (c) see ([**2**], pp. 47–53). In the proof of (c) the authors mentioned have used the additional assumption that μ_2 is a simple eigenvalue.

1.5. Combining the inequalities (8) and (9) we can assert that for an arbitrary domain $\mathfrak{D}$:

$$\mu_2/\lambda_1 \leqq p/j. \tag{10}$$

In particular we have $\mu_2 < \lambda_1$. A rigorous proof for the latter inequality (without assuming (9)) has been offered by Professor G. Pólya [**4**]; he compares the quantities A, λ_1, and μ_2 with the polar moment of inertia of the domain $\mathfrak{D}$. His argument furnishes incidentally the inequality

$$\mu_2 \leqq 2(A/\pi)^{-\frac{1}{2}}, \tag{11}$$

where the "true" constant is of course $p = 1.8412$ instead of 2.

1.6. Our principal purpose is to present a formal proof for the assertion of E. T. Kornhauser & I. Stakgold. In order to arrange the argument in a clear way, it is advisable to deal first with some special cases characterized by certain symmetry conditions. Naturally it would be possible to discuss the general case directly, omitting these preparations.

The proof of the assertion of E. T. Kornhauser & I. Stakgold to be given in §2 forms the essential content of the report [**7**]. A further part of this report is devoted to the study of "nearly circular" domains $\mathfrak{D}$. For such domains the second variation of the eigenvalues λ_1, λ_2, λ_3, μ_2, μ_3 is computed and compared with the area and other functionals of $\mathfrak{D}$. In the present paper, in view of space considerations, only the principal formulas for these eigenvalues of nearly circular domains are reproduced; concerning details of the proofs of these formulas and some applications we refer to the report [**7**].

§2. Proof of an assertion of E. T. Kornhauser & I. Stakgold

2.1. Preliminaries. Let $\mathfrak{D}$ be a domain in the complex $z = x + iy$–plane bounded by the analytic curve $\mathfrak{C}$. We consider in $\mathfrak{D}$ the differential equation

$$\nabla^2 u + \mu^2 u = 0 \tag{1}$$

with the boundary condition $\partial u/\partial n = 0$ on the curve $\mathfrak{C}$. Obviously $\mu = \mu_1 = 0$ is a trivial eigenvalue with the trivial eigenfunction $u = 1$. The first non-trivial (positive) eigenvalue μ_2 can be characterized by the following minimum property:

$$\mu_2^2 = \min \frac{\iint_{\mathfrak{D}} |\operatorname{grad} u|^2 \, d\sigma}{\iint_{\mathfrak{D}} u^2 \, d\sigma}, \tag{2}$$

admitting in this problem all functions u defined in $\mathfrak{D}$ which satisfy the side condition

$$\iint_{\mathfrak{D}} u \, d\sigma = 0. \tag{3}$$

Here $d\sigma$ is the area–element of $\mathfrak{D}$. In the case of the unit circle, μ_2 is the least positive zero $p = 1.8412$ of the Bessel function $J_1'(r)$ and the corresponding eigenfunctions are

$$u_1 = J_1(pr) \cos \phi, \qquad u_2 = J_1(pr) \sin \phi \tag{4}$$

(r, ϕ are polar coordinates).

Our purpose is to prove the inequality:

$$\mu_2 \leqq p(A/\pi)^{-\frac{1}{2}}. \tag{5}$$

To this end we establish first a one-to-one correspondence $z = f(\zeta)$ between the given domain $\mathfrak{D}$ of the z–plane and the unit circle $|\zeta| \leqq 1$ of the complex plane $\zeta = re^{i\phi}$. We then employ the minimum principle (2), (3), inserting for u one of the functions u_1 and u_2, where these functions u_1 and u_2 have to be interpreted as functions of $z = x + iy$. The resulting "Rayleigh quotients" appearing on the right of (2) yield upper bounds for μ_2^2.

It is advisable to choose for $z = f(\zeta)$ a conformal (schlicht) mapping. It is known that such a mapping is possible in infinitely many ways. If $f(\zeta)$ is one of these mappings, the most general mapping can be represented in the form

$$z = f\left(\epsilon \frac{\zeta + \alpha}{1 + \bar{\alpha}\zeta}\right), \tag{6}$$

where $|\epsilon| = 1$ and $|\alpha| < 1$. Our purpose is to determine the free parameters ϵ and α in such a way that u_1 and u_2, as functions of $z = x + iy$, satisfy the side condition (3).

This method of "transplantation" of the solutions of the problem for the unit circle to the general domain is a basic idea frequently used in the investigations of [**5**]. However, in most of the cases dealt with in [**5**] the transplanted functions possess circular symmetry, *i.e.*, they depend only on r. This is different in the present problem.

2.2. First special case. We assume first that the domain $\mathfrak{D}$ is "symmetrical of order m" with respect to the origin, *i.e.*, it remains invariant under the rotation $z' = ze^{2\pi i/m}$, m a positive integer, $m > 1$. In this case the special mapping leaving both the origin and the line-element there unchanged can be written as follows:

$$z = f(\zeta) = c_1 \zeta + c_{m+1} \zeta^{m+1} + c_{2m+1} \zeta^{2m+1} + \cdots, \quad c_1 > 0. \tag{7}$$

We assume first that $m \geqq 3$.

We insert in (2) one of the functions (4), say u_1. The side condition (3) is satisfied since

$$\iint_{\mathfrak{D}} J_1(pr) \cos\phi \, d\sigma = \iint_{|\zeta|<1} J_1(pr) \cos\phi \, |f'(\zeta)|^2 r \, dr \, d\phi = 0. \tag{8}$$

Indeed, $f'(\zeta)$ contains only those powers of $\zeta = re^{i\phi}$ which have exponents divisible by m; hence, when we expand $|f'(re^{i\phi})|^2$ in a Fourier series in ϕ no terms of the form $e^{\pm i\phi}$ will appear. (This is true even for $m \geqq 2$, and naturally all this holds also for $u = u_2$.)

Consequently, $u = u_1$ or u_2,

$$\begin{aligned} \mu_2^2 &\leqq \iint_{\mathfrak{D}} |\operatorname{grad} u|^2 \, d\sigma : \iint_{\mathfrak{D}} u^2 \, d\sigma \\ &= \iint_{|\zeta|\leqq 1} |\operatorname{grad} u|^2 r \, dr \, d\phi : \iint_{|\zeta|\leqq 1} u^2 \, |f'(\zeta)|^2 r \, dr \, d\phi \\ &= p^2 \iint_{|\zeta|\leqq 1} u^2 r \, dr \, d\phi : \iint_{|\zeta|\leqq 1} u^2 \, |f'(\zeta)|^2 r \, dr \, d\phi. \end{aligned} \tag{9}$$

In the step from the first to the second line we make use of the invariance of Dirichlet's integral under conformal mapping. In the next step we take into account the fact that the functions u_1 and u_2 are eigenfunctions of the unit circle.

The last integral occurring in (9) involves $\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi)$ (or $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$), and for $m \geqq 3$ the expansion of $|f'(re^{i\phi})|^2$ does not produce any terms $\cos 2\phi$ (or $\sin 2\phi$). Hence

$$\iint_{|\zeta|\leqq 1} u^2 |f'(\zeta)|^2 r\,dr\,d\phi = \pi \int_0^1 [J_1(pr)]^2 \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2}\cdot r\,dr. \tag{10}$$

Here we have inserted

$$f(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^n, \qquad f'(\zeta) = \sum_{n=1}^{\infty} nc_n \zeta^{n-1}$$

and integrated with respect to ϕ. (In fact the expansion of $f(\zeta)$ has the special form (7).) We can write (10) as follows:

$$\iint_{|\zeta|\leqq 1} u^2 |f'(\zeta)|^2 r\,dr\,d\phi = \pi \sum_{n=1}^{\infty} n |c_n|^2 \cdot M_n$$

where

$$M_n = n \int_0^1 [J_1(pr)]^2 r^{2n-1}\,dr. \tag{11}$$

Thus we have

$$\mu_2^2 \leqq p^2 M_1 : \sum_{n=1}^{\infty} n |c_n|^2 \cdot M_n. \tag{12}$$

From (11) we obtain by integrating by parts:

$$M_n = \left\{\frac{1}{2} [J_1(pr)]^2 r^{2n}\right\}_0^1 - \frac{1}{2}\int_0^1 \frac{d}{dr}[J_1(pr)]^2 \cdot r^{2n}\,dr. \tag{13}$$

The first term on the right is independent of n. Since $[J_1(pr)]^2$ is an increasing function of r, $0 \leqq r \leqq 1$, the derivative appearing in the integral is positive. Hence the integral decreases with increasing n, consequently *M_n is increasing with increasing n.* In particular $M_n \geqq M_1$.

This yields in view of (12):

$$\mu_2^2 \leqq p^2 : \sum_{n=1}^{\infty} n |c_n|^2 = p^2 : (A/\pi), \tag{14}$$

i.e., the assertion.

2.3. Second special case. Let us assume now that $m = 2$. We use the same argument as in §2.2, taking into account that both functions (4), as functions of $z = x + iy$, are eligible for our purpose. In both cases the side condition is satisfied (see the remark in 2.2). Hence

$$\mu_2^2 \leqq p^2 \iint_{|\zeta| \leqq 1} u_i^2 r\, dr\, d\phi : \iint_{|\zeta| \leqq 1} u_i^2 \,|\, f'(\zeta) \,|^2 r\, dr\, d\phi, \qquad i = 1, 2. \tag{15}$$

Let a, b, a', b' be positive numbers. The fraction

$$\frac{a + a'}{b + b'} = \frac{b(a/b) + b'(a'/b')}{b + b'}$$

is a mean value of the fractions a/b and a'/b'. Taking this trivial remark into account we conclude from (15) that

$$\mu_2^2 \leqq p^2 \iint_{|\zeta| \leqq 1} (u_1^2 + u_2^2) r\, dr\, d\phi : \iint_{|\zeta| \leqq 1} (u_1^2 + u_2^2) \,|\, f'(\zeta) \,|^2 r\, dr\, d\phi. \tag{16}$$

Now $u_1^2 + u_2^2 = [J_1(pr)]^2$ is independent of ϕ, hence after we expand $|\, f'(re^{i\phi}) \,|^2$ and integrate with respect to ϕ, only the terms independent of ϕ will survive. This yields again (14).

2.4. Third special case. Let the domain $\mathfrak{D}$ be symmetrical with respect to the real axis. We now choose the mapping $z = f(\zeta)$ of the unit circle $|\, \zeta \,| < 1$ onto the domain $\mathfrak{D}$ in such a manner that $\zeta = 0$ is carried into a real point and $f'(0) > 0$. Thus the power series expansion of $f(\zeta)$ around $\zeta = 0$ has real coefficients and the real axis remains unchanged.

If $z = f(\zeta)$ is a fixed mapping of this kind, the mapping

$$z = f\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \tag{17}$$

where α is real, $-1 < \alpha < 1$, will serve the same purpose. We now transplant the functions (4) into the z-plane by means of the mapping (17) and seek to determine the parameter α so as to satisfy the side condition (3), $u = u_1$ and $u = u_2$.

This condition appears in the following form:

$$\iint_{|\zeta| \leqq 1} J_1(pr) \begin{matrix}\cos \phi \\ \sin \phi\end{matrix} \left| f'\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \right|^2 \frac{(1 - \alpha^2)^2}{|\, 1 + \alpha\zeta \,|^4} r\, dr\, d\phi = 0. \tag{18}$$

Since f' assumes conjugate complex values at $\zeta = re^{i\phi}$ and $\bar{\zeta} = re^{-i\phi}$, the condition involving $\sin \phi$ is trivially satisfied for all real α. As to the condition involving $\cos \phi$, we denote the integral appearing on the left hand side of (18) (with $\cos \phi$ as second factor in the integrand) by $H(\alpha)$. Since, independently of α,

$$\iint_{|\zeta| \leqq 1} \left| f'\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \right|^2 \frac{(1 - \alpha^2)^2}{|\, 1 + \alpha\zeta \,|^4} r\, dr\, d\phi = A$$

(the area of the domain $\mathfrak{D}$), we find that

$$\frac{H(\alpha)}{A} - J_1(p) = \iint_{|\zeta| \leqq 1} \{J_1(pr) \cos \phi - J_1(p)\} \left| f'\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \right|^2 \cdot \frac{(1 - \alpha^2)^2}{|1 + \alpha\zeta|^4} r\, dr\, d\phi : \iint_{|\zeta| \leqq 1} \left| f'\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \right|^2 \frac{(1 - \alpha^2)^2}{|1 + \alpha\zeta|^4} r\, dr\, d\phi. \tag{19}$$

This ratio can be considered as a special case of "singular integrals," in the sense of Lebesgue. We prove that

$$\lim_{\alpha \to -1} \frac{H(\alpha)}{A} = J_1(p). \tag{20}$$

Indeed, let ϵ be an arbitrary positive number and let us denote by $\Delta = \Delta(\epsilon)$ that neighborhood of $\zeta = re^{i\phi} = +1$ in which

$$| J_1(pr) \cos \phi - J_1(p) | < \epsilon.$$

We observe that in $| \zeta | \leqq 1$ and outside of Δ

$$\left| f'\left(\frac{\zeta + \alpha}{1 + \alpha\zeta}\right) \right|^2 \frac{(1 - \alpha^2)^2}{|1 + \alpha\zeta|^4} < M(1 - \alpha^2)^2,$$

where $M = M(\epsilon)$ is independent of α; we assume that $-1 < \alpha < 0$. Decomposing the first integral in (19) into two parts, one extended over Δ and the other over the remaining part of the unit circle, we find by a familiar argument:

$$\left| \frac{H(\alpha)}{A} - J_1(p) \right| < \epsilon + 2\pi J_1(p) \cdot \frac{M(1 - \alpha^2)^2}{A}.$$

The second term on the right is $< \epsilon$ provided α is sufficiently near to -1, and this yields (20).

In a similar manner we can prove that $H(\alpha)/A$ tends to $-J_1(p)$ as $\alpha \to +1$. Since $J_1(p)$ is different from zero ($J_1(p) = J_1(1.8412) = 0.5819$), $H(\alpha)$ assumes values of opposite sign when α is sufficiently near to $+1$ and -1, respectively; thus there exists at least one value of α, $\alpha = \alpha_0$, $-1 < \alpha_0 < 1$, for which $H(\alpha_0) = 0$.

We choose for α that particular value and use the mapping (17) for the purpose of transplanting the functions (4). The side conditions will be then satisfied and we can repeat without change the argument of 2.3.

This establishes the assertion.

2.5. General Case. Finally we assume that $\mathfrak{D}$ is an arbitrary domain bounded by an analytic curve $\mathfrak{C}$. The only remaining difficulty is to choose the mapping (6) in such a manner that the transplanted functions satisfy the side condition

(3), $u = u_1$ and $u = u_2$. We write $\epsilon = 1$. The condition in question can be formulated now as follows:

$$(21) \qquad K(\alpha) = \iint_{|\zeta| \leq 1} J_1(pr)e^{i\phi} \left| f'\left(\frac{\zeta + \alpha}{1 + \bar{\alpha}\zeta}\right)\right|^2 \frac{(1 - |\alpha|^2)^2}{|1 + \bar{\alpha}\zeta|^4} r\, dr\, d\phi = 0.$$

This complex-valued function $K(\alpha)$ is defined and continuous in the whole unit circle $|\alpha| < 1$, $\alpha = \rho e^{i\gamma}$. We wish to prove the existence of a root $\alpha = \alpha_0$, $|\alpha_0| < 1$, of the equation $K(\alpha) = 0$.

The integrand in (21) is periodic in ϕ, so that we can replace ϕ by $\phi + \gamma$. We obtain this way

$$K(\alpha) = e^{i\gamma} \iint_{|\zeta| \leq 1} J_1(pr)e^{i\phi} \left| f'\left(e^{i\gamma} \frac{\zeta + \rho}{1 + \rho\zeta}\right)\right|^2 \frac{(1 - \rho^2)^2}{|1 + \rho\zeta|^4} r\, dr\, d\phi = e^{i\gamma} K_1(\alpha).$$

Let $|\alpha| = \rho \to 1$. We can prove by an argument quite similar to that used in §2.4 that

$$\lim_{\rho \to 1} \frac{K_1(\alpha)}{A} = -J_1(p).$$

Moreover, this relation holds uniformly in γ. Hence, if ρ is sufficiently near to 1 and α describes the circle $|\alpha| = \rho$, the complex number $K_1(\alpha)$ describes a curve arbitrarily close to $-AJ_1(p)$.

If α describes the circle $\alpha = \rho e^{i\gamma}$ in the positive direction, the quantity $\alpha' = K(\alpha)$ describes in the complex plane an oriented, continuous, closed curve not passing through the origin. (This curve will intersect itself in general.) In the usual manner we can define the "index-number" of this curve with respect to the origin $\alpha' = 0$. The index number of a product is the sum of the index numbers of the factors. Obviously, the index number of the circle described by $e^{i\gamma}$ is $+1$; the index number of the curve described by $K_1(\alpha)$ is zero provided ρ is sufficiently near to 1, since $J_1(p) \neq 0$. Hence the index number corresponding to $K(\alpha)$ is $+1$.

If $K(\alpha)$ is a complex-valued, continuous function in $|\alpha| < 1$ and for $|\alpha| = \rho$ sufficiently near 1 the index number of the curve corresponding to the circle $\alpha = \rho e^{i\gamma}$ is different from zero, the function $K(\alpha)$ must have at least one zero in $|\alpha| < 1$.

This well-known lemma follows, of course, trivially by shrinking the circle $|\alpha| = \rho$ continuously to $\alpha = 0$ and taking into account that the index number of the image curves corresponding to the circles $|\alpha| =$ const. are all defined provided $K(\alpha) \neq 0$; these numbers being integers, they cannot change when ρ decreases.

If we choose for α the value for which the side condition (21) is satisfied, the rest of the argument remains the same as in the special case dealt with in §2.3.

This completes the proof of the assertion of E. T. Kornhauser & I. Stakgold.

§3. Nearly circular domains. Problems and results

3.1. Problem. In ([5], p. 33) various interesting quantities occurring in mathematical physics were evaluated for "nearly circular" domains. We define a domain of this kind in polar coordinates r, ϕ by the condition

$$(1) \qquad r \leqq 1 + \rho(\phi) = 1 + a_0 + 2 \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi),$$

where the Fourier coefficients of the function $\rho(\phi)$ are considered as infinitesimals of the *first* order. The purpose is to compute certain functionals of this domain up to the *second* order, inclusive. It is often convenient to use the Fourier expansion of $\rho(\phi)$ in the complex form

$$(2) \quad \rho(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi}, \quad c_0 = a_0, \quad c_n = a_n - ib_n; \quad c_{-n} = \bar{c}_n, \quad n = 1, 2, 3, \cdots.$$

Following the notation introduced in §1, we have no difficulty in computing λ_1 in this sense. The result is due to Lord Rayleigh ([6], p. 341; *cf.* [5], p. 30, (3)) and can be formulated as follows:

$$(3) \qquad \bar{\lambda}_1 = \frac{j}{\lambda_1} = 1 + c_0 - \sum_{n=1}^{\infty} \left(1 + \frac{2jJ_n'(j)}{J_n(j)}\right) | c_n |^2.$$

The left-hand expression $\bar{\lambda}_1$ has a simple meaning: it represents the radius of the circle for which the principal frequency coincides with that of the given domain.

The next problem arising in a natural manner is the analogous evaluation of the eigenvalues

$$(4) \qquad \lambda_2, \lambda_3, \mu_2, \mu_3.$$

In the case of a circle we have $\lambda_2 = \lambda_3 = k/a$ where k denotes the least positive root of the equation $J_1(r) = 0$, $k = 3.8317$, and $\mu_2 = \mu_3 = p/a$ where p is the least positive root of the equation $J_1'(r) = 0$, $p = 1.8412$; a is the radius of the circle. Thus the eigenvalues for the circle are in both cases multiple, and the method to be used for the evaluation of the quantities (4) for nearly circular domains must be different from that employed in the case of λ_1. The results will have a different character also. The first order term in this case is not c_0 as in (3) (and in all other cases enumerated in [5], p. 33; *cf.* p. 31, (3)) but involves besides c_0 the Fourier coefficient c_2. As to the second order terms, the character of the result depends on whether c_2 vanishes or not. In both cases the second order terms are not linear combinations of the squares $| c_n |^2$ but are of a more complicated character.

The mean values $\frac{1}{2}(\lambda_2 + \lambda_3)$ and $\frac{1}{2}(\mu_2 + \mu_3)$ are much simpler; in structure they are similar to the quantities treated in ([**5**], p. 33).

3.2. Results for λ_2, λ_3 (boundary condition $u = 0$). The following formulas hold for the eigenvalues λ_2, λ_3 and for the corresponding "radii" $\bar{\lambda}_2 = k/\lambda_2$, $\bar{\lambda}_3 = k/\lambda_3$; $\bar{\lambda}_2$ and $\bar{\lambda}_3$ represent the radii of those circles for which the second and third eigenvalues are the quantities λ_2 and λ_3, respectively. We list also the mean value $\frac{1}{2}(\lambda_2 + \lambda_3)$ and the corresponding "radius."

Let $c_2 \neq 0$, $c_2 = | c_2 | e^{i\gamma}$. Then

$$\lambda_2 = k - k(c_0 + | c_2 |) + k(c_0 + | c_2 |)^2 + \frac{k}{4} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n+1} + e^{i\gamma} c_{n-1} |^2, \tag{5}$$

$$\lambda_3 = k - k(c_0 - | c_2 |) + k(c_0 - | c_2 |)^2 + \frac{k}{4} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n+1} - e^{i\gamma} c_{n-1} |^2, \tag{6}$$

$$\bar{\lambda}_2 = 1 + c_0 + | c_2 | - \frac{1}{4} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n+1} + e^{i\gamma} c_{n-1} |^2, \tag{7}$$

$$\bar{\lambda}_3 = 1 + c_0 - | c_2 | - \frac{1}{4} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n+1} - e^{i\gamma} c_{n-1} |^2, \tag{8}$$

$$\tfrac{1}{2}(\lambda_2 + \lambda_3) = k - kc_0 + k(c_0^2 + | c_2 |^2) + \frac{k}{2} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n-1} |^2, \tag{9}$$

$$\overline{\tfrac{1}{2}(\lambda_2 + \lambda_3)} = 1 + c_0 - | c_2 |^2 - \frac{1}{2} \sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) | c_{n-1} |^2. \tag{10}$$

In all these cases, as well as in (11)–(13), the arguments of J_n and J_n' are equal to k. In all summations n runs from $-\infty$ to $+\infty$ with the exception of the values $n = \pm 1$. The terms in (5)–(8) corresponding to n and $-n$ are the same. In (9) and (10) the quantity $| c_{n-1} |^2$ can be replaced by $| c_{n+1} |^2$. There is no difficulty in rewriting these series in terms of a_n, b_n. We note that (10) has the same form as (3) or the other quantities in the table ([**5**], p. 33).

Let $c_2 = 0$. We assume now that the sum

$$-\sum_{n \neq \pm 1} \left(1 + \frac{2kJ_n'}{J_n}\right) c_{n+1} \bar{c}_{n-1} = Qe^{i\delta}, \qquad Q > 0, \tag{11}$$

does not vanish. The formulas (5)–(10) remain valid provided we replace γ (which has no meaning in this case) by δ. We note the alternate expressions

$$\left.\begin{matrix}\lambda_3\\ \lambda_2\end{matrix}\right\} = k - kc_0 + kc_0^2 + \frac{k}{2}\sum_{n\neq\pm1}\left(1 + \frac{2kJ_n'}{J_n}\right)|c_{n-1}|^2 \pm \frac{k}{2}\left|\sum_{n\neq\pm1}\left(1 + \frac{2kJ_n'}{J_n}\right)c_{n+1}\bar{c}_{n-1}\right|, \tag{12}$$

$$\left.\begin{matrix}\bar{\lambda}_3\\ \bar{\lambda}_2\end{matrix}\right\} = 1 + c_0 - \frac{1}{2}\sum_{n\neq\pm1}\left(1 + \frac{2kJ_n'}{J_n}\right)|c_{n-1}|^2 \mp \frac{1}{2}\left|\sum_{n\neq\pm1}\left(1 + \frac{2kJ_n'}{J_n}\right)c_{n+1}\bar{c}_{n-1}\right| \tag{13}$$

where the upper signs correspond to λ_3, $\bar{\lambda}_3$ and the lower signs to λ_2, $\bar{\lambda}_2$. Formulas (9) and (10) hold without change for $c_2 = 0$.

3.3. Results for μ_2, μ_3 (boundary condition $\partial u/\partial n = 0$). The following formulas hold for the eigenvalues μ_2, μ_3 and for the corresponding "radii" $\bar{\mu}_2 = p/\mu_2$, $\bar{\mu}_3 = p/\mu_3$.

Let $c_2 \neq 0$, $c_2 = |c_2|e^{i\gamma}$. Then

$$\begin{aligned}\left.\begin{matrix}\mu_3\\ \mu_2\end{matrix}\right\} &= p + p(-c_0 \pm \beta|c_2|) + p(-c_0 \pm \beta|c_2|)^2 - \frac{2p(3p^2-1)}{(p^2-1)^3}|c_2|^2\\ &+ \frac{p}{4(p^2-1)}\sum_{n\neq\pm1}\left(2n + 1 - p^2 + \frac{2(n^2-p^4)}{p}\frac{J_n}{J_n'}\right)|c_{n+1} \mp e^{i\gamma}c_{n-1}|^2\\ &- \frac{p}{p^2-1}\sum_{n\neq\pm1} n\left(1 + \frac{2(n-p^2)}{p}\frac{J_n}{J_n'}\right)|c_{n-1}|^2.\end{aligned} \tag{14}$$

The upper signs yield μ_3, the lower signs μ_2. Here $\beta = (p^2 + 1)/(p^2 - 1)$. The arguments of J_n and J_n' are p. Moreover, in a similar arrangement

$$\begin{aligned}\left.\begin{matrix}\bar{\mu}_3\\ \bar{\mu}_2\end{matrix}\right\} &= 1 + c_0 \mp \beta|c_2| + \frac{2(3p^2-1)}{(p^2-1)^3}|c_2|^2\\ &- \frac{1}{4(p^2-1)}\sum_{n\neq\pm1}\left(2n + 1 - p^2 + \frac{2(n^2-p^4)}{p}\frac{J_n}{J_n'}\right)|c_{n+1} \mp e^{i\gamma}c_{n-1}|^2\\ &+ \frac{1}{p^2-1}\sum_{n\neq\pm1} n\left(1 + \frac{2(n-p^2)}{p}\frac{J_n}{J_n'}\right)|c_{n-1}|^2.\end{aligned} \tag{15}$$

Finally,

$$(16)\quad \begin{aligned} \tfrac{1}{2}(\mu_2 + \mu_3) &= p - pc_0 + p(c_0^2 + \beta^2 \mid c_2 \mid^2) - \frac{2p(3p^2 - 1)}{(p^2 - 1)^3} \mid c_2 \mid^2 \\ &\quad - \frac{p}{2(p^2 - 1)} \sum_{n \neq \pm 1} \left(2n - 1 + p^2 + \frac{2(n - p^2)^2}{p} \frac{J_n}{J_n'}\right) \mid c_{n-1} \mid^2, \end{aligned}$$

$$(17)\quad \begin{aligned} \overline{\tfrac{1}{2}(\mu_2 + \mu_3)} &= 1 + c_0 - \beta^2 \mid c_2 \mid^2 + \frac{2(3p^2 - 1)}{(p^2 - 1)^3} \mid c_2 \mid^2 \\ &\quad + \frac{1}{2(p^2 - 1)} \sum_{n \neq \pm 1} \left(2n - 1 + p^2 + \frac{2(n - p^2)^2}{p} \frac{J_n}{J_n'}\right) \mid c_{n-1} \mid^2. \end{aligned}$$

Let $c_2 = 0$. We assume now that the expression

$$(18)\quad -\sum_{n \neq \pm 1} \left(2n + 1 - p^2 + \frac{2(n^2 - p^4)}{p} \frac{J_n}{J_n'}\right) c_{n+1} \bar{c}_{n-1} = Qe^{i\delta}, \qquad Q > 0,$$

does not vanish. With this notation (14) and (15) remain valid provided we replace γ by δ. Formulas (16) and (17) hold whether $c_2 \neq 0$ or $c_2 = 0$. We note also the alternate formulas ($c_2 = 0$):

$$(19)\quad \begin{aligned} \left.\begin{matrix}\mu_3\\ \mu_2\end{matrix}\right\} &= p - pc_0 + pc_0^2 \\ &\quad - \frac{p}{2(p^2 - 1)} \sum_{n \neq \pm 1} \left(2n - 1 + p^2 + \frac{2(n - p^2)^2}{p} \frac{J_n}{J_n'}\right) \mid c_{n-1} \mid^2 \\ &\quad \pm \frac{p}{2(p^2 - 1)} \left| \sum_{n \neq \pm 1} \left(2n + 1 - p^2 + \frac{2(n^2 - p^4)}{p} \frac{J_n}{J_n'}\right) c_{n+1} \bar{c}_{n-1} \right|, \end{aligned}$$

$$(20)\quad \begin{aligned} \left.\begin{matrix}\bar{\mu}_3\\ \bar{\mu}_2\end{matrix}\right\} &= 1 + c_0 + \frac{1}{2(p^2 - 1)} \sum_{n \neq \pm 1} \left(2n - 1 + p^2 + \frac{2(n - p^2)^2}{p} \frac{J_n}{J_n'}\right) \mid c_{n-1} \mid^2 \\ &\quad \mp \frac{1}{2(p^2 - 1)} \left| \sum_{n \neq \pm 1} \left(2n + 1 - p^2 + \frac{2(n^2 - p^4)}{p} \frac{J_n}{J_n'}\right) c_{n+1} \bar{c}_{n-1} \right|. \end{aligned}$$

For the proofs of these formulas and for their applications we refer to the report [7].

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