*Inequalities for Eigenvalues of Membranes and Plates**

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Introduction. In the first section of this paper we shall obtain relationships involving the eigenvalues of a membrane fixed on a given boundary and those of a membrane with vanishing normal derivative on the boundary. In the second section we shall prove a conjecture of A. WEINSTEIN which relates certain membrane eigenvalues with those of the buckling problem for a clamped plate of the same shape.

In the first section, $\mathfrak{D}$ is assumed to be a closed convex domain with boundary C in the xy-plane, and $u(x, y)$ and $v(x, y)$ satisfy the differential equations

$$\Delta u + \lambda u = 0 \tag{1a}$$

$$\Delta v + \mu v = 0 \tag{1b}$$

in $\mathfrak{D}$. We are concerned with the two different boundary conditions

$$u = 0 \qquad \text{on } C, \tag{2}$$

and

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } C, \tag{3}$$

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where ν denotes the outward normal on C. In each of these cases there are infinitely many eigenvalues λ_i, μ_i. We denote them by

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots, \tag{4}$$

and

$$0 = \mu_1 < \mu_2 \leqq \mu_3 \leqq \cdots. \tag{5}$$

The corresponding eigenfunctions are expressed as $u_1, u_2, u_3, \cdots$ and $v_1, v_2, v_3, \cdots$.

It has been shown by PÓLYA [1] that $\mu_2 < \lambda_1$. In fact the isoperimetric inequality $\mu_2 \leqq j^2 p^{-2} \lambda_1$, ($j = 2.4048$, and $p = 1.8412$), which was formulated originally by KORNHAUSER & STAKGOLD [2], has just recently been rigorously proven by SZEGÖ [3]. The equality sign is valid only if C is a circle. In this report we shall obtain relationships involving the higher eigenvalues λ_i and μ_i. In particular we shall show that for $\mathfrak{D}$ convex

$$\mu_2 < \lambda_1 - \frac{2}{(\rho h)_{\max}}, \tag{6a}$$

$$\mu_3 < \lambda_1 - \frac{1}{(\rho h)_{\max}}, \tag{6b}$$

$$\mu_{n+2} < \lambda_n, \qquad n > 1, \tag{6c}$$

where ρ is the radius of curvature on C, and the value of h at a point P on C is the distance from an arbitrary origin in $\mathfrak{D}$ to the line tangent to C through P.

The problems characterized by (1a) and (1b) with conditions (2) and (3) are classical membrane problems which have been extensively treated in the literature (see for instance [4]).

In the second section we assume $\mathfrak{D}$ to be any closed domain with boundary C in the xy-plane. Let $W(x, y)$ satisfy the differential equation

$$\Delta^2 W + \Lambda \Delta W = 0 \tag{7}$$

in $\mathfrak{D}$, and the boundary conditions

$$W = \frac{\partial W}{\partial \nu} = 0, \qquad \text{on } C. \tag{8}$$

The function W is a solution to the buckling problem for a clamped plate (see [4]). In this case it is again known that infinitely many eigenvalues Λ_i exist. We denote them by

$$\Lambda_1 \leqq \Lambda_2 \leqq \Lambda_3 \leqq \cdots, \tag{9}$$

and the corresponding eigenfunctions by $W_1, W_2, W_3, \cdots$. The conjecture of WEINSTEIN, which we shall prove, is the following: *The first eigenvalue in the* *buckling problem for a clamped plate is not less than the second eigenvalue of the membrane of the same shape which is fixed on the boundary, i.e.*

$$\lambda_2 \leqq \Lambda_1. \tag{10}$$

WEINSTEIN's conjecture was based on the known solutions for the circle and on the lower bounds he obtained for the eigenvalues of a rectangular plate.

In addition to proving (10) we show further that equality exists only in case $\mathfrak{D}$ is a simply connected region with circular boundary. The proof of this last statement was obtained jointly by Professor WEINBERGER and the author, and can be considered as an interesting application of the theory of nodal lines.

1. Inequalities for certain eigenvalues of a membrane. In obtaining the inequalities (6) we make use of the following minimum principle for the eigenvalues μ_i:

$$\mu_i = \min. \frac{D(\varphi)}{\iint_{\mathfrak{D}} \varphi^2 \, dA}, \tag{11}$$

where $D(\varphi)$ denotes the Dirichlet integral on $\mathfrak{D}$, and φ is any sufficiently smooth function defined in $\mathfrak{D}$ which satisfies the condition

$$\iint_{\mathfrak{D}} v_k \, \varphi \, dA = 0, \qquad k = 1, 2, \cdots, i - 1. \tag{12}$$

We employ the following trial functions

$$\begin{aligned} \varphi_1 &= \frac{\partial u_1}{\partial x}, \\ \varphi_1' &= \frac{\partial u_1}{\partial y}, \\ \varphi_2 &= \frac{\partial u_1}{\partial x} + a_0 \frac{\partial u_1}{\partial y}, \\ \varphi_{n+2} &= \sum_{j=1}^{n} a_j u_j + a_{n+1} \frac{\partial u_n}{\partial x} + \frac{\partial u_n}{\partial y}, \qquad n \geqq 1. \end{aligned} \tag{13}$$

Inequalities (6a), (6b) and (6c) will be established in (A), (B) and (C) respectively.

A. It is well known that v_1 = constant and hence φ_1 and φ_1' satisfy (12) for $k = 1$. It follows then from (11) that

$$\mu_2 \leqq \frac{D(\varphi_1)}{\iint_{\mathfrak{D}} \varphi_1^2 \, dA}, \tag{14}$$

and

$$\mu_2 \leqq \frac{D(\varphi_1')}{\iint_{\mathfrak{D}} (\varphi_1')^2 \, dA}. \tag{15}$$

If the x and y directions are chosen in such a way that $\iint_{\mathfrak{D}} \varphi_1^2 \, dA = \iint_{\mathfrak{D}} (\varphi_1')^2 \, dA$, then from addition of (14) and (15)

$$\begin{aligned} 2\mu_2 &\leqq \frac{D(\varphi_1) + D(\varphi_1')}{\iint_{\mathfrak{D}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA} \\ &= 2 \frac{\left[D\left(\frac{\partial u_1}{\partial x}\right) + D\left(\frac{\partial u_1}{\partial y}\right)\right]}{D(u_1)}. \end{aligned} \tag{16}$$

An application of Green's theorem gives

$$\mu_2 \leqq \lambda_1 + \frac{\frac{1}{2} \oint_C \frac{\partial}{\partial \nu} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] ds}{D(u_1)}. \tag{17}$$

The term in brackets may be replaced by the quantity $[(\partial u_1/\partial \nu)^2 + (\partial u_1/\partial s)^2]$, where the second integral is taken in the tangential direction on the boundary. Since u_1 vanishes on C

$$\mu_2 \leqq \lambda_1 + \frac{\oint_C \frac{\partial u_1}{\partial \nu} \frac{\partial^2 u_1}{\partial \nu^2} \, ds}{D(u_1)}, \tag{18}$$

or, from the differential equation satisfied by u_1

$$\mu_2 \leqq \lambda_1 - \frac{\oint_C \frac{1}{\rho} \left(\frac{\partial u_1}{\partial \nu}\right)^2 ds}{D(u_1)}, \tag{19}$$

where ρ is the radius of curvature of C. We now make use of an identity due to RELLICH [5], which in two dimensions has the form

$$\lambda_i \iint_{\mathfrak{D}} u_i^2 \, dA = \frac{1}{2} \oint_C h \left(\frac{\partial u_i}{\partial \nu}\right)^2 ds. \tag{20}$$

The quantity h is defined in (6). Thus for convex $\mathfrak{D}$

$$\mu_2 \leqq \lambda_1 - \frac{2}{(\rho h)_{\max}}. \tag{21}$$

We now show that (21) is a strict inequality; for in order that the equality sign in (21) be valid φ_1 and φ_1' must have vanishing normal derivatives on C, *i.e.* $(\partial/\partial\nu)$ $(\partial u_1/\partial x)$ and $(\partial/\partial\nu)$ $(\partial u_1/\partial y) = 0$. It follows from (1a) and (2) that $(\partial/\partial s)$ $(\partial u_1/\partial x)$ and $(\partial/\partial s)$ $(\partial u_1/\partial y)$ must also vanish on C. But the functions $\partial u_1/\partial x$ and $\partial u_1/\partial y$ each vanish at points on the boundary; hence they must vanish identically on C. For the equality sign to be valid then both u_1 and $\partial u_1/\partial\nu$ must vanish on the boundary. But this would imply that u_1 must vanish identically (Equation (20)), which it does not do. Hence (21) is a strict inequality and (6a) is proven.

B. In order to verify the inequality (6b) we introduce φ_2 into (11), with the constant a_0 so chosen that (12) is satisfied for $k = 2$. Thus

$$\mu_3 \leqq \frac{D(\varphi_2)}{\iint_{\mathfrak{D}} \varphi_2^2 \, dA} = \lambda_1 + \frac{\oint_C \varphi_2 \dfrac{\partial\varphi_2}{\partial\nu}\, ds}{D(\varphi_2)}. \tag{22}$$

It is easily seen that

$$\oint_C \frac{\partial u_i}{\partial x}\frac{\partial}{\partial\nu}\left(\frac{\partial u_i}{\partial y}\right) ds = 0,$$

for

$$\begin{aligned}\oint_C \frac{\partial u_i}{\partial x}\frac{\partial}{\partial\nu}\left(\frac{\partial u_i}{\partial y}\right) ds &= \iint_{\mathfrak{D}} \frac{\partial u_i}{\partial x}\Delta\left(\frac{\partial u_i}{\partial y}\right) dA + D\left(\frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial y}\right)\\ &= -\lambda_i \iint_{\mathfrak{D}} \left[\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial y} + u_i \frac{\partial^2 u_i}{\partial x\partial y}\right] dA\\ &= 0\end{aligned} \tag{23}$$

upon integration by parts. Similarly

$$\oint_C \frac{\partial u_i}{\partial y}\frac{\partial}{\partial\nu}\left(\frac{\partial u_i}{\partial x}\right) ds = 0$$

and

$$\begin{aligned}\oint_C \frac{\partial}{\partial\nu}\left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial u_i}{\partial y}\right)^2\right] ds &= \iint_{\mathfrak{D}} \Delta\left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial u_i}{\partial y}\right)^2\right] dA\\ &= -2\lambda_i \iint_{\mathfrak{D}} \left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial u_i}{\partial y}\right)^2 + u_i\left(\frac{\partial^2 u_i}{\partial x^2} - \frac{\partial^2 u_i}{\partial y^2}\right)\right] dA\\ &= 0\end{aligned} \tag{24}$$

by partial integration. In (23) and (24) we have used the differential equation satisfied by u_i at step two. We now proceed to evaluate the boundary integral in (22).

$$\oint_C \varphi_2 \frac{\partial \varphi_2}{\partial \nu} ds = \oint_C \left(\frac{\partial u_i}{\partial x} + a_0 \frac{\partial u_1}{\partial y}\right) \frac{\partial}{\partial \nu}\left(\frac{\partial u_1}{\partial x} + a_0 \frac{\partial u_1}{\partial y}\right) ds$$
$$(25) \qquad = \oint_C \left[\frac{\partial u_i}{\partial x}\frac{\partial}{\partial \nu}\left(\frac{\partial u_i}{\partial x}\right) + a_0^2 \frac{\partial u_i}{\partial y}\frac{\partial}{\partial \nu}\left(\frac{\partial u_i}{\partial y}\right)\right] ds,$$

the other terms vanishing as seen from (23); or

$$\oint_C \varphi_2 \frac{\partial \varphi_2}{\partial \nu} ds = \tfrac{1}{2}(1 + a_0^2) \oint_C \frac{\partial}{\partial \nu}\left[\left(\frac{\partial u_i}{\partial x}\right)^2 + \left(\frac{\partial u_i}{\partial y}\right)^2\right] ds$$
$$(26) \qquad + \tfrac{1}{2}(1 - a_0^2) \oint_C \frac{\partial}{\partial \nu}\left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial u_i}{\partial y}\right)^2\right] ds.$$

The last integral in (26) vanishes by (24); hence

$$\mu_3 \leqq \lambda_1 + \tfrac{1}{2}(1 + a_0^2) \frac{\oint_C \frac{\partial u_i}{\partial \nu} \frac{\partial^2 u_i}{\partial \nu^2} ds}{D(\varphi_2)}$$
$$(27) \qquad = \lambda_1 - \tfrac{1}{2}(1 + a_0^2) \frac{\oint_C \frac{1}{\rho}\left(\frac{\partial u_i}{\partial \nu}\right)^2 ds}{D(\varphi_2)}$$
$$\leqq \lambda_1.$$

In fact from (20)

$$(28) \quad \mu_3 \leqq \lambda_1 \left\{1 - \frac{(1 + a_0^2)}{(\rho h)_{\max}} \frac{\iint_{\mathfrak{D}} u_1^2 \, dA}{\iint_{\mathfrak{D}} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + 2a_0 \frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y} + a_0^2 \left(\frac{\partial u_1}{\partial y}\right)^2\right] dA}\right\}.$$

But

$$\iint_{\mathfrak{D}} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + 2a_0 \frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y} + a_0^2 \left(\frac{\partial u_1}{\partial y}\right)^2\right] dA$$
$$(29) \qquad \leqq \left[\sqrt{\iint_{\mathfrak{D}} \left(\frac{\partial u_1}{\partial x}\right)^2 dA} + |a_0| \sqrt{\iint_{\mathfrak{D}} \left(\frac{\partial u_1}{\partial y}\right)^2 dA}\right]^2$$
$$= \tfrac{1}{2}(1 + |a_0|)^2 D(u_1),$$

the last reduction arising from the fact that the x and y directions are again so chosen that the integrals under the radical signs are equal. Hence

$$\mu_3 \leqq \lambda_1 - \frac{1 + a_0^2}{(1 + |a_0|)^2} \frac{2}{(\rho h)_{\max}} \leqq \lambda_1 - \frac{1}{(\rho h)_{\max}}. \tag{30}$$

We show again that the equality sign in (30) is never valid. Inequality is introduced in (29) unless $a_0 = \pm 1$ and at the same time $\partial u_1/\partial x = \pm(\partial u_1/\partial y)$ (the plus or minus signs corresponding in either case). But for this to be true the eigenfunction u_1 must possess nodal lines. Since this is not the case (30) becomes a strict inequality and (6b) is established.

C. For proof of (6c) we use φ_{n+2} as the trial function in (11). The constants a_j and a_{n+1} are so chosen that (12) is satisfied for $k = 1, 2, 3, \cdots n - 1$. Then

$$\begin{aligned}
\left(\iint_{\mathfrak{D}} \varphi_{n+2}^2 \, dA\right) \mu_{n+2} &\leqq D(\varphi_{n+2}) \\
&= \sum_{j=1}^{n} a_j^2 \lambda_j \iint_{\mathfrak{D}} u_j^2 \, dA + 2\lambda_n \sum_{j=1}^{n} a_j \iint_{\mathfrak{D}} u_j \left(a_{n+1} \frac{\partial u_n}{\partial x} + \frac{\partial u_n}{\partial x}\right) dA \\
&\qquad + \iint_{\mathfrak{D}} \left| \operatorname{grad} \left(a_{n+1} \frac{\partial u_n}{\partial x} + \frac{\partial u_n}{\partial y}\right)\right|^2 dA \\
&\leqq \lambda_n \iint_{\mathfrak{D}} \varphi_{n+2}^2 \, dA + \oint_C \left(a_{n+1} \frac{\partial u_n}{\partial x} + \frac{\partial u_n}{\partial y}\right) \\
&\qquad \cdot \frac{\partial}{\partial \nu} \left(a_{n+1} \frac{\partial u_n}{\partial x} + \frac{\partial u_n}{\partial y}\right) ds \\
&= \lambda_n \iint_{\mathfrak{D}} \varphi_{n+2}^2 \, dA + \tfrac{1}{4}(1 + a_{n+1}^2) \oint_C \frac{\partial}{\partial \nu} \left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2\right] ds \\
&< \lambda_n \iint_{\mathfrak{D}} \varphi_{n+2}^2 \, dA.
\end{aligned} \tag{31}$$

This establishes the third of the inequalities (6).

2. Proof of Weinstein's conjecture. In this section we establish the inequality

$$\lambda_2 \leqq \Lambda_1. \tag{32}$$

To this end we make use of the following minimum principle for the estimation of λ_2:

$$\lambda_2 = \min \frac{D(\psi)}{\iint_{\mathfrak{D}} \psi^2 \, dA}, \tag{33}$$

where ψ is any sufficiently smooth function defined in $\mathfrak{D}$ which vanishes on C and satisfies the condition

$$\iint_{\mathfrak{D}} u_1 \psi \, dA = 0. \tag{34}$$

We choose as trial functions ψ_1 and ψ_2 the expressions

$$\begin{aligned} \psi_1 &= a_1 W_1 + \frac{\partial W_1}{\partial x}, \\ \psi_2 &= a_2 W_1 + \frac{\partial W_1}{\partial y}, \end{aligned} \tag{35}$$

where W_1 is the first eigenfunction in the buckling problem for the clamped plate of the same shape (satisfying (7) and (8)), and the x and y directions are as yet unspecified. The constants a_1 and a_2 are so chosen that (34) is satisfied.

We shall establish the inequality (32) in (A) and prove in (B), (C) and (D) that the equality sign in (32) is valid if and only if D is a simply connected region with circular boundary C.

A. The insertion of ψ_1 and ψ_2 from (35) into (33) gives rise to the inequalities

$$\lambda_2 \leqq \frac{a_1^2 D(W_1) + D\left(\dfrac{\partial W_1}{\partial x}\right)}{a_1^2 \displaystyle\iint_{\mathfrak{D}} W_1^2 \, dA + \iint_{\mathfrak{D}} \left(\frac{\partial W_1}{\partial x}\right)^2 dA}, \tag{36}$$

and

$$\lambda_2 \leqq \frac{a_2^2 D(W_1) + D\left(\dfrac{\partial W_1}{\partial y}\right)}{a_2^2 \displaystyle\iint_{\mathfrak{D}} W_1^2 \, dA + \iint_{\mathfrak{D}} \left(\frac{\partial W_1}{\partial y}\right)^2 dA}. \tag{37}$$

We use now a well known arithmetic result (see WEINSTOCK [6]), *i.e.* if m, n, m', and n' are all positive, then

$$\left.\begin{aligned} h &\leqq m/n \\ h &\leqq m'/n' \end{aligned}\right\} \text{ implies } h \leqq \frac{m + m'}{n + n'}. \tag{38}$$

This follows from the fact that the expression

$$\frac{m + m'}{n + n'} = \frac{n(m/n) + n'(m'/n')}{n + n'} \tag{39}$$

is a mean value of m/n and m'/n'. Hence

$$\frac{m + m'}{n + n'} \geqq \min\left[\frac{m}{n}, \frac{m'}{n'}\right]. \tag{40}$$

Thus (36) and (37) imply

$$\lambda_2 \leqq \frac{(a_1^2 + a_2^2)D(W_1) + D\left(\frac{\partial W_1}{\partial x}\right) + D\left(\frac{\partial W_1}{\partial y}\right)}{(a_1^2 + a_2^2)\iint_{\mathfrak{D}} W_1^2\, dA + D(W_1)}. \tag{41}$$

Since $W_1 = (\partial W_1/\partial \nu) = 0$ on C it is easily shown from Green's theorem that

$$D\left(\frac{\partial W_1}{\partial x}\right) + D\left(\frac{\partial W_1}{\partial y}\right) = \iint_{\mathfrak{D}} (\Delta W_1)^2\, dA. \tag{42}$$

Also from the Schwarz inequality

$$D(W_1) \leqq \frac{\iint_{\mathfrak{D}} (\Delta W_1)^2\, dA}{D(W_1)} \iint_{\mathfrak{D}} W_1^2\, dA. \tag{43}$$

Since

$$\iint_{\mathfrak{D}} (\Delta W_1)^2\, dA - \Lambda_1 D(W_1) = 0, \tag{44}$$

upon insertion of (42) and (43) in (41) and making use of (44) we obtain the desired inequality,

$$\lambda_2 \leqq \Lambda_1. \tag{45}$$

In the case of a circular region the eigenvalues λ_2 and Λ_1 are known to be equal. We shall now prove that this is the only case in which equality exists.

B. It is convenient at this point to make a few preliminary observations. The first of these is the fact that if equality exists then a_1 and a_2 in (35) must be zero. If this is not the case, it follows from (41) that W_1 must be a solution of the membrane equation which vanishes together with its first derivative on the boundary C. But according to (20) such a function must vanish identically. Thus a_1 and a_2 in (35) must be zero if the equality sign in (45) is to hold. It follows then that, if λ_2 and Λ_1 are equal, $\partial W_1/\partial x$ and $\partial W_1/\partial y$ are membrane eigenfunctions for the second eigenvalue λ_2. Since the x and y directions are completely arbitrary the partial derivative of W_1 in any direction must give an eigenfunction with eigenvalue λ_2.

It is well known (see WEINSTEIN [7]) that the function W_1 admits the following decomposition

$$W_1 = u_1^* + h, \tag{46}$$

where

$$\Delta u_1^* + \Lambda_1 u_1^* = 0, \tag{47}$$

$$\Delta h = 0, \qquad \text{in } \mathfrak{D}.$$

But in case of equality one has

$$\Delta u_1^* + \lambda_2 u_1^* = 0, \tag{48}$$
$$\Delta\left(\frac{\partial u_1^*}{\partial s}\right) + \lambda_2\left[\frac{\partial u_1^*}{\partial s} + \frac{\partial h}{\partial s}\right] = 0,$$

where s denotes any direction. It is clear then that if $\lambda_2 = \Lambda_1$

$$h = \text{constant}. \tag{49}$$

Since W_1 is a solution to the clamped plate buckling problem it follows that u_1^* is a solution to the membrane equation (1) which satisfies the condition

$$u_1^* = \text{constant on } C, \tag{50}$$
$$\frac{\partial u_1^*}{\partial \nu} = 0 \text{ on } C.$$

Thus we have from (3): $\Lambda_1 = \lambda_2 = \mu_i$, where tbe value of i is undetermined.

Now if $\partial W_1/\partial x$ and $\partial W_1/\partial y$ are eigenfunctions with eigenvalue λ_2, so is the quantity

$$\frac{\partial W_1}{\partial \theta} = x\frac{\partial W_1}{\partial y} - y\frac{\partial W_1}{\partial x}, \tag{51}$$

where the choice of origin is arbitrary. The quantities $\partial W_1/\partial x$, $\partial W_1/\partial y$, and $\partial W_1/\partial \theta$ are independent unless the boundary C is circular, for otherwise

$$x\frac{\partial W_1}{\partial y} - y\frac{\partial W_1}{\partial x} = a\frac{\partial W_1}{\partial x} - b\frac{\partial W_1}{\partial y}. \tag{52}$$

With a new choice of origin at $x = a$, $y = b$, this reduces to

$$\frac{\partial W_1}{\partial \theta'} = 0, \tag{53}$$

which cannot be true unless C is a circle. However, the region $\mathfrak{D}$ may still be ring-shaped. Hence we shall treat in (C) the case where C is not a circle and shall handle the case where $\mathfrak{D}$ is ring-shaped in section (D).

Assuming that C is not circular we shall now proceed to prove that $\partial W_1/\partial x$, $\partial W_1/\partial y$, and $\partial W_1/\partial \theta$ cannot all be eigenfunctions for λ_2. We do this by showing that a particular combination of the three results in an eigenfunction which contradicts the nodal line theorem (see [4, p. 393]). From this it follows that $\partial W_1/\partial x$ and $\partial W_1/\partial y$ are not eigenfunctions for λ_2, and hence $\lambda_2 < \Lambda_1$.

C. Let us for convenience choose $u_2 = \partial W_1/\partial \theta$, with the origin taken at the point where W_1 has a stationary value. (This is equivalent to taking for u_2 a linear combination of $\partial W_1/\partial x'$, $\partial W_1/\partial y'$, and $\partial W_1/\partial \theta'$, where the primes correspond to any translation of the origin and rotation of the coordinate system). Around this point we expand the quantity W_1 in a series of the type:

$$(54) \qquad W_1 = \text{Const.} + A_0 J_0(\lambda_2 r) + \sum_{n=1}^{\infty} A_n J_n(\lambda_2 r) \cos n(\theta - \alpha_n),$$

where A_0, A_n and α_n are constants. Then

$$(55) \qquad u_2 = \frac{\partial W_1}{\partial \theta} = -\sum_{n=1}^{\infty} n A_n J_n(\lambda_2 r) \sin n(\theta - \alpha_n).$$

From the choice of the origin it is clear that the quantity $\partial u_2/\partial r$ also vanishes at the origin; hence

$$(56) \qquad \partial u_2/\partial r = -\lambda_2 \sum_{n=1}^{\infty} n A_n J_n'(\lambda_2 r) \sin n(\theta - \alpha_n) = 0, \qquad r = 0.$$

But this means that $A_1 \equiv 0$. Thus

$$(57) \qquad u_2 = -\sum_{n=2}^{\infty} n A_n J_n(\lambda_2 r) \sin n(\theta - \alpha_n).$$

But as $r \rightarrow 0$

$$(58) \qquad u_2 = -\frac{A_2}{4} r^2 \sin 2(\theta - \alpha_2) + O(r^3).$$

This shows that the function u_2 has at least two intersecting nodal lines, or a nodal line intersecting itself, at the origin. In either case the eigenfunction u_2 must divide $\mathfrak{D}$ into at least three regions each of which is bounded by a portion of a nodal line and/or a portion of C. (Note that if $A_2 = 0$ there will be at least three intersecting nodal lines, *etc.*) But this contradicts the well known nodal line theorem [4, p. 393] which, for the membrane equation, states that if the eigenfunctions are ordered according to increasing eigenvalues the eigenfunction corresponding to the n^{th} eigenvalue can divide the region into at most n parts. Thus $\partial W_1/\partial \theta$ cannot be a second eigenfunction and hence, if C is not a circle, neither can $\partial W_1/\partial x$ and $\partial W_1/\partial y$.

D. We must now show that if $\mathfrak{D}$ is ring-shaped $\lambda_2 < \Lambda_1$. We simply observe that if W_1 is the first eigenfunction in the clamped plate buckling problem for the ring-shaped region, then the quantity $\partial W_1/\partial x$ (or $\partial W_1/\partial y$) must divide $\mathfrak{D}$ into four portions (this derivative gives a function with a ring nodal line and two portions of a diameter). Hence, by the nodal line theorem $\partial W_1/\partial x$ (or $\partial W_1/\partial y$) cannot be the second eigenfunction for the membrane.

It follows then that $\lambda_2 = \Lambda_1$ if and only if $\mathfrak{D}$ is a simply connected region with circular boundary C.

We list in (*E*) some other inequalities which either follow directly from (45) or are derived by the same methods as those used in obtaining (45).

E. It was shown by PÓLYA & SZEGÖ [8, p. 230] that if Ω_1 is the lowest eigenvalue for the vibrating clamped plate then

$$\Omega_1 \geqq \lambda_1 \Lambda_1. \tag{59}$$

From (32) we see that

$$\Omega_1 \geqq \lambda_1 \lambda_2. \tag{60}$$

One can show in a manner similar to that used in WEINSTEIN's conjecture that the successive introduction of the quantities,

$$\begin{aligned} \psi_2^1 &= a_1 W_1 + b_1 W_2 + \partial W_1/\partial x, \\ \psi_2^2 &= a_2 W_1 + b_2 W_2 + \partial W_1/\partial y, \\ \psi_2^3 &= a_3 W_1 + b_3 W_2 + \partial W_2/\partial x, \\ \psi_2^4 &= a_4 W_1 + b_4 W_2 + \partial W_2/\partial y, \end{aligned} \tag{61}$$

where the a_i and b_i are so chosen that the quantities ψ_2^i are orthogonal to both u_1 and u_2, results in the additional inequality

$$\lambda_3 \leqq \Lambda_2. \tag{62}$$

It follows then from the easily proven inequality (an extension of that of PÓLYA & SZEGÖ for the first eigenvalues [8])

$$\Omega_n \geqq \lambda_1 \Lambda_n \tag{63}$$

that

$$\Omega_2 \geqq \lambda_1 \lambda_3, \tag{64}$$

where Ω_2 is the second eigenvalue for the vibrating clamped plates, according to the ordering $\Omega_1 \leqq \Omega_2 \leqq \Omega_3 \leqq \cdots$.

Concluding Remarks. From formulas (45) and (62) one is led to the following conjecture: *The n^{th} eigenvalue in the buckling problem for a clamped plate is not less than the $n+1^{\text{st}}$ eigenvalue for the membrane of the same shape which is fixed on the boundary.* If the conjecture is true, then

$$\Omega_n \geqq \lambda_1 \lambda_{n+1}. \tag{65}$$

This conjecture is borne out in the few cases in which solutions or sufficiently close bounds are known. In fact known results seem to indicate that the third eigenvalue for the membrane is still lower than the first eigenvalue in the clamped plate buckling problem.

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