Discrete Comput Geom 13:217–231 (1995)

Discrete & Computational Geometry

Inequalities for Convex Bodies and Polar Reciprocal Lattices in R^n

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Abstract. Let L be a lattice and let U be an o-symmetric convex body in $\boldsymbol{R}^n$. The Minkowski functional $\|\,\|_U$ of U, the polar body U^0, the dual lattice L^*, the covering radius $\mu(L,U)$, and the successive minima $\lambda_i(L,U)$, $i = 1,\dots,n$, are defined in the usual way. Let $\mathscr{L}_n$ be the family of all lattices in $\boldsymbol{R}^n$. Given a pair U, V of convex bodies, we define

$$\mathrm{mh}(U,V) = \sup_{L\in\mathscr{L}_n} \max_{1\le i\le n} \lambda_i(L,U)\lambda_{n-i+1}(L^*,V),$$

$$\mathrm{lh}(U,V) = \sup_{L\in\mathscr{L}_n} \mu(L,U)\lambda_1(L^*,V),$$

and $\mathrm{kh}(U,V)$ is defined as the smallest positive number s for which, given arbitrary $L\in\mathscr{L}_n$ and $u\in\boldsymbol{R}^n\setminus(L+U)$, some $v\in L^*$ with $\|v\|_V\le s\,d(uv,\boldsymbol{Z})$ can be found. Upper bounds for $\mathrm{jh}(U,U^0)$, j = k, l, m, belong to the so-called transference theorems in the geometry of numbers. The technique of Gaussian-like measures on lattices, developed in an earlier paper [4] for euclidean balls, is applied to obtain upper bounds for $\mathrm{jh}(U,V)$ in the case when U, V are n-dimensional ellipsoids, rectangular parallelepipeds, or unit balls in l_p^n, $1\le p\le\infty$. The gaps between the upper bounds obtained and the known lower bounds are, roughly speaking, of order at most $\log n$ as $n\to\infty$. It is also proved that if U is symmetric through each of the coordinate hyperplanes, then $\mathrm{jh}(U,U^0)$ are less than $Cn\log n$ for some numerical constant C.

Introduction

A lattice in $\boldsymbol{R}^n$ is an additive subgroup of $\boldsymbol{R}^n$ generated by n linearly independent vectors. The family of all lattices in $\boldsymbol{R}^n$ is denoted by $\mathscr{L}_n$. Given a lattice $L\in\mathscr{L}_n$, we define the dual lattice L^* in the usual way:

$$L^* = \{u \in \boldsymbol{R}^n\colon\ uv \in \boldsymbol{Z} \text{ for each } v \in L\},$$

where uv is the canonical inner product in $\boldsymbol{R}^n$. We have $L^{**} = L$.

A convex body in $\boldsymbol{R}^n$ is a compact convex subset of $\boldsymbol{R}^n$ containing interior points. The family of all convex bodies in $\boldsymbol{R}^n$ which are symmetric with respect to zero is denoted by $\mathscr{C}_n$. Given a convex body $U \in \mathscr{C}_n$, we define the polar body U^0 in the usual way:

$$U^0 = \{u \in \boldsymbol{R}^n\colon\ |uv| \le 1 \text{ for each } v \in U\}.$$

We have $U^{00} = U$. By $\|\ \|_U$ we denote the norm on $\boldsymbol{R}^n$ induced by U (the Minkowski functional of U). By d_U we denote the metric induced by $\|\ \|_U$.

By span A we denote the linear subspace of $\boldsymbol{R}^n$ spanned over a subset A. Given a lattice $L \in \mathscr{L}_n$ and a convex body $U \in \mathscr{C}_n$, we write

$$\mu(L,U) = \max\{d_U(u,L)\colon\ u \in \boldsymbol{R}^n\} = \min\{r > 0\colon\ L + rU = \boldsymbol{R}^n\},$$

$$\lambda_i(L,U) = \min\{r > 0\colon\ \dim \operatorname{span}(L \cap rU) \ge i\} \qquad (i = 1,\dots,n).$$

The quantities $\mu(L,U)$ and $\lambda_i(L,U)$ are called, respectively, the *covering radius* and the *successive minima* of L with respect to U.

For each $U \in \mathscr{C}_n$ let us consider the quantities

$$\operatorname{kh}(U) = \sup_{L \in \mathscr{L}_n} \sup_{u \in \boldsymbol{R}^n \setminus L} \inf_{\substack{v \in L^* \\ uv \notin \boldsymbol{Z}}} \frac{d_U(u,L)\|v\|_{U^0}}{d(uv,\boldsymbol{Z})},$$

$$\operatorname{lh}(U) = \sup_{L \in \mathscr{L}_n} \mu(L,U)\lambda_1(L^*,U^0),$$

$$\operatorname{mh}(U) = \sup_{L \in \mathscr{L}_n} \max_{1 \le i \le n} \lambda_i(L,U)\lambda_{n-i+1}(L^*,U^0).$$

It is convenient to denote them by $\operatorname{jh}(U)$, j = k, l, m; it is clear that they are affine invariants of U. Upper bounds for $\operatorname{jh}(U)$ belong to the so-called *transference theorems* in the geometry of numbers. For $\operatorname{kh}(U)$, see, e.g., Chapter XI, Section 3.3 of [8]; for $\operatorname{lh}(U)$ and $\operatorname{mh}(U)$, see, e.g., Section 5 of [10].

Denote by $\boldsymbol{R}^n_U$ and $\boldsymbol{R}^n_2$ the space $\boldsymbol{R}^n$ with the norm $\|\ \|_U$ and with the canonical euclidean norm, respectively. Let $d(\boldsymbol{R}^n_U, \boldsymbol{R}^n_2)$ be the corresponding Banach–Mazur distance and let B^n_2 be the closed unit ball in $\boldsymbol{R}^n_2$. It is clear that

$$\operatorname{jh}(U) \le \operatorname{jh}(B^n_2)\, d(\boldsymbol{R}^n_U, \boldsymbol{R}^n_2), \qquad \text{j} = \text{k}, \text{l}, \text{m}.$$

Since $d(\boldsymbol{R}^n_U, \boldsymbol{R}^n_2) \le n^{1/2}$, it follows that $\operatorname{jh}(U) \le n^{1/2}\operatorname{jh}(B^n_2)$ for j = k, l, m.

Upper bounds for $\operatorname{jh}(U)$ were investigated in several papers; a detailed specification is given in the introduction to [4]. Let us only mention here the bounds $\operatorname{kh}(B^n_2) \le Cn^2$, $\operatorname{lh}(B^n_2) \le Cn^{3/2}$, and $\operatorname{mh}(B^n_2) \le Cn^2$ obtained in [11] and [13] by means of Korkin–Zolotarev bases. Here and below, C is some numerical constant which may vary from line to line.

Let U be a convex body in $\boldsymbol{R}^n$, symmetric or not, and let $L \in \mathscr{L}_n$. The number

$$\mathrm{w}_L(U) = \min_{\substack{v \in L^* \\ v \neq 0}} \left(\max_{u \in U} uv - \min_{u \in U} uv \right)$$

is called the L-width of U; its reciprocal is known as the first covering minimum $\mu_1(U, L)$. Consider the quantity

$$\mathrm{wh}(U) = \sup_{L \in \mathscr{L}_n} \mathrm{w}_L(U) \mu(L, U).$$

Investigating the so-called flatness problem, Kannan and Lovász [12] proved that $\mathrm{wh}(U) \leq Cn^2$ (see (3.12) and (3.13) of [9]). It is clear that $\mathrm{w}_L(U) = 2\lambda_1(L^*, U^0)$ for $U \in \mathscr{C}_n$; then $\mathrm{wh}(U) = 2\mathrm{lh}(U)$.

Applying a probability argument based on gaussian-like measures on lattices, the author proved in [4] that $\mathrm{jh}(B_2^n) \leq Cn$; hence it follows that $\mathrm{jh}(U) \leq Cn^{3/2}$. In this paper, by modifying the method of [4], we show that $\mathrm{jh}(U) \leq Cn \log n$ if U is symmetric through the coordinate hyperplanes (i.e., if $\boldsymbol{R}_U^n$ has a 1-unconditional basis), and that $\mathrm{jh}(U) \leq Cn(\log n)^{1/2}$ if U is the unit ball in l_p^n, $1 \leq p \leq \infty$.

For lower bounds, there is the result of Conway and Thompson (see Chapter II, Theorem 9.5 of [14]), which implies that $\mathrm{jh}(B_2^n) \geq C^{-1}n$. A standard argument shows that $\mathrm{jh}(U) \geq C^{-1}n$ for any $U \in \mathscr{C}_n$; see (3.10) below.

Recently, the author has shown that $\mathrm{jh}(U) \leq Cn \log n$ for any $U \in \mathscr{C}_n$. The proof, however, requires more sophisticated results of the local theory of Banach spaces and is given in [5].

The proofs of the upper bounds become more lucid if, instead of considering just one convex body U and the polar body U^0, a pair of independent convex bodies U, $V \in \mathscr{C}_n$ are considered. For each such a pair, let us denote

$$\mathrm{kh}(U, V) = \sup_{l \in \mathscr{L}_n} \sup_{u \in \boldsymbol{R}^n \setminus L} \inf_{\substack{v \in L^* \\ uv \notin \boldsymbol{Z}}} \frac{d_U(u, L)\|v\|_V}{d(uv, \boldsymbol{Z})},$$

$$\mathrm{lh}(U, V) = \sup_{L \in \mathscr{L}_n} \mu(L, U)\lambda_1(L^*, V),$$

$$\mathrm{mh}(U, V) = \sup_{L \in \mathscr{L}_n} \max_{1 \leq i \leq n} \lambda_i(L, U)\lambda_{n-i+1}(L^*, V).$$

The structure of the paper is as follows. In Section 1 we introduce certain numbers $\alpha(U)$, $\beta(U) \in (0, 1)$ for $U \in \mathscr{C}_n$. Then we show that $\mathrm{jh}(U, V)$ are small provided that $\alpha(U)$, $\alpha(V)$ and $\beta(U)$, $\beta(V)$ are. In Section 2 we give upper bounds for $\alpha(U)$ and $\beta(U)$ where U is a body of the form

$$\{(x_1, \ldots, x_n) \in \boldsymbol{R}^n \colon |a_1 x_1|^p + \cdots + |a_n x_n|^p \leq 1\} \qquad (a_1, \ldots, a_n > 0; 1 \leq p \leq \infty). \tag{1}$$

In Section 3 we apply the results of Sections 1 and 2 to obtain bounds for $\mathrm{jh}(U, V)$.

There is also another source of motivation for considering the quantities $\mathrm{jh}(U, V)$. Let D be an n-dimensional ellipsoid in $\boldsymbol{R}^n$ with center at zero and principal semiaxes $\xi_1 \leq \cdots \leq \xi_n$. The theory of additive subgroups of topological vector spaces presented in the monograph [2] is, in fact, based on the bounds

$$\mathrm{kh}(B_2^n, D) \leq C \sum_{k=1}^{n} k\xi_k^{-1}, \tag{2}$$

$$\mathrm{lh}(B_2^n, D) \leq C\left(\sum_{k=1}^{n} k^2\xi_k^{-2} \right)^{1/2} \tag{3}$$

(see the final remarks in Section 3 of [2]). Inequality (2) is also the basic tool in [3], [6], and [7]. In this paper we show that

$$\mathrm{jh}(B_2^n, D) \leq C \sum_{k=1}^{n} \xi_k^{-1}, \qquad \mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}. \tag{4}$$

Inequality (4) with $\mathrm{j} = \mathrm{k}$ implies that a nuclear operator acting between Hilbert spaces has the property that inverse images of closed additive subgroups are weakly closed (see Remark 1.9 of [1]). Bounds for $\mathrm{kh}(U, V)$, where U, V are of the form (1), yield similar information for diagonal operators between spaces l_p, $1 \leq p \leq \infty$.

Another set of questions is connected with the possibility of extending continuous characters and unitary representations defined on additive subgroups of nuclear spaces, and with the generalization of the Minlos theorem on positive-definite functions to subgroups of nuclear spaces (see Chapter 4 of [2]). The problem of characterizing the corresponding classes of linear operators acting between Banach spaces (in the case of the Minlos theorem, an analogue of radonifying operators) leads in a natural way to upper bounds for $\mathrm{lh}(U, V)$. The quantity $\mathrm{lh}(U, V)$ itself is directly related to extending characters defined on discrete subgroups of Banach spaces (see p. 43 of [2]).

1. Preliminaries

The inner product of vectors x, $y \in \boldsymbol{R}^n$ is denoted by xy. We write x^2 instead of xx. It is convenient to denote

$$\varrho(A) = \sum_{x \in A} e^{-\pi x^2} \qquad (A \subset \boldsymbol{R}^n).$$

Let L be a lattice in $\boldsymbol{R}^n$. By σ_L we denote the probability measure on L given by the formula

$$\sigma_L(A) = \frac{\varrho(A)}{\varrho(L)} \qquad (A \subset L).$$

The Fourier transform $\hat{\sigma}_L$ of σ_L is given by the formula

$$\hat{\sigma}_L(u) = \int_{\boldsymbol{R}^n} e^{2\pi i u x}\, d\sigma_L(x) = \sum_{x \in L} \cos 2\pi u x \sigma_L(\{x\}) \qquad (u \in \boldsymbol{R}^n).$$

By φ_L we denote the function on $\boldsymbol{R}^n$ defined by the formula

$$\varphi_L(u) = \frac{\varrho(L+u)}{\varrho(L)} \qquad (u \in \boldsymbol{R}^n).$$

We sometimes write L_u instead of $L + u$ for $L \in \mathscr{L}_n$ and $u \in \boldsymbol{R}^n$.

Lemma 1.1. *One has $\hat{\sigma}_L = \varphi_{L^*}$ for each $L \in \mathscr{L}_n$.*

This is Corollary (1.2) of [4].

Corollary 1.2. *One has $\varphi_L(u) \leq \varphi_L(0)$ for all $L \in \mathscr{L}_n$ and $u \in \boldsymbol{R}^n$.*

Proof. The function φ_L is positive-definite being, due to (1.1), the Fourier transform of the positive measure σ_{L^*}. □

Let U be an o-symmetric convex body in $\boldsymbol{R}^n$. We denote

$$\alpha(U) = \sup_{L \in \mathscr{L}_n} \frac{\varrho(L \setminus U)}{\varrho(L)} = \sup_{L \in \mathscr{L}_n} \sigma_L(L \setminus U),$$

$$\beta(U) = \sup_{L \in \mathscr{L}_n} \sup_{u \in \boldsymbol{R}^n} \frac{\varrho(L_u \setminus U)}{\varrho(L)}.$$

Lemma 1.3. *Let $L \in \mathscr{L}_n$, $U \in \mathscr{C}_n$, and $u \in \boldsymbol{R}^n$. If $u \notin L + U$, then $\varphi_L(u) \leq \beta(U)$.*

Proof. If $u \notin L + U$, then $L + u = (L + u) \setminus U$ and, according to our definitions, we may write

$$\varphi_L(u) = \frac{\varrho(L+u)}{\varrho(L)} = \frac{\varrho(L_u \setminus U)}{\varrho(L)} \leq \beta(U). \qquad \square$$

Lemma 1.4. *If $U, V \in \mathscr{C}_n$ and $2\beta(U) + 3\alpha(V) \leq 1$, then* $\mathrm{kh}(U, V) \leq 6$.

Proof. Take arbitrary $L \in \mathscr{L}_n$, $u \in \boldsymbol{R}^n \setminus L$, and $\varepsilon > 0$. We have to find some $v \in L^*$, with $uv \notin \boldsymbol{Z}$, such that

$$\frac{d_U(u, L)\|v\|_V}{d(uv, \boldsymbol{Z})} < 6(1 + \varepsilon). \tag{5}$$

We may assume that $d_U(u, L) = 1 + \varepsilon$, otherwise we would replace u by tu and L by tL for a suitably chosen $t > 0$. Then $u \notin L + U$, and Lemma 1.3 implies that $\varphi_L(u) \le \beta(U)$. Denote $s = \min_{x \in L^* \cap V} \cos 2\pi ux$. Then we may write

$$\begin{aligned}\hat{\sigma}_{L^*}(u) &= \sum_{x \in L^*} \sigma_{L^*}(\{x\}) \cos 2\pi ux \\ &= \sum_{x \in L^* \cap V} + \sum_{x \in L^* \setminus V} \sigma_{L^*}(\{x\}) \cos 2\pi ux > s\sigma_{L^*}(L^* \cap V) - \sigma_{L^*}(L^* \setminus V) \\ &= s - (1+s)\sigma_{L^*}(L^* \setminus V) \ge s - (1+s)\alpha(V).\end{aligned}$$

Lemma 1.1 says that $\hat{\sigma}_{L^*}(u) = \varphi_L(u)$. Thus $s[1 - \alpha(V)] < \alpha(V) + \beta(U)$. Since $2\beta(U) + 3\alpha(V) \le 1$, it follows that $s < \frac{1}{2}$. So, there is some $v \in L^* \cap V$ with $\cos 2\pi uv < \frac{1}{2}$. Then $\|v\|_V \le 1$ and $d(uv, \mathbf{Z}) > \frac{1}{6}$, which yields (5). □

Lemma 1.5. *If $U, V \in \mathscr{C}_n$ and $\beta(U) + 2\alpha(V) \le 1$, then* $\mathrm{lh}(U, V) \le 1$.

Proof. Suppose that $\mathrm{lh}(U, V) > 1$. Then we can find a lattice $L \in \mathscr{L}_n$ with $\mu(L, U) > 1$ and $\lambda_1(L^*, V) > 1$. The first condition means that there is some $u \in \mathbf{R}^n \setminus (L + U)$. Hence $\varphi_L(u) \le \beta(U)$ due to Lemma 1.3. On the other hand, the condition $\lambda_1(L^*, V) > 1$ implies that $L^* \cap V = \{0\}$, and then

$$\begin{aligned}\hat{\sigma}_{L^*}(u) &= \sum_{x \in L^*} \sigma_{L^*}(\{x\}) \cos 2\pi ux \\ &= \sum_{x \in L^* \cap V} + \sum_{x \in L^* \setminus V} \sigma_{L^*}(\{x\}) \cos 2\pi ux > \sigma_{L^*}(L^* \cap V) - \sigma_{L^*}(L^* \setminus V) \\ &= 1 - 2\sigma_{L^*}(L^* \setminus V) \ge 1 - 2\alpha(V).\end{aligned}$$

Thus, by Lemma 1.1, we have $1 - 2\alpha(V) < \hat{\sigma}_{L^*}(u) = \varphi_{L^{**}}(u) = \varphi_L(u) > \beta(U)$. □

Lemma 1.6. *Let B be the euclidean unit ball in $\mathbf{R}^n$. If $U, V \in \mathscr{C}_n$ and $2\alpha(U) + \beta(V) \le 1 - e^{-\pi}$, then* $\mathrm{mh}(U, V + B) \le 1$.

Proof. Suppose that $\mathrm{mh}(U, V + B) > 1$. Then we can find some $L \in \mathscr{L}_n$ and $i = 1, \ldots, n$ with $\lambda_i(L, U) > 1$ and $\lambda_{n-i+1}(L^*, V + B) > 1$. Denote $M = \mathrm{span}(U \cap L)$ and $N = \mathrm{span}((V + B) \cap L^*)$; then $\dim M \le i - 1$ and $\dim N \le n - i$. So, denoting the orthogonal complements of M and N in $\mathbf{R}^n$ by $M^\perp$ and $N^\perp$, respectively, we have $\dim M^\perp + \dim N^\perp \ge n + 1$. Consequently, there is some $u \in M^\perp \cap N^\perp$ with $u^2 = 1$. Then

$$\begin{aligned}\sum_{x \in L_u^*} e^{-\pi x^2} = \sum_{x \in L^*} e^{-\pi(x+u)^2} &= \sum_{x \in L^* \cap N} + \sum_{x \in L^* \setminus N} e^{-\pi(x+u)^2} \\ &= \sum_{x \in L^* \cap N} e^{-\pi x^2} e^{-\pi u^2} + \sum_{x \in (L^* \setminus N) + u} e^{-\pi x^2} \\ &< e^{-\pi} \sum_{x \in L^*} e^{-\pi x^2} + \sum_{x \in L_u^* \setminus V} e^{-\pi x^2}\end{aligned}$$

because $(L^* \setminus N) + u \subset L_u^* \setminus V$. Hence

$$\varphi_{L^*}(u) = \frac{\varrho(L^* + u)}{\varrho(L^*)} < e^{-\pi} + \frac{\varrho(L_u^* \setminus V)}{\varrho(L^*)} \leq e^{-\pi} + \beta(V). \tag{6}$$

On the other hand, as $u \in M^{\perp}$ and $L \setminus M \subset L \setminus U$, we have

$$\begin{aligned}\hat{\sigma}_L(u) &= \sum_{x \in L} \sigma_L(\{x\}) \cos 2\pi ux \\ &= \sum_{x \in L \cap M} + \sum_{x \in L \setminus M} \sigma_L(\{x\}) \cos 2\pi ux > \sigma_L(L \cap M) - \sigma_L(L \setminus M) \\ &= 1 - 2\sigma_L(L \setminus M) \geq 1 - 2\sigma_L(L \setminus U).\end{aligned}$$

In view of (6) and Lemma 1.1, this implies that $1 - 2\alpha(U) < e^{-\pi} + \beta(V)$. □

2. Bounds for $\alpha(U)$ and $\beta(U)$.

In this section n is a fixed positive integer. It is convenient to denote the kth coordinate of a vector $x \in \boldsymbol{R}^n$ by x_k, i.e., to write $x = (x_1, \ldots, x_n)$. Let us denote

$$A = \{a \in \boldsymbol{R}^n \colon\ a_k > 0 \text{ for } k = 1, \ldots, n\}.$$

For each $a \in A$, we define

$$U_p^a = \left\{x \in \boldsymbol{R}^n \colon\ \sum_{k=1}^{n} |a_k x_k|^p \leq 1\right\} \qquad (1 \leq p < \infty),$$

$$U_\infty^a = \{x \in \boldsymbol{R}^n \colon\ |a_k x_k| \leq 1 \text{ for } k = 1, \ldots, n\}.$$

Lemma 2.1. *Let L be a lattice and let u be an arbitrary vector in $\boldsymbol{R}^n$. Then*

$$\sum_{x \in L_u} x_k^2 e^{-tx^2} \leq \frac{1}{t} \sum_{x \in L} e^{-tx^2} \qquad (t > 0; k = 1, \ldots, n).$$

If $u = 0$, the coefficient $1/t$ may be replaced by $1/2t$.

This is Lemma 1.3 of [4].

Corollary 2.2. *For each $a \in A$, one has:*

(i) $\alpha(U_2^a) \leq a^2/2\pi$.
(ii) $\beta(U_2^a) \leq a^2/\pi$.

Proof. Let us take arbitrary $L \in \mathscr{L}_n$, $u \in \boldsymbol{R}^n$, and $a \in A$. By Lemma 2.1, we have

$$\varrho(L_u \setminus U_2^a) = \sum_{x \in L_u \setminus U_2^a} e^{-\pi x^2} < \sum_{x \in L_u \setminus U_2^a} \left(\sum_{k=1}^{n} a_k^2 x_k^2 \right) e^{-\pi x^2}$$

$$\leq \sum_{k=1}^{n} a_k^2 \sum_{x \in L_u} x_k^2 e^{-\pi x^2} \leq \frac{a^2}{\pi} \sum_{x \in L} e^{-\pi x^2} = \frac{a^2}{\pi} \varrho(L).$$

This proves (ii). The proof of (i) differs in the factor $\frac{1}{2}$. □

Remark 2.3. An argument similar to that used in the proof of (1.4) in [4] allows it to be shown that

$$\alpha(U_2^a) \leq \sqrt{\frac{2\pi e}{a^2}}\, e^{-\pi/a^2}, \qquad \beta(U_2^a) \leq 2\sqrt{\frac{2\pi e}{a^2}}\, e^{-\pi/a^2},$$

provided that $a^2 \leq 2\pi$; if $a_1 = \cdots = a_n$, see Lemma 2.8 below.

Lemma 2.4. *Let L be a lattice and let u be an arbitrary vector in $\boldsymbol{R}^n$. Then*

$$\sum_{\substack{x \in L_u \\ |x_k| \geq t}} e^{-\pi x^2} < 2e^{-\pi t^2} \sum_{x \in L} e^{-\pi x^2} \qquad (t \geq 0;\, k = 1, \ldots, n).$$

Proof. Fix arbitrary $t \geq 0$ and $k = 1, \ldots, n$. Let $v \in \boldsymbol{R}^n$ be the vector given by $xv = x_k$ for $x \in \boldsymbol{R}^n$. We may write

$$\sum_{x \in L_u} e^{-\pi x^2} \cosh 2\pi t x_k = \tfrac{1}{2} e^{\pi t^2} \left[\sum_{x \in L_u} e^{-\pi(x - tv)^2} + \sum_{x \in L_u} e^{-\pi(x + tv)^2} \right]$$

$$= \tfrac{1}{2} e^{\pi t^2} \left[\sum_{x \in L_u - tv} e^{-\pi x^2} + \sum_{x \in L_u + tv} e^{-\pi x^2} \right]$$

$$= e^{\pi t^2} \frac{\varrho(L + u - tv) + \varrho(L + u + tv)}{2} \leq e^{\pi t^2} \varrho(L)$$

due to (1.2). Denote $Q = \{x \in \boldsymbol{R}^n \colon |x_k| \geq t\}$. Then

$$\sum_{x \in L_u} e^{-\pi x^2} \cosh 2\pi t x_k \geq \sum_{x \in Q \cap L_u} e^{-\pi x^2} \cosh 2\pi t x_k$$

$$> \cosh 2\pi t^2 \sum_{x \in Q \cap L_u} e^{-\pi x^2} = \varrho(Q \cap L_u) \cosh 2\pi t^2.$$

Consequently, we derive

$$\frac{\varrho(Q \cap L_u)}{\varrho(L)} < \frac{e^{\pi t^2}}{\cosh 2\pi t^2} < 2e^{-\pi t^2}. \qquad \square$$

Corollary 2.5. *For each $a \in A$, one has*

$$\beta(U_\infty^a) \le 2 \sum_{k=1}^{n} e^{-\pi/a_k^2}.$$

Proof. Let us take arbitrary $L \in \mathscr{L}_n$, $u \in \boldsymbol{R}^n$, and $a \in A$. We have to show that

$$\frac{\varrho(L_u \setminus U_\infty^a)}{\varrho(L)} \le 2 \sum_{k=1}^{n} e^{-\pi/a_k^2}.$$

Denote

$$Q_k = \{x \in \boldsymbol{R}^n \colon |x_k| \ge a_k^{-1}\} \qquad (k = 1, \dots, n).$$

It follows from Lemma 2.4 that

$$\varrho(L_u \cap Q_k) < 2\varrho(L) e^{-\pi/a_k^2} \qquad (k = 1, \dots, n).$$

We have

$$L_u \setminus U_\infty^a = L_u \cap (Q_1 \cup \cdots \cup Q_n) = \bigcup_{k=1}^{n} (L_u \cap Q_k).$$

Thus

$$\varrho(L_u \setminus U_\infty^a) \le \sum_{k=1}^{n} \varrho(L_u \cap Q_k) < 2\varrho(L) \sum_{k=1}^{n} e^{-\pi/a_k^2}. \qquad \square$$

Lemma 2.6. *Let L be a lattice and let u be an arbitrary vector in $\boldsymbol{R}^n$. Then*

$$\sum_{x \in L_u} |x_k|^p e^{-\pi x^2} < p\pi^{-p/2} \Gamma\left(\frac{p}{2}\right) \sum_{x \in L} e^{-\pi x^2} \qquad (p > 0; k = 1, \dots, n).$$

For $p = 2$, see Lemma 2.1.

Proof. Choose arbitrary $p > 0$ and $k = 1, \dots, n$. By virtue of Lemma 2.4, we may write

$$\begin{aligned}\sum_{x \in L_u} |x_k|^p e^{-\pi x^2} &= \sum_{x \in L_u} |x_k|^p \varrho(\{x\}) \\ &= p\int_0^\infty t^{p-1} \varrho(\{x \in L_u \colon |x_k| \ge t\})\, dt < 2p \sum_{x \in L} e^{-\pi x^2} \int_0^\infty t^{p-1} e^{-\pi t^2}\, dt \\ &= p\pi^{-p/2} \Gamma\left(\frac{p}{2}\right) \sum_{x \in L} e^{-\pi x^2}. \qquad \square\end{aligned}$$

Corollary 2.7. *For arbitrary $a \in A$ and $p \in [1,\infty)$, one has*

$$\beta(U_p^a) \le p\pi^{-p/2}\Gamma\left(\frac{p}{2}\right)\sum_{k=1}^{n} a_k^p.$$

Proof. Let us take arbitrary $L \in \mathscr{L}_n$, $u \in \boldsymbol{R}^n$, $a \in A$, and $p \in [1,\infty)$. We have to show that

$$\frac{\varrho(L_u \setminus U_p^a)}{\varrho(L)} \le p\pi^{-p/2}\Gamma\left(\frac{p}{2}\right)\sum_{k=1}^{n} a_k^p.$$

By Lemma 2.6, we may write

$$\varrho(L_u \setminus U_p^a) = \sum_{x \in L_u \setminus U_p^a} e^{-\pi x^2} < \sum_{x \in L_u \setminus U_p^a}\left(\sum_{k=1}^{n} a_k^p |x_k|^p\right) e^{-\pi x^2}$$

$$\le \sum_{k=1}^{n} a_k^p \sum_{x \in L_u} |x_k|^p e^{-\pi x^2} < p\pi^{-p/2}\Gamma\left(\frac{p}{2}\right)\varrho(L)\sum_{k=1}^{n} a_k^p. \qquad \square$$

Let us denote

$$B_p^n = \left\{x \in \boldsymbol{R}^n\colon \sum_{k=1}^{n} |x_k|^p \le 1\right\} \qquad (1 \le p < \infty),$$

$$B_\infty^n = \{x \in \boldsymbol{R}^n\colon |x_k| \le 1 \text{ for } k = 1,\dots,n\}.$$

Lemma 2.8. *For reach $r \ge \sqrt{n/2\pi}$, one has*

$$\alpha(rB_2^n) < \left(\frac{2\pi e}{n}\right)^{n/2} r^n e^{-\pi r^2}, \qquad \beta(rB_2^n) < 2\left(\frac{2\pi e}{n}\right)^{n/2} r^n e^{-\pi r^2}.$$

This is Lemma 1.5 of [4].

Lemma 2.9. *For arbitrary $r > 0$ and $p \in [1,\infty)$, one has*

$$\beta(rB_p^n) < pn\pi^{-p/2}\Gamma\left(\frac{p}{2}\right)r^{-p}.$$

This is a direct consequence of Corollary 2.7.

Lemma 2.10. *For each $r > 0$, one has*

$$\beta(rB_\infty^n) < 2ne^{-\pi r^2}$$

This is a direct consequence of Corollary 2.5.

3. Transference Theorems

The results of Sections 1 and 2 allow upper bounds for $\mathrm{jh}(U, V)$, for various pairs U, $V \in \mathscr{C}_n$, to be obtained. Here we confine ourselves to consideration of a few most important cases.

Theorem 3.1. *Let D be an n-dimensional o-symmetric ellipsoid in $\boldsymbol{R}^n$ with principal semiaxes $d_1, \dots, d_n$. Denote $d = (d_1^{-1} + \cdots + d_n^{-1})^{-1}$. Then:*

(i) $\mathrm{kh}(B_2^n, D) \leq 21/\pi d$.
(ii) $\mathrm{lh}(B_2^n, D) \leq 2/\pi d$.
(iii) $\mathrm{mh}(B_2^n, D) \leq 3/2d$.

Proof. We may assume that

$$D = \left\{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon \frac{x_1^2}{d_1^2} + \cdots + \frac{x_n^2}{d_n^2} \leq 1\right\}.$$

Consider the ellipsoid

$$C = \left\{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon \frac{x_1^2}{d_1} + \cdots + \frac{x_n^2}{d_n} \leq 1\right\}.$$

It is clear that $\mathrm{kh}(B_2^n, D) = \mathrm{kh}(C, C)$ and $\mathrm{lh}(B_2^n, D) = \mathrm{lh}(C, C)$. By Corollary 2.2, we have $\alpha(C) \leq 1/2\pi d$ and $\beta(C) \leq 1/\pi d$.

If $d \geq 7/2\pi$, then $2\beta(C) + 3\alpha(C) \leq 1$, and Lemma 1.4 yields $\mathrm{kh}(C, C) \leq 6$. Thus, if $d \geq 7/2\pi$, then $\mathrm{kh}(B_2^n, D) \leq 6$; this proves (i).

If $d \geq 2/\pi$, then $\beta(C) + 2\alpha(C) \leq 1$, and Lemma 1.5 implies that $\mathrm{lh}(C, C) \leq 1$. So, if $d \geq 2/\pi$, then $\mathrm{lh}(B_2^n, D) \leq 1$; this proves (ii).

To prove (iii), assume that $d = \frac{3}{2}$. Denote $U = \frac{2}{3}C$ and $W = \frac{3}{2}C$. Then $\mathrm{mh}(B_2^n, D) = \mathrm{mh}(U, W)$. The principal semiaxes of D are greater than d. Consequently, those of C are greater than $d^{1/2}$, so that $B_2^n \subset d^{-1/2}C$. Then

$$U + B_2^n \subset \left(\frac{2}{3} + \frac{1}{\sqrt{d}}\right)C \subset \tfrac{3}{2}C = W.$$

Thus $\mathrm{mh}(B_2^n, D) \leq \mathrm{mh}(U, U + B_2^n)$. By Corollary 2.2, we have $\alpha(U) \leq 3/4\pi$ and $\beta(U) \leq 3/2\pi$, whence $2\alpha(U) + \beta(U) \leq 3/\pi < 1 - \mathrm{e}^{-\pi}$, and Lemma 1.6 implies that $\mathrm{mh}(U, U + B_2^n) \leq 1$.

We have shown that if $d = \frac{3}{2}$, then $\mathrm{mh}(B_2^n, D) \leq 1$; this proves (iii). □

Theorem 3.2. *Let $a_1, \dots, a_n$ be arbitrary positive numbers. Denote*

$$P = \{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon |x_k| \leq a_k \text{ for } k = 1, \dots, n\}.$$

Let t_1, t_2, t_3 be the roots of the equations

$$\sum_{k=1}^{n} e^{-\pi t a_k} = \tfrac{1}{10}, \qquad \sum_{k=1}^{n} e^{-\pi t a_k} = \tfrac{1}{6}, \qquad \sum_{k=1}^{n} e^{-\pi(\sqrt{t a_k + 1/4} - 1/2)^2} = \frac{1 - e^{-\pi}}{6},$$

respectively. Then

$$\mathrm{kh}(B_\infty^n, P) \le 6t_1, \qquad \mathrm{lh}(B_\infty^n, P) \le t_2, \qquad \mathrm{mh}(B_n^\infty, P) \le t_3.$$

Proof. Let us denote

$$Q_i = \left\{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon |x_k| \le \sqrt{t_i p_k} \text{ for } k = 1, \dots, n\right\}$$

for $i = 1, 2$. It is clear that

$$\mathrm{kh}(B_\infty^n, t_1 P) = \mathrm{kh}(Q_1, Q_1), \qquad \mathrm{lh}(B_\infty^n, t_2 P) = \mathrm{lh}(Q_2, Q_2).$$

From Corollary 2.5 and our definitions of t_1 and t_2 we get

$$\alpha(Q_1) \le \beta(Q_1) \le \tfrac{1}{5}, \qquad \alpha(Q_2) \le \beta(Q_2) \le \tfrac{1}{3}.$$

Now, from Lemma 1.4 we obtain $\mathrm{kh}(Q_1, Q_1) \le 6$, while Lemma 1.5 yields $\mathrm{lh}(Q_2, Q_2) \le 1$. Thus

$$\mathrm{kh}(B_\infty^n, P) = t_1\,\mathrm{kh}(B_\infty^n, t_1 P) = t_1\,\mathrm{kh}(Q_1, Q_1) \le 6t_1,$$

$$\mathrm{lh}(B_\infty^n, P) = t_2\,\mathrm{lh}(B_\infty^n, t_2 P) = t_2\,\mathrm{lh}(Q_2, Q_2) \le t_2.$$

Next, let us define

$$u_k = \sqrt{t_3 a_k + \tfrac{1}{4}} - \tfrac{1}{2}, \qquad w_k = \sqrt{t_3 a_k + \tfrac{1}{4}} + \tfrac{1}{2} \qquad (k = 1, \dots, n),$$

$$U = \{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon |x_k| \le u_k \text{ for } k = 1, \dots, n\},$$

$$W = \{(x_1, \dots, x_n) \in \boldsymbol{R}^n \colon |x_k| \le w_k \text{ for } k = 1, \dots, n\}.$$

We have $u_k w_k = t_3 p_k$ for every k, therefore $\mathrm{mh}(B_\infty^n, t_3 P) = \mathrm{mh}(U, W)$. As $w_k = u_k + 1$, it follows that $U + B_2^n \subset U + B_\infty^n \subset W$. From Corollary 2.5 and our definition of t_3 we get $\alpha(U) \le \beta(U) \le (1 - e^{-\pi})/6$, and Lemma 1.6 implies that $\mathrm{mh}(U, U + B_2^n) \le 1$. Thus

$$\mathrm{mh}(B_\infty^n, P) = t_3\,\mathrm{mh}(B_\infty^n, t_3 P) = t_3\,\mathrm{mh}(U, W) \le t_3\,\mathrm{mh}(U, U + B_2^n) \le t_3. \qquad \square$$

For each pair U, $V \in \mathscr{C}_n$, let us denote

$$\mathrm{nh}(U, V) = \max\{\mathrm{jh}(U, V) \colon \mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}\},$$

$$\mathrm{ph}(U, V) = \max(\mathrm{nh}(U, V), \mathrm{nh}(V, U)).$$

Proposition 3.3. *A numerical constant C exists such that*

$$\mathrm{ph}(B_p^n, B_q^n) \le C\sqrt{pq}\, n^{1/p+1/q} \qquad (1 \le p, q < \infty;\ n = 1,2,\dots).$$

This follows directly from Lemmas 1.4–1.6 and 2.9.

Corollary 3.4. *A numerical constant C exists such that*

$$\mathrm{ph}\left(B_p^n, (B_p^n)^0\right) \le Cn\sqrt{\frac{p^2}{p-1}} \qquad (1 < p < \infty;\ n = 1,2,\dots).$$

Proof. It is enough to observe that $(B_p^n)^0 = B_q^n$ where $q = p/(p-1)$. □

Proposition 3.5. *A numerical constant C exists such that*

$$\mathrm{ph}(B_p^n, B_\infty^n) \le C\sqrt{p}\, n^{1/p}(\log n)^{1/2} \qquad (1 \le p < \infty;\ n = 1,2,\dots).$$

This is a direct consequence of Lemmas 1.4–1.6, 2.9, and 2.10.

Proposition 3.6. *A numerical constant C exists such that*

$$\mathrm{ph}\left(B_p^n, (B_p^n)^0\right) \le Cn(\log n)^{1/2} \qquad (1 \le p \le \infty;\ n = 1,2,\dots).$$

Proof. Take an arbitrary $p \in [1, \infty]$ and let $q = p/(p-1)$. Then $(B_p^n)^0 = B_q^n$. We may assume that $p \le 2 \le q$. Let $r = n^{1/2-1/p}$ and $s = n^{-1/q}$. Then $rB_2^n \subset B_p^n$ and $sB_\infty^n \subset B_q^n$. Due to Proposition 3.5, a numerical constant C exists such that

$$\mathrm{ph}(B_2^n, B_\infty^n) \le Cn^{1/2}(\log n)^{1/2}$$

for every n. Thus

$$\begin{aligned}\mathrm{ph}(B_p^n, B_q^n) &\le \mathrm{ph}(rB_2^n, sB_\infty^n) = \frac{1}{rs}\mathrm{ph}(B_2^n, B_\infty^n)\\ &\le Cn^{1/2}(\log n)^{1/2} n^{1/p-1/2} n^{1/q} = Cn(\log n)^{1/2}\end{aligned}$$ □

Proposition 3.7. *A numerical constant C exists such that*

$$\mathrm{ph}(B_\infty^n, B_\infty^n) \le C \log n \qquad (n = 1,2,\dots).$$

This is a consequence of Theorem 3.2.

Corollary 3.8. *Let U be a convex body in $\boldsymbol{R}^n$ symmetric with respect to the coordinate hyperplanes. Then*

$$\mathrm{ph}(U, U^0) \le Cn \log n,$$

where C is a numerical constant.

Proof. It is a standard fact that a linear isomorphism $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ exists such that $TU, (TU)^0 \supset n^{-1/2}B_\infty^n$. Thus by Proposition 3.7, we have

$$\mathrm{ph}(U, U^0) = \mathrm{ph}(TU, (TU)^0) \le \mathrm{ph}(n^{-1/2}B_\infty^n, n^{-1/2}B_\infty^n) \le Cn \log n. \qquad \square$$

Remark 3.9. To each $U \in \mathscr{C}_n$ there corresponds a linear isomorphism $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that

$$B_\infty^n \subset T(U) \subset n\,\text{ké}(\boldsymbol{R}_U^n)B_1^n,$$

where $\text{ké}(\boldsymbol{R}_U^n)$ is the so-called ké constant of $\boldsymbol{R}_U^n$, introduced and investigated in [15]. Then

$$T(U)^0 \supset (n\,\text{ké}(\boldsymbol{R}_U^n)B_1^n)^0 = [n\,\text{ké}(\boldsymbol{R}_U^n)]^{-1}B_\infty^n,$$

and, by Proposition 3.7, we have

$$\mathrm{ph}(U, U^0) = \mathrm{ph}(T(U), T(U^0)) \le n\,\text{ké}(\boldsymbol{R}_U^n)\,\mathrm{ph}(B_\infty^n, B_\infty^n) \le Cn \log n\,\text{ké}(\boldsymbol{R}_U^n).$$

Thus

$$\mathrm{jh}(U) \le Cn \log n\,\text{ké}(\boldsymbol{R}_U^n), \qquad \mathrm{j} = \mathrm{k, l, m}.$$

Remark 3.10. A standard argument based on Siegel's mean value theorem shows that to each pair $U, V \in \mathscr{C}_n$ there corresponds a lattice $L \in \mathscr{L}_n$ such that

$$\lambda_1(L, U)\lambda_1(L^*, V) \ge [\mathrm{vol}_n(U)\mathrm{vol}_n(V)]^{-1/n},$$

where vol_n is the n-dimensional Lebesgue measure on $\boldsymbol{R}^n$ (the proof is given in [5]). This yields lower bounds for $\mathrm{jh}(U, V)$ which are not very far from the upper bounds given in Theorems 3.1 and 3.2, Propositions 3.3 and 3.5–3.7, and Corollaries 3.4 and 3.8. For instance, under the notation of Theorem 3.1, we obtain

$$\mathrm{jh}(B_2^n, D) \ge Cn(d_1 \cdots d_n)^{-1/n}.$$

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Received May 25, 1993, and in revised form August 29, 1994.