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Inequalities for Convex Bodies and Polar Reciprocal Lattices in R^n II: Application of K-Convexity*

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Abstract. The paper is a supplement to [2]. Let L be a lattice and U an o-symmetric convex body in $\boldsymbol{R}^n$. The Minkowski functional $\| \|_U$ of U, the polar body U^0, the dual lattice L^*, the covering radius $\mu(L, U)$, and the successive minima $\lambda_i(L, U)$, $i = 1, \dots, n$, are defined in the usual way. Let $\mathcal{L}_n$ be the family of all lattices in $\boldsymbol{R}^n$. Given a convex body U, we define

$$\mathrm{mh}(U) = \sup_{L\in\mathcal{L}_n} \max_{1\le i\le n} \lambda_i(L, U)\,\lambda_{n-i+1}(L^*, U^0),$$

$$\mathrm{lh}(U) = \sup_{L\in\mathcal{L}_n} \lambda_1(L, U)\cdot\mu(L^*, U^0),$$

and $\mathrm{kh}(U)$ is defined as the smallest positive number s for which, given arbitrary $L \in \mathcal{L}_n$ and $x \in \boldsymbol{R}^n\backslash(L + U)$, some $y \in L^*$ with $\|y\|_{U^0} \le s\,d(xy, \boldsymbol{Z})$ can be found. It is proved that

$$C_1 n \le \mathrm{jh}(U) \le C_2 n K(\boldsymbol{R}^n_U) \le C_3 n(1 + \log n),$$

for j = k, l, m, where C_1, C_2, C_3 are some numerical constants and $K(\boldsymbol{R}^n_U)$ is the K-convexity constant of the normed space $(\boldsymbol{R}^n, \| \|_U)$. This is an essential strengthening of the bounds obtained in [2]. The bounds for $\mathrm{lh}(U)$ are then applied to improve the results of Kannan and Lovász [5] estimating the lattice width of a convex body U by the number of lattice points in U.

This paper is a supplement to the earlier paper [2]. We recall briefly the notation introduced there. By $\mathcal{L}_n$ and $\mathcal{C}_n$ we denote, respectively, the family of all n-dimensional lattices and the family of all symmetric convex bodies in $\boldsymbol{R}^n$. Let $L \in \mathcal{L}_n$ and $U \in \mathcal{C}_n$. By L^* and U^0 we denote, respectively, the dual lattice and the polar body, defined in

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the usual way. By $\mu(L, U)$ and $\lambda_i(L, U)$, $i = 1, \ldots, n$, we denote, respectively, the covering radius and the successive minima of L with respect to U.

Consider a pair of convex bodies $U, V \in \mathcal{C}_n$. Let d_U be the metric on $\boldsymbol{R}^n$ induced by U and let $\| \|_V$ be the norm on $\boldsymbol{R}^n$ induced by V. Let xy be the euclidean inner product of vectors $x, y \in \boldsymbol{R}^n$ and let $d(xy, \boldsymbol{Z})$ be the usual one-dimensional distance of xy to $\boldsymbol{Z}$. In [2] we obtained upper bounds for the following quantities:

$$\begin{aligned} \mathrm{kh}(U, V) &= \sup_{L \in \mathcal{L}_n} \sup_{\substack{x \in \boldsymbol{R}^n \\ x \notin L}} \inf_{\substack{y \in L^* \\ xy \notin \mathbb{Z}}} \frac{d_U(x, L) \cdot \|y\|_V}{d(xy, \mathbb{Z})}, \\ \mathrm{lh}(U, V) &= \sup_{L \in \mathcal{L}_n} \lambda_1(L, U) \cdot \mu(L^*, V), \\ \mathrm{mh}(U, V) &= \sup_{L \in \mathcal{L}_n} \max_{1 \le i \le n} \lambda_i(L, U) \cdot \lambda_{n-i+1}(L^*, V). \end{aligned}$$

It is convenient to denote them by $\mathrm{jh}(U, V)$, $\mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}$. In this paper, applying the notion of K-convexity, we derive upper bounds for $\mathrm{jh}(U, V)$ which are essentially stronger than those obtained in [2]. We are interested mainly in the case $V = U^0$. Thus, we denote $\mathrm{jh}(U) = \mathrm{jh}(U, U^0)$ for $\mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}$ and $U \in \mathcal{C}_n$. Naturally, $\mathrm{jh}(U)$ are affine invariants of U. Upper bounds for $\mathrm{jh}(U)$ belong to the so-called transference theorems in the geometry of numbers; for motivations and earlier results we refer the reader to [2].

Let B_p^n denote the unit ball of the normed space l_p^n, $1 \le p \le \infty$ (we identify vector spaces l_p^n and $\boldsymbol{R}^n$). It was proved in [1] that

$$\mathrm{jh}(B_2^n) \le Cn. \tag{1}$$

Here and below, C is some numerical constant which may vary from line to line. Next, it was proved in [2] that

$$\mathrm{jh}(B_p^n) \le C_p n, \qquad 1 < p < \infty, \tag{2}$$

where C_p depends on p only, that

$$\mathrm{jh}(B_p^n) \le Cn(1 + \log n)^{1/2}, \qquad 1 \le p \le \infty,$$

and that $\mathrm{jh}(U) \le Cn(1 + \log n)$ provided that U is symmetric with respect to the coordinate hyperplanes. In this paper we prove that

$$C^{-1} n \le \mathrm{jh}(U) \le Cn(1 + \log n) \tag{3}$$

for any $U \in \mathcal{C}_n$. Then we apply upper bounds for $\mathrm{lh}(U)$ to improve the results of Kannan and Lovász [5] estimating the width of a convex body U by the number of lattice points in U.

The inequality on the left in (3) was announced in [1]; for $U = B_2^n$, it had been known earlier (see Chapter II, Theorem 9.5, of [8]). It is a direct consequence of Siegel's mean value theorem. The proof of the inequality on the right makes use of some results of [2], of dual properties of ℓ-norm, and of the theorem of M. Talagrand on majorizing measure.

Let A be a discrete subset of $\boldsymbol{R}^n$. It is convenient to write $\varrho(A) = \sum_{x \in A} e^{-\pi x^2}$. For $U \in \mathcal{C}_n$, we define

$$\beta(U) = \sup_{L \in \boldsymbol{R}^n} \sup_{a \in \boldsymbol{R}^n} \frac{\varrho((L + a) \backslash U)}{\varrho(L)}.$$

The following fact is a direct consequence of Lemmas 1.4, 1.5, and 1.6 of [2]:

Lemma 1. *There exists a numerical constant C_1 such that if $U, V \in \mathcal{C}_n$ and $\beta(U)$, $\beta(V) \le C_1^{-1}$, then* $\mathrm{jh}(U, V) \le C_1$ *for* $\mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}$.

For $U \in \mathcal{C}_n$, by $\boldsymbol{R}^n_U$ we denote the space $\boldsymbol{R}^n$ endowed with the norm $\| \, \|_U$. Endowed with the euclidean norm, $\boldsymbol{R}^n$ is denoted by $\boldsymbol{R}^n_2$. By T_U we denote the identity operator from $\boldsymbol{R}^n_2$ to $\boldsymbol{R}^n_U$, and $\ell(T_U)$ is the ℓ-norm of T_U (see Section 2 of [3], Chapter 3 of [10], (12.2) of [12], or (2.3.16) and (2.3.17) of [6]).

Lemma 2. *To each $\varepsilon > 0$ there corresponds some $\delta > 0$ such that if $U \in \mathcal{C}_n$ and $\ell(T_U) < \delta$, then $\beta(U) < \varepsilon$.*

Proof. Fix δ and take any $U \in \mathcal{C}_n$ with $\ell(T_U) < \delta$. The result of Talagrand [11] on majorizing measure implies that there is a sequence $x_k^* \in (\boldsymbol{R}^n_2)^*$ such that

$$\|x_k^*\| \le C\ell(T_U)\,(1 + \log k)^{-1/2}, \qquad k = 1, 2, \ldots,$$

and, denoting

$$W_k = \{x \in \boldsymbol{R}^n : |\langle x, x_k^* \rangle| \le 1\}, \qquad k = 1, 2, \ldots,$$

we have $W = \bigcap_{k=1}^\infty W_k \subset U$ (see pp. 128–129 of [11] and p. 85 of [10]). Here C is some numerical constant. Choose any $L \in \mathcal{L}_n$ and $a \in \boldsymbol{R}^n$. It is convenient to write L_a instead of $L + a$. Lemma 2.4 of [2] says that

$$\varrho(\{x \in L_a : |\langle x, x^* \rangle| \ge r\|x^*\|\}) < 2\,e^{-\pi r^2}\varrho(L), \qquad r > 0, \quad x^* \in (\boldsymbol{R}^n_2)^*.$$

Hence

$$\varrho(L_a \backslash W_k) < 2e^{-\pi \|x_k^*\|^{-2}}\varrho(L) \le 2\varrho(L) \cdot (ke)^{-\pi C^{-2}\delta^{-2}}$$

for $k = 1, 2, \ldots$, which implies that

$$\begin{aligned} \varrho(L_a \backslash U) &\le \varrho(L_a \backslash W) = \varrho\left(L_a \backslash \bigcap_{k=1}^{\infty} W_k\right) = \varrho\left(\bigcup_{k=1}^{\infty}(L_a \backslash W_k)\right) \\ &\le \sum_{k=1}^{\infty} \varrho(L_a \backslash W_k) \le 2\varrho(L) \cdot \sum_{k=1}^{\infty}(ke)^{-\pi C^{-2}\delta^{-2}} = 2\varrho(L) \cdot f(\delta). \end{aligned}$$

As $L \in \mathcal{L}_n$ and $a \in \boldsymbol{R}^n$ were arbitrary, it follows that $\beta(U) \le 2f(\delta)$. Now it remains to observe that $f(\delta) \to 0$ as $\delta \to 0$. □

Lemma 3. *There exists a universal constant C such that*

$$\mathrm{jh}(U, V) \le C\ell(T_U) \cdot \ell(T_V)$$

for all $U, V \in \mathcal{C}_n$ and $\mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}$.

Proof. Let C_1 be the constant from Lemma 1. By Lemma 2, there exists some $\delta > 0$ such that if $W \in \mathcal{C}_n$ and $\ell(T_W) \leq \delta$, then $\beta(W) \leq C_1^{-1}$. Choose arbitrary $U, V \in \mathcal{C}_n$ and $\mathrm{j} = \mathrm{k, l, m}$. Let $s = \delta^{-1}\ell(T_U)$ and $t = \delta^{-1}\ell(T_V)$. Then $\ell(T_{sU}) = s^{-1}\ell(T_U) = \delta$ and, similarly, $\ell(T_{tV}) = \delta$. Hence $\beta(sU), \beta(tV) \leq C_1^{-1}$, which implies that $\mathrm{jh}(sU, tV) \leq C_1$. Thus

$$\mathrm{jh}(U, V) = st \cdot \mathrm{jh}(sU, tV) \leq stC_1 = C_1\delta^{-2}\ell(T_U) \cdot \ell(T_V). \qquad \square$$

Let X be an n-dimensional real normed space. By $d(X, l_2^n)$ we denote the Banach–Mazur distance of X to l_2^n; note that $d(X, l_2^n) \leq n^{1/2}$. By $K(X)$ we denote the K-convexity constant of X (see, e.g., Chapter 2 of [10] or (2.2.19) of [6]). We recall some basic facts about $K(X)$:

(i) $K(X^*) = K(X)$.
(ii) $K(l_2^n) = 1$.
(iii) $K(X) \leq C(1 + \log d(X, l_2^n))$.
(iv) $K(X) \leq C(1 + \log d(X, l_2^n))^{1/2}$ if X has a 1-unconditional basis.

For (i)–(iii) see p. 20 of [10]; assertion (iv) was proved in [9].

Lemma 4. *To each $U \in \mathcal{C}_n$ there corresponds a linear isomorphism S of $\boldsymbol{R}^n$ such that*

$$\ell(T_{S(U)}) \cdot \ell(T_{S(U)^0}) \leq nK(\boldsymbol{R}_U^n).$$

This fact was proved by Figiel and Tomczak-Jaegermann in [3], by using a general theorem of D. R. Lewis. The isomorphism S describes the so-called ℓ-ellipsoid for U. See also (4.1.9) of [6], Theorem 3.11 of [10], or Section 12 of [12] for a detailed analysis.

Theorem 1. *There exists a universal constant C such that*

$$\mathrm{jh}(U) \leq CnK(\boldsymbol{R}_U^n)$$

for all $U \in \mathcal{C}_n$ and $\mathrm{j} = \mathrm{k, l, m}$.

Proof. Choose any $U \in \mathcal{C}_n$. Due to Lemma 4, we can find an affine image W of U with $\ell(T_W) \cdot \ell(T_{W^0}) \leq nK(\boldsymbol{R}_U^n)$. Let C be the constant from Lemma 3. Then, for each $\mathrm{j} = \mathrm{k, l, m}$, we have

$$\mathrm{jh}(U) = \mathrm{jh}(W) = \mathrm{jh}(W, W^0) \leq C\ell(T_W) \cdot \ell(T_{W^0}) \leq CnK(\boldsymbol{R}_U^n). \qquad \square$$

The following result is a direct consequence of Theorem 1 and (iii):

Corollary 1. *There exists a universal constant C such that*

$$\mathrm{jh}(U) \leq Cn(1 + \log n) \tag{4}$$

for all $U \in \mathcal{C}_n$ and $\mathrm{j} = \mathrm{k, l, m}$.

It follows from (i)–(iv) that for many convex bodies U the logarithmic factor in (4) can be improved. See also (1) and (2). It should be pointed out that the proof of (1) in [1] gives quite good numerical values of C; see (9) below.

Let Q be a convex body in $\boldsymbol{R}^n$, symmetric or not, and let $L \in \mathcal{L}_n$. Denote by s the number of points of L in Q and let $w_L(Q)$ be the L-width of Q:

$$w_L(Q) = \min_{y \in L^* \setminus \{0\}} \left(\max_{x \in Q} xy - \min_{x \in Q} xy \right).$$

Kannan and Lovász [5] proved that

$$w_L(Q) \leq c_0 \lceil (s+1)^{1/n} \rceil n^2, \tag{5}$$

where c_0 is the constant which comes from the Bourgain–Milman inequality:

$$\mathrm{vol}_n(U) \cdot \mathrm{vol}_n(U^0) \geq \left(\frac{4}{c_0 n} \right)^n, \qquad U \in \mathcal{C}_n.$$

It is assumed here that $c_0 \geq 1$. Furthermore, it was proved in [5] that

$$w_L(Q) \leq c_0 n^2 + 2 c_0 n s^{1/n} \tag{6}$$

provided that Q has a center of symmetry. See also (3.12) and (3.13) of [4].

Now, suppose that Q is symmetric with respect to some point p and let $U = Q - p$. It is clear that $w_L(Q) = 2\lambda_1(L^*, U^0)$ (see Lemma (2.3) of [5]). We denote

$$c_U = 4n^{-1} (\mathrm{vol}_n(U) \cdot \mathrm{vol}_n(U^0))^{-1/n}.$$

The proof of (6) in [5] shows actually that

$$w_L(Q) < 2\mu(L, U) \cdot \lambda_1(L^*, U^0) + 2 c_U n s^{1/n}. \tag{7}$$

By Corollary 1, we have

$$\mu(L, U) \cdot \lambda_1(L^*, U^0) \leq \mathrm{lh}(U^0) \leq Cn(1 + \log n).$$

Since, by definition, $c_U \leq c_0$, it follows that

$$w_L(Q) < Cn(1 + \log n) + 2 c_0 n s^{1/n}, \tag{8}$$

which is better than (6) at least for large n.

For convex bodies Q satisfying some additional conditions, inequality (7) allows us to obtain further improvements of (8); see the remarks following Corollary 1. Consider, in particular, an n-dimensional ellipsoid D with center at some point p. Let $B = D - p$; then $c_B < 2(\pi e)^{-1}$ due to the Santalo inequality. Let t denote the number of points of L in D. It was proved in [1] that $\mathrm{lh}(B) \leq \frac{1}{2} n$ and

$$\mathrm{lh}(B) \leq (2\pi)^{-1} n + O(n^{1/2}) \qquad \text{as} \quad n \to \infty.$$

This yields

$$w_L(D) < (1+4(\pi e)^{-1}t^{1/n})n, \tag{9}$$
$$w_L(D) < (1+4e^{-1}t^{1/n})\pi^{-1}n + O(n^{1/2}) \qquad \text{as} \quad n \to \infty.$$

Now, suppose that Q is an arbitrary convex body in $\boldsymbol{R}^n$, not necessarily centrally symmetric. Let D be the ellipsoid of maximal volume in Q and let p, B, and t be defined as above. Then $D \subset Q \subset nB + p$ due to the John theorem. It is obvious that $t \le s$ and

$$w_L(Q) \le w_L(nB+p) = nw_L(D).$$

Now, using (9), we obtain upper bounds for $w_L(Q)$ which differ from (5) only in numerical constants.

A very interesting problem is to improve the factor n^2 in (5). In our proof of (8), the central symmetry of Q or U is essential only in Lemma 4. However, to give a nonsymmetric analogue of Lemma 4 seems to be a difficult task.

Let $\mathrm{vol}_n(U)$ denote the n-dimensional euclidean volume of a convex body $U \in \mathcal{C}_n$.

Lemma 5. *To each pair $U, V \in \mathcal{C}_n$ there corresponds a lattice $L \in \mathcal{L}_n$ such that*

$$\lambda_1(L, U) \cdot \lambda_1(L^*, V) > [\mathrm{vol}_n(U) \cdot \mathrm{vol}_n(V)]^{-1/n}.$$

Proof. (Sketch) For $W \in \mathcal{C}_n$ and $L \in \mathcal{L}_n$, let $N_W(L)$ be the number of nonzero points of L in W, and let $N_W^*(L) = N_W(L^*)$. Siegel's mean value theorem says that the mean value $\mathfrak{M}(N_W)$ of $N_W(L)$ over all lattices L with determinant 1 is equal to $\mathrm{vol}_n(W)$ (the best description of this averaging process is given in [7]). It is not very hard to see that $\mathfrak{M}(N_W^*) = \mathfrak{M}(N_W)$ (very loosely speaking, there is a fundamental domain invariant under the transformation $L \mapsto L^*$).

Now, take arbitrary $U, V \in \mathcal{C}_n$, and let $s = \mathrm{vol}_n(U)^{-1/n}$, $t = \mathrm{vol}_n(V)^{-1/n}$. Then

$$\begin{aligned}\mathfrak{M}(N_{sU} + N_{tV}^*) &= \mathfrak{M}(N_{sU}) + \mathfrak{M}(N_{tV}^*) = \mathfrak{M}(N_{sU}) + \mathfrak{M}(N_{tV}) \\ &= \mathrm{vol}_n(sU) + \mathrm{vol}_n(tV) = s^n \,\mathrm{vol}_n(U) + t^n \,\mathrm{vol}_n(V) = 2.\end{aligned}$$

Since the function $L \mapsto (N_{sU} + N_{tV}^*)(L)$ assumes nonnegative even values only, there must be some $L \in \mathcal{L}_n$ with $N_{sU}(L) = 0$ and $N_{tV}^*(L) = 0$; in other words, $\lambda_1(L, sU) > 1$ and $\lambda_1(L^*, tV) > 1$. Then

$$\lambda_1(L, U) \cdot \lambda_1(L^*, V) = st\lambda_1(L, sU) \cdot \lambda_1(L^*, tV) > st. \qquad \square$$

Theorem 2. *There exists a numerical constant C such that* $\mathrm{jh}(U) \ge C^{-1}n$ *for all* $U \in \mathcal{C}_n$ *and* $\mathrm{j} = \mathrm{k}, \mathrm{l}, \mathrm{m}$.

Proof. Let $U \in \mathcal{C}_n$. By Lemma 5, we can find a lattice $L \in \mathcal{L}_n$ with

$$\lambda_1(L, U) \cdot \lambda_1(L^*, U^0) > [\mathrm{vol}_n(U) \cdot \mathrm{vol}_n(U^0)]^{-1/n},$$

which is greater than $(2\pi e)^{-1}n$ due to the Santalo inequality. Then

$$\mathrm{mh}(U) \ge \lambda_i(L, U) \cdot \lambda_{n-i+1}(L^*, U^0) \ge \lambda_1(L, U) \cdot \lambda_1(L^*, U^0) > (2\pi e)^{-1}n.$$

Next, we have the obvious inequality $\mu(L^*, U^0) \geq \frac{1}{2}\lambda_1(L^*, U^0)$, whence

$$\text{lh}(U) \geq \lambda_1(L, U) \cdot \mu(L^*, U^0) \geq \frac{1}{2}\lambda_1(L, U) \cdot \lambda_1(L^*, U^0) > (4\pi e)^{-1} n.$$

Finally, it remains to observe that $\text{kh}(U) \geq 2\ \text{lh}(U)$. □

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