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Inequalities for eigenvalues of Laplacian on domains and compact complex hypersurfaces in complex projective spaces

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Abstract. It is well known that the spectrum of Laplacian on a compact Riemannian manifold M is an important analytic invariant and has important geometric meanings. There are many mathematicians to investigate properties of the spectrum of Laplacian and to estimate the spectrum in term of the other geometric quantities of M. When M is a bounded domain in Euclidean spaces, a compact homogeneous Riemannian manifold, a bounded domain in the standard unit sphere or a compact minimal submanifold in the standard unit sphere, the estimates of the $k+1$-th eigenvalue were given by the first k eigenvalues (see [**9**], [**12**], [**19**], [**20**], [**22**], [**23**], [**24**] and [**25**]). In this paper, we shall consider the eigenvalue problem of the Laplacian on compact Riemannian manifolds. First of all, we shall give a general inequality of eigenvalues. As its applications, we study the eigenvalue problem of the Laplacian on a bounded domain in the standard complex projective space $\boldsymbol{CP}^n(4)$ and on a compact complex hypersurface without boundary in $\boldsymbol{CP}^n(4)$. We shall give an explicit estimate of the $k+1$-th eigenvalue of Laplacian on such objects by its first k eigenvalues.

1. Introduction.

In this paper, we consider the eigenvalue problem of the Laplacian on a compact Riemannian manifold M with boundary (possible empty):

$$\begin{cases} \Delta u = -\lambda u, \text{ in } M, \\ u|_{\partial M} = 0, \end{cases} \tag{1.1}$$

where Δ is the Lapalacian on M.

This problem has a real and purely discrete spectrum

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty. \tag{1.2}$$

We put $\lambda_0 = 0$ if $\partial M = \varnothing$. Here each eigenvalue is repeated from its multiplicity.

It is well known that the spectrum of Laplacian on M is an important analytic invariant and has important geometric meanings (cf. Chavel [**7**]). There are many

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mathematicians to investigate properties of spectrum of Laplacian and to estimate the spectrum in term of the other geometric quantities of M. First of all, we will review several important results on estimates of eigenvalues.

(1). When $M = \bar{\Omega}$, where Ω is a connected bounded domain in the Euclidean space $\boldsymbol{R}^n$, the main contributions on estimates of higher eigenvalues are obtained by Payne, Pólya and Weinberger [**22**] and Thompson [**23**], Hile and Protter [**14**] and Yang [**24**]. The following very sharper inequality was obtained by Yang [**24**]

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big(\lambda_{k+1}-\Big(1+\frac{4}{n}\Big)\lambda_i\Big) \le 0, \text{ for } k=1,2,\cdots. \tag{1.3}$$

According to the inequality, he also obtained (cf. [**24**])

$$\lambda_{k+1} \le \frac{1}{k}\Big(1+\frac{4}{n}\Big)\sum_{i=1}^{k}\lambda_i, \text{ for } k=1,2,\cdots. \tag{1.4}$$

We must remark that when $k=1$, Ashbaugh and Benguria in [**3**], [**4**] and [**5**] gave an optimal estimate of λ_2 by λ_1.

(2). When M is an n-dimensional compact homogeneous Riemannian manifold, Li [**20**] estimated the difference of any two consecutive eigenvalues of Laplacian on M. He proved

$$\lambda_{k+1}-\lambda_k \le \frac{2}{k+1}\left(\sqrt{\Big(\sum_{i=1}^{k}\lambda_i\Big)^2+(k+1)\sum_{i=1}^{k}\lambda_i\lambda_1}+\sum_{i=1}^{k}\lambda_i\right)+\lambda_1. \tag{1.5}$$

Furthermore, Harrel II and Michel [**12**] improved the estimates introduced by Li [**20**]. Namely, they obtained that

$$\lambda_{k+1}-\lambda_k \le \frac{4}{k+1}\sum_{i=1}^{k}\lambda_i+\lambda_1. \tag{1.6}$$

Recently, the authors [**9**] obtained

$$\begin{aligned}\lambda_{k+1} \le {}& \frac{3}{k+1}\sum_{i=1}^{k}\lambda_i+\frac{\lambda_1}{2}\\ &+\frac{1}{2}\sqrt{\Big(\frac{4}{k+1}\sum_{i=1}^{k}\lambda_i+\lambda_1\Big)^2-\frac{20}{k+1}\sum_{i=0}^{k}\Big(\lambda_i-\frac{1}{k+1}\sum_{j=1}^{k}\lambda_j\Big)^2},\end{aligned} \tag{1.7}$$

where $\lambda_0=0$. Since $\lambda_k \ge \frac{1}{k}\sum_{i=1}^{k}\lambda_i$ holds, we infered, from the above inequality,

$$\lambda_{k+1}-\lambda_k \le \frac{2k-1}{k(k+1)}\sum_{i=1}^k \lambda_i + \frac{\lambda_1}{2} + \frac{1}{2}\sqrt{\left(\frac{4}{k+1}\sum_{i=1}^k \lambda_i + \lambda_1\right)^2 - \frac{20}{k+1}\sum_{i=0}^k\left(\lambda_i - \frac{1}{k+1}\sum_{j=1}^k \lambda_j\right)^2}. \quad (1.8)$$

From $\left(\frac{\sum_{i=1}^k \lambda_i}{k+1}\right)^2 \le \frac{k}{k+1}\frac{\sum_{i=1}^k \lambda_i^2}{k+1}$ and the inequality above, we obtained, in [**9**],

$$\lambda_{k+1}-\lambda_k < \frac{4k-1}{k(k+1)}\sum_{i=1}^k \lambda_i + \lambda_1 < \frac{4}{k+1}\sum_{i=1}^k \lambda_i + \lambda_1.$$

(3). When M is a compact minimal submanifold in the standard unit sphere $S^N(1)$, P. C. Yang and Yau [**25**] obtained

$$\lambda_{k+1}-\lambda_k \le n + \frac{2}{n(k+1)}\left(\sqrt{\left(\sum_{i=1}^k \lambda_i\right)^2 + n^2(k+1)\sum_{i=1}^k \lambda_i} + \sum_{i=1}^k \lambda_i\right). \quad (1.9)$$

(cf. Leung [**19**]). Furthermore, Harrel II and Michel [**12**] proved

$$\lambda_{k+1}-\lambda_k \le n + \frac{4}{n(k+1)}\sum_{i=1}^k \lambda_i. \quad (1.10)$$

In [**9**], we gave an estimate of the $k+1$-th eigenvalue in terms of the first k eigenvalues:

$$\lambda_{k+1} \le \left(1+\frac{2}{n}\right)\frac{1}{k+1}\sum_{i=0}^k \lambda_i + \frac{n}{2} + \left[\left(\frac{n}{2}+\frac{2}{n}\frac{1}{k+1}\sum_{i=0}^k \lambda_i\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^k\left(\lambda_j - \frac{1}{k+1}\sum_{i=0}^k \lambda_i\right)^2\right]^{1/2},$$

where $\lambda_0 = 0$. From $\lambda_k \ge \frac{1}{k}\sum_{i=1}^k \lambda_i$, it is obvious that our inequality above is much sharper than these inequalities (1.9) and (1.10) of P. C. Yang and Yau [**25**] and Harrel II and Michel [**12**].

(4). When M is a connected bounded domain of $S^n(1)$, in [**9**], we obtained

$$\lambda_{k+1} \le \left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^k \lambda_i + \frac{n}{2} + \left[\left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^k \lambda_i + \frac{n}{2}\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{j=1}^k\left(\lambda_j - \frac{1}{k}\sum_{i=1}^k \lambda_i\right)^2\right]^{1/2}$$

and

$$\lambda_{k+1}-\lambda_k \le 2\left[\left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^k \lambda_i+\frac{n}{2}\right)^2-\left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{j=1}^k\left(\lambda_j-\frac{1}{k}\sum_{i=1}^k\lambda_i\right)^2\right]^{1/2}.$$

In this paper, we shall consider the eigenvalue problem of the Laplacian on compact Riemannian manifolds. In section 2, we shall give a general inequality of eigenvalues for the eigenvalue problem (1.1). Namely, when M is a compact Riemannian manifold with boundary ∂M, for any function $g \in C^3(M)\cap C^2(\partial M)$ and any integer k, we obtained (see Proposition 1)

$$\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla g\|^2 \le (\lambda_{k+1}-\lambda_i)\|2\nabla g\cdot\nabla u_i+u_i\Delta g\|^2,$$

where u_i is the orthonormal eigenfunction corresponding to eigenvalue λ_i of (1.1). As applications of the above inequality, in section 3, we study the eigenvalue problem of the Laplacian on a connected bounded domain of the standard complex projective space $\boldsymbol{CP}^n(4)$ with holomorphic sectional curvature 4. We proved (see Theorem 1)

$$\begin{aligned}\lambda_{k+1} \le& \left(1+\frac{1}{n}\right)\frac{1}{k}\sum_{i=1}^k\lambda_i+2(n+1)\\ &+\left\{\left[\frac{1}{n}\frac{1}{k}\sum_{i=1}^k\lambda_i+2(n+1)\right]^2-\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{j=1}^k\left(\lambda_j-\frac{1}{k}\sum_{i=1}^k\lambda_i\right)^2\right\}^{1/2}.\end{aligned}$$

In section 4, for a compact complex hypersurface without boundary in $\boldsymbol{CP}^{n+1}(4)$, we derived (see Theorem 2)

$$\begin{aligned}\lambda_{k+1} \le& \left(1+\frac{1}{n}\right)\frac{1}{k+1}\sum_{i=1}^k\lambda_i+2(n+1)\\ &+\left\{\left[\frac{1}{n}\frac{1}{k+1}\sum_{i=1}^k\lambda_i+2(n+1)\right]^2-\left(1+\frac{2}{n}\right)\frac{1}{k+1}\sum_{j=0}^k\left(\lambda_j-\frac{1}{k+1}\sum_{i=1}^k\lambda_i\right)^2\right\}^{1/2}.\end{aligned}$$

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2. A general Inequality of Eigenvalues.

Let M be an n-dimensional compact Riemannian manifold with boundary ∂M (possible empty). We consider the following eigenvalue problem:

$$\begin{cases}\Delta u=-\lambda u, \text{ in } M,\\ u|_{\partial M}=0,\end{cases} \tag{2.1}$$

where Δ is the Laplacian of M. We shall prove a general inequality of eigenvalues for the above eigenvalue problem which will be applied to prove Theorems 1 and 2.

PROPOSITION 1. *Let λ_i be the i-th eigenvalue of the above eigenvalue problem* (2.1) *and u_i be the orthonormal eigenfunction corresponding to λ_i, that is, u_i satisfies*

$$\begin{cases} \Delta u_i = -\lambda_i u_i, \ \textit{in } M, \\ u_i|_{\partial M} = 0, \\ \displaystyle\int_M u_i u_j = \delta_{ij}, \ \textit{for any } i, j = 1, 2, \cdots. \end{cases}$$

Then, for any function $g \in C^3(M) \cap C^2(\partial M)$ and any integer k, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla g\|^2 \leq (\lambda_{k+1} - \lambda_i) \|2\nabla g \cdot \nabla u_i + u_i \Delta g\|^2,$$

where

$$\|f\|^2 = \int_M f^2.$$

PROOF. Assume that u_i is an orthonormal eigenfunction corresponding to the i-th eigenvalue λ_i, i.e. u_i satisfies

$$\begin{cases} \Delta u_i = -\lambda_i u_i, \ \text{in } M, \\ u_i|_{\partial M} = 0, \\ \displaystyle\int_M u_i u_j = \delta_{ij}. \end{cases} \tag{2.2}$$

We have

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k+1} \leq \cdots. \tag{2.3}$$

Define φ_i, a_{ij} and b_{ij}, for $i, j = 1, \cdots, k$, by

$$\begin{cases} a_{ij} = \displaystyle\int_M g u_i u_j, \\ \varphi_i = g u_i - \displaystyle\sum_{j=1}^{k} a_{ij} u_j, \\ b_{ij} = \displaystyle\int_M u_j \left(\nabla u_i \cdot \nabla g + \frac{1}{2} u_i \Delta g \right), \end{cases} \tag{2.4}$$

where ∇ denotes the gradient operator. Then, we have, from (2.2),

$$a_{ij} = a_{ji}, \quad \int_M \varphi_i u_j = 0, \text{ for } j = 1, 2, \cdots, k. \tag{2.5}$$

Thus, we have

$$\lambda_{k+1} \leq \frac{\int_M |\nabla \varphi_i|^2}{\int_M \varphi_i^2}. \tag{2.6}$$

According to Stokes formula, we infer

$$\begin{aligned}\lambda_j a_{ij} &= \int_M g(-\Delta u_j) u_i = \int_M (-2u_j \nabla u_i \cdot \nabla g - \Delta g u_i u_j - g u_j \Delta u_i) \\ &= -2b_{ij} + \lambda_i a_{ij}.\end{aligned}$$

Hence, we obtain

$$2b_{ij} = -2b_{ji} = (\lambda_i - \lambda_j) a_{ij}. \tag{2.7}$$

From the definition of φ_i and a simple calculation, we have

$$\begin{aligned}\Delta \varphi_i &= g \Delta u_i + 2\nabla g \cdot \nabla u_i + u_i \Delta g - \sum_{j=1}^k a_{ij} \Delta u_j \\ &= -\lambda_i g u_i + 2\nabla g \cdot \nabla u_i + u_i \Delta g + \sum_{j=1}^k a_{ij} \lambda_j u_j.\end{aligned}$$

Therefore, from (2.5) and the above equality, we have

$$\begin{aligned}\int_M |\nabla \varphi_i|^2 &= -\int_M \varphi_i (\Delta \varphi_i) \\ &= \lambda_i \int_M \varphi_i^2 - \int_M \varphi_i (2\nabla g \cdot \nabla u_i + u_i \Delta g).\end{aligned} \tag{2.8}$$

From (2.6) and (2.8), we infer

$$(\lambda_{k+1} - \lambda_i) \|\varphi_i\|^2 \leq -\int_M \varphi_i (2\nabla g \cdot \nabla u_i + u_i \Delta g) \equiv w_i. \tag{2.9}$$

Since

$$2 \int_M u_j \left(\nabla u_i \cdot \nabla g + \frac{1}{2} u_i \Delta g \right) = 2b_{ij} = -2b_{ji} = (\lambda_i - \lambda_j) a_{ij}, \tag{2.10}$$

from $\varphi_i = g u_i - \sum_{j=1}^k a_{ij} u_j$ and the definition of a_{ij}, we have

$$w_i = -\int_M \left(gu_i - \sum_{j=1}^{k} a_{ij}u_j\right)(2\nabla g \cdot \nabla u_i + u_i \Delta g)$$
$$= \|u_i \nabla g\|^2 + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2. \tag{2.11}$$

On the other hand, from the definition of w_i and (2.5), we have

$$w_i = -\int_M \varphi_i \left(2\nabla g \cdot \nabla u_i + u_i \Delta g - 2\sum_{j=1}^{k} b_{ij}u_j\right).$$

From Schwarz inequality, we obtain

$$(\lambda_{k+1} - \lambda_i) w_i^2 \le (\lambda_{k+1} - \lambda_i)\|\varphi_i\|^2 \|2\nabla g \cdot \nabla u_i + u_i \Delta g - 2\sum_{j=1}^{k} b_{ij}u_j\|^2. \tag{2.12}$$

From (2.9), (2.12) and the definition of b_{ij}, we infer

$$(\lambda_{k+1} - \lambda_i) w_i^2 \le w_i \left(\|2\nabla g \cdot \nabla u_i + u_i \Delta g\|^2 - 4\sum_{j=1}^{k} b_{ij}^2\right).$$

Hence,

$$(\lambda_{k+1} - \lambda_i) w_i \le \|2\nabla g \cdot \nabla u_i + u_i \Delta g\|^2 - 4\sum_{j=1}^{k} b_{ij}^2. \tag{2.13}$$

Multiplying (2.13) by $(\lambda_{k+1} - \lambda_i)$ and taking sum on i from 1 to k, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i \le -4\sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) b_{ij}^2$$
$$+ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\|2\nabla g \cdot \nabla u_i + u_i \Delta g\|^2. \tag{2.14}$$

In terms of $2b_{ij} = (\lambda_i - \lambda_j)a_{ij}$, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i \le \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\|2\nabla g \cdot \nabla u_i + u_i \Delta g\|^2$$
$$- \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 a_{ij}^2. \tag{2.15}$$

Further, from (2.11) and the symmetry of a_{ij}, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 w_i = \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla g\|^2$$
$$+\sum_{i,j=1}^{k}\frac{1}{2}\big[(\lambda_{k+1}-\lambda_i)^2-(\lambda_{k+1}-\lambda_j)^2\big](\lambda_i-\lambda_j)a_{ij}^2$$
$$=\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla g\|^2-\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2.$$

Therefore, we infer, from (2.15) and the above equality,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla g\|^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla g\cdot\nabla u_i+u_i\Delta g\|^2. \tag{2.16}$$

This completes the proof of Proposition 1. □

3. Eigenvalues of Laplacian on a domain in $\boldsymbol{CP}^n(4)$.

In this section, we shall consider the eigenvalue problem of the Laplacian on a domain in $\boldsymbol{CP}^n(4)$, that is, let M be a connected domain in $\boldsymbol{CP}^n(4)$ with boundary ∂M, we consider the following:

$$\begin{cases}\Delta u = -\lambda u, \text{ in } M,\\ u|_{\partial M}=0,\end{cases} \tag{3.1}$$

where Δ is the Laplacian of $\boldsymbol{CP}^n(4)$.

THEOREM 1. *Let M be a connected bounded domain in the standard n-dimensional complex projective space $\boldsymbol{CP}^n(4)$. Assume that λ_i is the i-th eigenvalue of the above eigenvalue problem* (3.1). *Then, we have*

$$\lambda_{k+1}\le\left(1+\frac{1}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2(n+1)$$
$$+\left\{\left[\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2(n+1)\right]^2-\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}. \tag{3.2}$$

COROLLARY 1. *Under the assumptions of Theorem* 1, *we have*

$$\lambda_{k+1}-\lambda_k\le 2\left\{\left[\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2(n+1)\right]^2-\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}.$$

PROOF OF THEOREM 1. Let $Z=(Z^0,Z^1,\dots,Z^n)$ be a homogeneous coordinate system of $\boldsymbol{CP}^n(4)$, $(Z^p\in\boldsymbol{C})$. Defining $f_{p\bar q}$, for $p,q=0,1,\cdots,n$, by

$$f_{p\bar{q}} = \frac{Z^p \overline{Z^q}}{\sum_{r=0}^{n} Z^r \overline{Z^r}}, \tag{3.3}$$

we have

$$f_{p\bar{q}} = \overline{f_{q\bar{p}}}, \qquad \sum_{p,q=0}^{n} f_{p\bar{q}} \overline{f_{p\bar{q}}} = 1. \tag{3.4}$$

For any fixed point $P \in M$, we can choose a new homogeneous coordinate system of $\boldsymbol{CP}^n(4)$ such that, at P

$$\widetilde{Z}^0 \neq 0, \widetilde{Z}^1 = \cdots = \widetilde{Z}^n = 0 \tag{3.5}$$

and

$$Z^p = \sum_{r=0}^{n} C_{pr} \widetilde{Z}^r, \tag{3.6}$$

where $C = (C_{pq}) \in U(n+1)$ is an $(n+1) \times (n+1)$-unitary matrix, that is, C_{pq} satisfies

$$\sum_{s=0}^{n} C_{sp} \overline{C}_{sq} = \sum_{s=0}^{n} C_{ps} \overline{C}_{qs} = \delta_{pq}. \tag{3.7}$$

Then, we know that $z = (z^1, \ldots, z^n)$, $z^p = \widetilde{Z}^p / \widetilde{Z}^0$, is a local holomorphic coordinate system of $\boldsymbol{CP}^n(4)$ in a neighborhood U of the point $P \in M$ and satisfies, at P,

$$z^1 = \cdots = z^n = 0. \tag{3.8}$$

Hence, we infer, for $p, q = 0, 1, \cdots, n$,

$$\begin{aligned} \widetilde{f}_{p\bar{q}} &= \frac{\widetilde{Z}^p \overline{\widetilde{Z}^q}}{\sum_{r=0}^{n} \widetilde{Z}^r \overline{\widetilde{Z}^r}} = \frac{z^p \overline{z^q}}{1 + \sum_{r=1}^{n} z^r \overline{z}^r}, \\ f_{p\bar{q}} &= \sum_{r,s=0}^{n} C_{pr} \overline{C_{qs}} \widetilde{f}_{r\bar{s}}, \quad z^0 \equiv 1. \end{aligned} \tag{3.9}$$

Putting $G_{p\bar{q}} = \mathrm{Re}(f_{p\bar{q}})$ and $F_{p\bar{q}} = \mathrm{Im}(f_{p\bar{q}})$, for $p, q = 0, 1, \ldots, n$, then, we have

$$\begin{aligned} &\sum_{p,q=0}^{n} \left(G_{p\bar{q}}^2 + F_{p\bar{q}}^2\right) = \sum_{p,q=0}^{n} f_{p\bar{q}} \overline{f_{p\bar{q}}} = \sum_{p,q=0}^{n} \widetilde{f}_{p\bar{q}} \overline{\widetilde{f}_{p\bar{q}}} = 1, \\ &\sum_{p,q=0}^{n} \left(G_{p\bar{q}} \nabla G_{p\bar{q}} + F_{p\bar{q}} \nabla F_{p\bar{q}}\right) = 0. \end{aligned} \tag{3.10}$$

Let $ds^2 = \sum_{p,q=1}^{n} g_{p\overline{q}} dz^p d\overline{z^q}$ is the Fubini-Study metric of $\boldsymbol{CP}^n(4)$. Then,

$$g_{p\overline{q}} = \frac{\delta_{p\overline{q}}}{1+\sum_{r=1}^{k}|z^r|^2} - \frac{z^q\overline{z^p}}{\left(1+\sum_{r=1}^{n}|z^r|^2\right)^2}. \tag{3.11}$$

$$\begin{aligned}(g_{p\overline{q}})^{-1} &= (g^{p\overline{q}}),\\ g^{p\overline{q}} &= \left(1+\sum_{r=1}^{n}|z^r|^2\right)(\delta^{p\overline{q}} + z^q\overline{z^p}).\end{aligned} \tag{3.12}$$

Under the local coordinate system, for any smooth function f, we have

$$\Delta f = \sum_{p,q=1}^{n} 4g^{p\overline{q}} \frac{\partial^2}{\partial z^p \partial \overline{z^q}} f. \tag{3.13}$$

And, at P,

$$\Delta = 4\sum_{r=1}^{n} \frac{\partial^2}{\partial z^r \partial \overline{z^r}},$$

$$\begin{aligned}&\nabla \widetilde{f}_{p\overline{q}} = 0, \quad \text{if } p \neq 0 \text{ and } q \neq 0,\\ &\nabla \widetilde{f}_{p\overline{p}} = 0,\\ &\Delta \widetilde{f}_{p\overline{q}} = 0, \quad \text{if } p \neq q\\ &\Delta \widetilde{f}_{0\overline{0}} = -4n, \quad \Delta \widetilde{f}_{r\overline{r}} = 4,\ r = 1, \cdots, n.\end{aligned} \tag{3.14}$$

Thus, we obtain from (3.10), at P,

$$\begin{aligned}\sum_{p,q=0}^{n}\left(\nabla G_{p\overline{q}} \cdot \nabla G_{p\overline{q}} + \nabla F_{p\overline{q}} \cdot \nabla F_{p\overline{q}}\right) &= -\sum_{p,q=0}^{n}\left(G_{p\overline{q}}\Delta G_{p\overline{q}} + F_{p\overline{q}}\Delta F_{p\overline{q}}\right)\\ &= -\mathrm{Re}\sum_{p,q=0}^{n} \overline{f_{p\overline{q}}}\Delta f_{p\overline{q}}\\ &= -\mathrm{Re}\sum_{p,q=0}^{n}\sum_{r,s=0}^{n} \overline{C_{pr}\overline{C_{qs}}\widetilde{f}_{r\overline{s}}} \sum_{u,v=0}^{n} C_{pu}\overline{C_{qv}}\Delta \widetilde{f}_{u\overline{v}}\\ &= -\sum_{p,q=0}^{n} \mathrm{Re}\overline{\widetilde{f}_{p\overline{q}}}\Delta \widetilde{f}_{p\overline{q}} = -\widetilde{f}_{0\overline{0}}\Delta \widetilde{f}_{0\overline{0}} = 4n.\end{aligned} \tag{3.15}$$

By a similar calculation, we have, at P,

$$\sum_{p,q=0}^{n} \big(\nabla G_{p\bar{q}}\Delta G_{p\bar{q}} + \nabla F_{p\bar{q}}\Delta F_{p\bar{q}}\big) = \mathrm{Re}\sum_{p,q=0}^{n} \nabla\overline{f_{p\bar{q}}}\Delta f_{p\bar{q}} = 0, \tag{3.16}$$

$$\begin{aligned}\sum_{p,q=0}^{n} \big(\Delta G_{p\bar{q}}\Delta G_{p\bar{q}} + \Delta F_{p\bar{q}}\Delta F_{p\bar{q}}\big) &= \mathrm{Re}\sum_{p,q=0}^{n} \overline{\Delta f_{p\bar{q}}}\Delta f_{p\bar{q}} = \mathrm{Re}\sum_{p,q=0}^{n} \overline{\Delta \widetilde{f}_{p\bar{q}}}\Delta \widetilde{f}_{p\bar{q}} \\ &= (-4n)\times(-4n) + 4\times 4\times n = 16n(n+1)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\sum_{p,q=0}^{n} \{(\nabla G_{p\bar{q}}\cdot\nabla u_i)^2 + (\nabla F_{p\bar{q}}\cdot\nabla u_i)^2\} &= \mathrm{Re}\sum_{p,q=0}^{n} \overline{(\nabla f_{p\bar{q}}\cdot\nabla u_i)}\big(\nabla f_{p\bar{q}}\cdot\nabla u_i\big) \\ = \mathrm{Re}\sum_{p,q=0}^{n} \overline{(\nabla \widetilde{f}_{p\bar{q}}\cdot\nabla u_i)}\big(\nabla \widetilde{f}_{p\bar{q}}\cdot\nabla u_i\big) &= 2|\nabla u_i|^2.\end{aligned} \tag{3.18}$$

Since P is arbitrary, we have at any point $x \in M$,

$$\begin{cases} \sum_{p,q=0}^{n} \big(\nabla G_{p\bar{q}}\cdot\nabla G_{p\bar{q}} + \nabla F_{p\bar{q}}\cdot\nabla F_{p\bar{q}}\big) = 4n, \\ \sum_{p,q=0}^{n} \big(\Delta G_{p\bar{q}}\Delta G_{p\bar{q}} + \Delta F_{p\bar{q}}\Delta F_{p\bar{q}}\big) = 16n(n+1), \\ \sum_{p,q=0}^{n} \big(\nabla G_{p\bar{q}}\Delta G_{p\bar{q}} + \nabla F_{p\bar{q}}\Delta F_{p\bar{q}}\big) = 0, \\ \sum_{p,q=0}^{n} \big\{\big(\nabla G_{p\bar{q}}\cdot\nabla u_i\big)^2 + \big(\nabla F_{p\bar{q}}\cdot\nabla u_i\big)^2\big\} = 2|\nabla u_i|^2. \end{cases} \tag{3.19}$$

By applying the Proposition 1 to the functions $G_{p\bar{q}}$ and $F_{p\bar{q}}$ and taking sum on p and q from 0 to n, we infer

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M \sum_{p,q=0}^{n} \big(\|u_i\nabla G_{p\bar{q}}\|^2 + \|u_i\nabla F_{p\bar{q}}\|^2\big) \\ &\quad\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\bigg\{4\int_M \sum_{p,q=0}^{n} \{(\nabla G_{p\bar{q}}\cdot\nabla u_i)^2 + (\nabla F_{p\bar{q}}\cdot\nabla u_i)^2\} \\ &\qquad\qquad + 4\int_M \sum_{p,q=0}^{n} \big(\nabla G_{p\bar{q}}\Delta G_{p\bar{q}}\cdot u_i\nabla u_i + \nabla F_{p\bar{q}}\Delta F_{p\bar{q}}\cdot u_i\nabla u_i\big) \\ &\qquad\qquad + \int_M \sum_{p,q=0}^{n} \big(\Delta G_{p\bar{q}}\Delta G_{p\bar{q}} + \Delta F_{p\bar{q}}\Delta F_{p\bar{q}}\big)u_i^2\bigg\}.\end{aligned} \tag{3.20}$$

From (2.2), (3.19) and (3.20), we obtain

$$4n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)[8\lambda_i+16n(n+1)],$$

namely,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)[\lambda_i+2n(n+1)]. \tag{3.21}$$

Letting $\mu_i = \lambda_i + 2n(n+1)$, we have

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \le \frac{2}{n}\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i. \tag{3.22}$$

Since (3.22) is a quadratic inequality of μ_{k+1}, we have

$$\begin{aligned}\lambda_{k+1} \le& \left(1+\frac{1}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + 2(n+1)\\ &+\left[\left(\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2(n+1)\right)^2-\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{3.23}$$

This finishes the proof of Theorem 1. □

PROOF OF COROLLARY 1. In the proof of Theorem 1, since k is any integer, we know that (3.22) is also true if we replace $k+1$ with k, namely, we have

$$\sum_{i=1}^{k-1}(\mu_k-\mu_i)^2 \le \frac{2}{n}\sum_{i=1}^{k-1}(\mu_k-\mu_i)\mu_i.$$

Hence, we infer

$$\sum_{i=1}^{k}(\mu_k-\mu_i)^2 \le \frac{2}{n}\sum_{i=1}^{k}(\mu_k-\mu_i)\mu_i. \tag{3.24}$$

Thus, μ_k also satisfies the same quadratic inequality. Hence, we have

$$\begin{aligned}\lambda_k \ge& \left(1+\frac{1}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + 2(n+1)\\ &-\left[\left(\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i+2(n+1)\right)^2-\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Therefore, we have

$$\lambda_{k+1} - \lambda_k \leq 2\left\{ \left[\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i + 2(n+1) \right]^2 - \left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j - \frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2 \right\}^{1/2}. \qquad \square$$

4. Eigenvalues of Laplacian on a complex hypersurface in $\boldsymbol{CP}^{n+1}(4)$.

In this section, we shall consider the eigenvalue problem of the Laplacian on a compact complex hypersurface M without boundary in $\boldsymbol{CP}^{n+1}(4)$:

$$\Delta u = -\lambda u, \text{ in } M, \tag{4.1}$$

where Δ is the Laplacian of M. We know that this eigenvalue problem has a discrete spectrum: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. As an application of the Proposition 1, we obtain the following Theorem 2. In this case, we must consider the eigenfunction u_0 corresponding to eigenvalue $\lambda_0 = 0$.

THEOREM 2. *Let M be a compact complex hypersurface without boundary in $\boldsymbol{CP}^{n+1}(4)$. Assume that λ_i is the i-th eigenvalue of the above eigenvalue problem* (4.1). *Then, we have*

$$\begin{aligned}\lambda_{k+1} \leq & \left(1+\frac{1}{n}\right)\frac{1}{k+1}\sum_{i=1}^{k}\lambda_i + 2(n+1) \\ & + \left\{ \left[\frac{1}{n}\frac{1}{k+1}\sum_{i=1}^{k}\lambda_i + 2(n+1) \right]^2 - \left(1+\frac{2}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\left(\lambda_j - \frac{1}{k+1}\sum_{i=1}^{k}\lambda_i\right)^2 \right\}^{1/2}.\end{aligned} \tag{4.2}$$

COROLLARY 2. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned}\lambda_{k+1} - \lambda_k \leq 2\Bigg\{ & \left[\frac{1}{n}\frac{1}{k+1}\sum_{i=1}^{k}\lambda_i + 2(n+1) \right]^2 \\ & - \left(1+\frac{2}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\left(\lambda_j - \frac{1}{k+1}\sum_{i=1}^{k}\lambda_i\right)^2 \Bigg\}^{1/2}.\end{aligned}$$

PROOF OF THEOREM 2. Since the method of proof is the same as in the proof of Theorem 1, we shall only give its outline. Let $Z = (Z^0, Z^1, \ldots, Z^{n+1})$ be a homogeneous coordinate system of $\boldsymbol{CP}^{n+1}(4)$. Defining $f_{p\bar{q}}$, for $p, q = 0, 1, \cdots, n, n+1$, by

$$f_{p\bar{q}} = \frac{Z^p \overline{Z^q}}{\sum_{r=0}^{n+1} Z^r \overline{Z^r}}, \tag{4.3}$$

we have

$$f_{p\bar{q}} = \overline{f_{q\bar{p}}}, \quad \sum_{p,q=0}^{n+1} f_{p\bar{q}}\overline{f_{p\bar{q}}} = 1. \tag{4.4}$$

For any fixed point $P \in M$, we can choose a new homogeneous coordinate system of $\boldsymbol{CP}^{n+1}(4)$ such that, at P

$$\widetilde{Z}^0 \neq 0, \widetilde{Z}^1 = \cdots = \widetilde{Z}^{n+1} = 0 \tag{4.5}$$

and

$$Z^p = \sum_{r=0}^{n+1} C_{pr}\widetilde{Z}^r, \tag{4.6}$$

where $C = (C_{pq}) \in U(n+2)$ is an $(n+2)\times(n+2)$-unitary matrix, that is, C_{pr} satisfies

$$\sum_{s=0}^{n+1} C_{sp}\overline{C}_{sq} = \sum_{s=0}^{n+1} C_{ps}\overline{C}_{qs} = \delta^{pq}. \tag{4.7}$$

Let

$$z^p = \widetilde{Z}^p/\widetilde{Z}^0, \text{ for } p = 0, 1, \cdots, n, n+1.$$

Then, we know that $z = (z^1, \ldots, z^n)$ is a local holomorphic coordinate system of M in a neighborhood U of the point $P \in M$ and $z^{n+1} = h(z^1, \ldots, z^n) \in \mathscr{O}(U)$ is a holomorphic function of $z^1, \ldots, z^n$ satisfying

$$\left.\frac{\partial h}{\partial z^p}\right|_P = 0, \quad \text{for } p = 1, \cdots, n, \tag{4.8}$$

and, at P,

$$z^1 = \cdots = z^n = z^{n+1} = 0. \tag{4.9}$$

Hence, we infer, for $p, q = 0, 1, \cdots, n, n+1$,

$$\begin{aligned} \widetilde{f}_{p\bar{q}} &= \frac{\widetilde{Z}^p\overline{\widetilde{Z}^q}}{\sum_{r=0}^{n+1}\widetilde{Z}^r\overline{\widetilde{Z}^r}} = \frac{z^p\overline{z^q}}{1+\sum_{r=1}^{n+1} z^r\overline{z}^r}, \\ f_{p\bar{q}} &= \sum_{r,s=0}^{n+1} C_{pr}\overline{C_{qs}}\widetilde{f}_{r\bar{s}}, \quad z^0 \equiv 1. \end{aligned} \tag{4.10}$$

Putting $G_{p\overline{q}} = \mathrm{Re}(f_{p\overline{q}})$ and $F_{p\overline{q}} = \mathrm{Im}(f_{p\overline{q}})$, for $p, q = 0, 1, \cdots, n, n+1$, then, for the $2(n+2)^2$ functions $G_{p\overline{q}}$ and $F_{p\overline{q}}$, we have

$$\begin{aligned}
&\sum_{p,q=0}^{n+1} \left(G_{p\overline{q}}^2 + F_{p\overline{q}}^2\right) = \sum_{p,q=0}^{n+1} f_{p\overline{q}}\overline{f_{p\overline{q}}} = \sum_{p,q=0}^{n+1} \widetilde{f}_{p\overline{q}}\overline{\widetilde{f}_{p\overline{q}}} = 1, \\
&\sum_{p,q=0}^{n+1} \left(G_{p\overline{q}}\nabla G_{p\overline{q}} + F_{p\overline{q}}\nabla F_{p\overline{q}}\right) = 0.
\end{aligned} \tag{4.11}$$

Since, under the local coordinate system, we have, for $z \in U$,

$$ds_M^2 = \sum_{p,q=1}^{n} (1 + O(|z|^2)) dz^p d\overline{z^q},$$

we obtain, at P,

$$\Delta = 4\sum_{r=1}^{n} \frac{\partial^2}{\partial z^r \partial \overline{z^r}},$$

$$\begin{aligned}
&\nabla \widetilde{f}_{p\overline{q}} = 0, \quad \text{if } p \neq 0 \text{ and } q \neq 0, \\
&\nabla \widetilde{f}_{p\overline{p}} = 0, \\
&\nabla \widetilde{f}_{\overline{p}(n+1)} = \nabla \widetilde{f}_{(n+1)\overline{p}} = \nabla \widetilde{f}_{(n+1)\overline{(n+1)}} = 0, \ \text{for } p = 1, \cdots, n, \\
&\Delta \widetilde{f}_{p\overline{q}} = 0, \quad \text{if } p \neq q, \quad \Delta \widetilde{f}_{n+1\overline{n+1}} = 0, \\
&\Delta \widetilde{f}_{0\overline{0}} = -4n, \quad \Delta \widetilde{f}_{r\overline{r}} = 4, \ r = 1, \cdots, n.
\end{aligned} \tag{4.12}$$

Thus, making use of the same arguments as in the proof of Theorem 1, we obtain, at any point $x \in M$,

$$\begin{cases}
\displaystyle\sum_{p,q=0}^{n+1} \left(\nabla G_{p\overline{q}} \cdot \nabla G_{p\overline{q}} + \nabla F_{p\overline{q}} \cdot \nabla F_{p\overline{q}}\right) = 4n, \\
\displaystyle\sum_{p,q=0}^{n+1} \left(\Delta G_{p\overline{q}} \Delta G_{p\overline{q}} + \Delta F_{p\overline{q}} \Delta F_{p\overline{q}}\right) = 16n(n+1), \\
\displaystyle\sum_{p,q=0}^{n+1} \left(\nabla G_{p\overline{q}} \Delta G_{p\overline{q}} + \nabla F_{p\overline{q}} \Delta F_{p\overline{q}}\right) = 0, \\
\displaystyle\sum_{p,q=0}^{n+1} \left\{\left(\nabla G_{p\overline{q}} \cdot \nabla u_i\right)^2 + \left(\nabla F_{p\overline{q}} \cdot \nabla u_i\right)^2\right\} = 2|\nabla u_i|^2.
\end{cases} \tag{4.13}$$

By applying the Proposition 1 to the functions $G_{p\bar{q}}$ and $F_{p\bar{q}}$ and taking sum on p and q from 0 to $n+1$, we infer

$$\lambda_{k+1} \leq \left(1+\frac{1}{n}\right)\frac{1}{k+1}\sum_{i=1}^{k}\lambda_i + 2(n+1) + \left[\left(\frac{1}{n}\frac{1}{k+1}\sum_{i=1}^{k}\lambda_i + 2(n+1)\right)^2 - \left(1+\frac{2}{n}\right)\frac{1}{k+1}\sum_{i=0}^{k}\left(\lambda_i - \frac{1}{k+1}\sum_{j=1}^{k}\lambda_j\right)^2\right]^{\frac{1}{2}}.$$

This finishes the proof of Theorem 2. □

Proof of Corollary 2. By using the same proof as in the Corollary 1, Corollary 2 is obvious. □

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