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Inequalities for q-h-Integrals via $\hbar$-Convex and m-Convex Functions

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Abstract: This paper investigates several integral inequalities held simultaneously for q and h-integrals in implicit form. These inequalities are established for symmetric functions using certain types of convex functions. Under certain conditions, Hadamard-type inequalities are deducible for q-integrals. All the results are applicable for $\hbar$-convex, m-convex and convex functions defined on the non-negative part of the real line.

Keywords: q-derivative; q-integral; q-h-integral; Jensen inequality; Hadamard inequality

MSC: 26A33; 26D15; 26A51; 33E12

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1. Introduction and Preliminaries

It is very interesting to think outside the box on ordinary concepts defined in the literature in classical ways. For example, fractional order derivatives as well as integrals are due to an unusual question of a fractional order derivative raised by Leibniz in the 16th century by writing a letter to L Hospital. At that time, it was considered as a stupid question, but later it was analyzed by several physicists and mathematicians to obtain an answer in the form of Riemann–Liouville fractional derivatives and integrals. In this era, the subject of fractional calculus has become an important tool for generalizing and solving the concepts of science and engineering related to ordinary calculus; for more detail please see [1,2].

In q-calculus, q-derivative and h-derivative are studied very frequently in place of ordinary derivatives. Several new theories and mathematical models of real world problems have been studied for these derivatives (see [3–5]).

The aim of this article is to construct some inequalities for q-h-derivatives and integrals of convex, $\hbar$-convex, m-convex and convex functions. For recent results on inequalities for q-calculus, see [6–13], and for (p,q)-calculus, see [14,15]. In the following, we define the aforementioned notions.

Definition 1. *Let a real function f be defined on some non-empty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if the following inequality holds:*

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b),$$

for $t \in [0,1]$, $a,b \in I$.

Definition 2 ([16])**.** *Let $\hbar : J \supseteq (0,1) \to \mathbb{R}^+{}_\circ := [0,\infty]$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}^+{}_\circ$. We say that f is $\hbar$-convex function if for all $x,y \in I$ and $t \in (0,1)$, we have*

$$f(tx+(1-t)y) \leq \hbar(t)f(x)+\hbar(1-t)f(y).$$

If $\hbar(t) = t$, we obtain the definition of a convex function. If $\hbar(t) = t^s$, $s \in [0,1]$, then we obtain the definition of an s-convex function.

Definition 3 ([17])**.** *A function $f : [0, b] \to \mathbb{R}$ is called m-convex function if for any $x, y \in [0, b]$ and $t \in [0, 1]$, we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

where $0 \leq m \leq 1$.

If $m = 1$, we obtain the definition of a convex function, and for $m = 0$, the definition of a star-shaped function is obtained. Next, we give the Hadamard inequality for a convex function, and the Fejér–Hadamard inequality for convex and symmetric functions.

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$ and $a, b \in I$ where $a < b$. Then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Theorem 2. *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and $a, b \in I$ where $a < b$. If g is a symmetric function about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2}$$

The Hadamard inequality for an $\hbar$-convex function is given in the following result.

Theorem 3 ([18])**.** *Let $f : I \to \mathbb{R}$ be an $\hbar$-convex function defined on an interval $I \subseteq \mathbb{R}$ and $a, b \in I$, where $a < b$. Then, the following Hadamard inequality for $\hbar$-convex function holds:*

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (f(a)+f(b))\int_0^1 \hbar(t)dt. \tag{3}$$

The goal of this paper is to find inequalities for q-h-integrals; in the following, we give definitions of q-derivative, q-h-derivative, q-integral and q-h-integral.

Definition 4 ([5])**.** *The q-derivative of a continuous function $f : I \to \mathbb{R}$ is defined by:*

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{4}$$

where $0 < q < 1$.

Definition 5 ([19])**.** *The q-h-derivative of a continuous function $f : I \to \mathbb{R}$ is defined by:*

$$C_h D_q f(x) = \frac{{}_h d_q f(x)}{{}_h d_q x} = \frac{f(q(x+h)) - f(x)}{(q-1)x + qh}, \tag{5}$$

where $0 < q < 1$, $h \in \mathbb{R}$.

For $h = 0$ in (5), we obtain (4), i.e.,

$$C_0 D_q f(x) = D_q f(x).$$

Definition 6 ([8])**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, the q-definite integral on $[a, b]$ is defined as follows:*

$$\int_a^x f(t) d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a). \tag{6}$$

In [6], the following q-Hadamard inequalities for differentiable convex functions are given:

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function. Then, it must satisfy the upcoming inequality for q_a-integrals:*

$$f\left(\frac{b+aq}{1+q}\right) \le \frac{\int_a^b f(x) d_q^a x}{b-a} \le \frac{qf(a)+f(b)}{1+q}. \tag{7}$$

Theorem 5. *The following inequality holds for q_a-integrals under the conditions of Theorem 4:*

$$f\left(\frac{a+bq}{1+q}\right) + f'\left(\frac{a+bq}{1+q}\right)\frac{(b-a)(1-q)}{1+q} \le \frac{\int_a^b f(x) d_q^a x}{b-a} \le \frac{qf(a)+f(b)}{1+q}. \tag{8}$$

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function. Then, it must satisfy the upcoming inequality for q_a-integrals:*

$$f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\frac{(b-a)(1-q)}{2(1+q)} \le \frac{\int_a^b f(x) d_q^a x}{b-a} \le \frac{qf(a)+f(b)}{1+q}. \tag{9}$$

Additionally, in [16], authors established the following q-Hadamrd inequality for convex functions:

Theorem 7. *A differentiable convex function $f : [a, b] \to \mathbb{R}$ must satisfy the upcoming inequality for q_b-integrals:*

$$f\left(\frac{a+bq}{1+q}\right) \le \frac{\int_a^b f(x) d_q^b x}{b-a} \le \frac{f(a)+qf(b)}{1+q}. \tag{10}$$

The above inequalities stated in Theorems 4–6 are further generalized for q-h-integrals in the article [20]. The definition of q-h-integrals is given as follows:

Definition 7 ([19])**.** *Let $0 < q < 1$ and $f : I = [a, b] \to \mathbb{R}$ be a continuous function. Then, the left q-h-integral and the right q-h-integral on I denoted by $I_{q-h}^{a^+} f$ and $I_{q-h}^{b^-} f$ are defined as follows:*

$$\begin{aligned} I_{q-h}^{a^+} f(x) &:= \int_a^x f(t)_h d_q t \\ &= ((1-q)(x-a)+qh) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x + nq^n h),\ x > a, \end{aligned} \tag{11}$$

$$\begin{aligned} I_{q-h}^{b^-} f(x) &:= \int_x^b f(t)_h d_q t \\ &= ((1-q)(b-x)+qh) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h),\ x < b. \end{aligned} \tag{12}$$

It is clear that $I_{q-h}^{a^+} f(b) = I_{q-h}^{b^-} f(a) = \int_a^b f(t)_h d_q t$.

In the next section, by using Jensen's inequality and applying q-h-integrals, Theorem 8 is proved for convex functions. By applying Definitions 2 and 3 and symmetry of functions, Theorems 10 and 12 are established. Special cases are discussed after the proof of each theorem.

2. Generalizations of the q-Hadamard Inequalities

In this section, we prove inequalities for $\hbar$-convex, m-convex and convex functions by using q-h-integrals. Some special cases are also given at the end of each theorem.

Theorem 8. *Let $f : I \to \mathbb{R}$ be a convex function and $q \in (0,1)$; also let $\sum_{k=0}^{\infty} kq^{2k} = S$, then for $a, b \in I$, $a < b$, we have the following inequality that holds for q-h-integrals:*

$$\begin{aligned} &f\left(\frac{a+qx}{1+q} + (1-q)hS\right) + f\left(\frac{x+qb}{1+q} + (1-q)hS\right) \\ &\leq \frac{1-q}{(1-q)(x-a)+qh} \int_a^x f(x)_h d_q x + \frac{1-q}{(1-q)(b-x)+qh} \int_x^b f(x)_h d_q x \\ &\leq \frac{(f(a)+qf(b))(b-a) + (1+q)(f(a)(b-x)+f(b)(x-a))}{(1+q)(b-a)} + \frac{2(f(b)-f(a))}{b-a} \\ &\times hS(1-q). \end{aligned} \tag{13}$$

Proof. One can easily see that

$$\frac{a+qx}{1+q} + hS(1-q) = \sum_{k=0}^{\infty} (1-q)q^k(q^k a + (1-q^k)x + kq^k h),$$

where $\sum_{k=0}^{\infty}(1-q)q^k = 1$. By using the Jensen's inequality and the definition of left q-h-integrals, one can have

$$\begin{aligned} &f\left(\frac{a+qx}{1+q} + hS(1-q)\right) \\ &\leq \frac{1-q}{(1-q)(x-a)+qh} \sum_{k=0}^{\infty} ((1-q)(x-a)+qh)q^k f(q^k a + (1-q^k)x + kq^k h) \\ &= \frac{1-q}{(1-q)(x-a)+qh} \int_a^x f(x)_h d_q t. \end{aligned} \tag{14}$$

Additionally, one can observe that

$$\frac{x+qb}{1+q} + hS(1-q) = \sum_{k=0}^{\infty} (1-q)q^k(q^k x + (1-q^k)b + kq^k h).$$

Again, by using the Jensen's inequality and the definition of right q-h-integrals, one can have

$$\begin{aligned} &f\left(\frac{x+qb}{1+q} + hS(1-q)\right) \\ &\leq \frac{1-q}{(1-q)(b-x)+qh} \sum_{k=0}^{\infty} ((1-q)(b-x)+qh)q^k f(q^k x + (1-q^k)b + kq^k h) \\ &= \frac{1-q}{(1-q)(b-x)+qh} \int_x^b f(t)_h d_q t. \end{aligned} \tag{15}$$

The first inequality in (13) is obtained by summing the inequalities (14) and (15).

Next, we prove the second inequality of (13). By using the convexity of f, we know that $f(x) \leq k(x)$, where $k(x)$ is the chord that joins the points $(a, f(a))$ and $(b, f(b))$, given as follows:

$$k(x) = \frac{f(b) - f(a)}{b - a} x + \frac{bf(a) - af(b)}{b - a}.$$

The following inequality for the left q-h-integral can be yielded:

$$\begin{aligned}
&\int_a^x f(x)_h d_q x \leq \int_a^x \left(\frac{f(b) - f(a)}{b - a} x + \frac{bf(a) - af(b)}{b - a} \right)_h d_q x \\
&= \frac{(1-q)(x-a) + qh}{(1-q)(b-a)} \left(\frac{f(a)(b-a) + qf(a)(b-x) + qf(b)(x-a)}{(1+q)} \right) \\
&+ (f(b) - f(a))hS(1-q).
\end{aligned}$$

This further takes the following form

$$\begin{aligned}
&\int_a^x f(x)_h d_q x \leq \frac{(1-q)(x-a) + qh}{(1-q)(b-a)} \left(\frac{f(a)(b-a) + qf(a)(b-x) + qf(b)(x-a)}{(1+q)} \right) \\
&+ (f(b) - f(a))hS(1-q).
\end{aligned} \tag{16}$$

Additionally, the following inequality for the right q-h-integral can be yielded:

$$\begin{aligned}
&\int_x^b f(x)_h d_q x \leq \int_x^b \left(\frac{f(b) - f(a)}{b - a} x + \frac{bf(a) - af(b)}{b - a} \right)_h d_q x \\
&= \frac{(1-q)(b-x) + qh}{1-q} \left(\frac{f(b)(x-a) + f(a)(b-x) + qf(b)(b-a)}{(1+q)(b-a)} \right. \\
&\left. + \frac{f(b) - f(a)}{b - a} hS(1-q) \right).
\end{aligned}$$

This further takes the following form

$$\begin{aligned}
&\int_x^b f(x)_h d_q x \leq \frac{(1-q)(b-x) + qh}{(1-q)(b-a)} \left(\frac{f(b)(x-a) + f(a)(b-x) + qf(b)(b-a)}{1+q} \right) \\
&+ (f(b) - f(a))hS(1-q).
\end{aligned} \tag{17}$$

From Equations (16) and (17), one can obtain the second inequality of (13). □

Corollary 1. *If $h = 0$, we obtain the following inequality, which holds for q-integrals:*

$$\begin{aligned}
&f\left(\frac{a + qx}{1+q}\right) + f\left(\frac{x + qb}{1+q}\right) \leq \left[\frac{1}{x-a} \int_a^x f(x) d_q x + \frac{1}{b-x} \int_x^b f(x) d_q x \right] \\
&\leq \frac{(f(a) + qf(b))(b-a) + (1+q)(f(a)(b-x) + f(b)(x-a))}{(1+q)(b-a)}.
\end{aligned} \tag{18}$$

Theorem 9. *Under the assumptions of above Theorem 8, one can have the following inequality:*

$$\begin{aligned}
&f\left(\frac{a + qb}{1+q} + hS(1-q)\right) \leq \frac{1-q}{(1-q)(b-a) + qh} \int_a^b f(t)_h d_q t \leq \frac{f(a) + qf(b)}{1+q} \\
&+ \frac{f(b) - f(a)}{b - a} hS(1-q).
\end{aligned} \tag{19}$$

Proof. By setting $x = b$ in (14) or $x = a$ in (15), we obtain the following inequality:

$$f\left(\frac{a+qb}{1+q}+hs(1-q)\right)\leq\frac{1-q}{(1-q)(b-a)+qh}\int_a^b f(t)_h d_qt. \tag{20}$$

Similarly, by setting $x=b$ in (16) or $x=a$ in (17), we obtain the following inequality:

$$\frac{1-q}{(1-q)(b-a)+qh}\int_a^b f(t)_h d_qt\leq\frac{f(a)+qf(b)}{1+q}+\frac{f(b)-f(a)}{b-a}hs(1-q). \tag{21}$$

Further, from (20) and (21), the inequality (19) can be obtained. □

Corollary 2. *If we put $h=0$ in (19), the following Hadamard type inequality holds for q-integrals:*

$$f\left(\frac{a+qb}{1+q}\right)\leq\frac{1}{b-a}\int_a^b f(x)d_qt\leq\frac{f(a)+qf(b)}{1+q}. \tag{22}$$

Theorem 10. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}^+{}_\circ$ be $\hbar$-convex function such that $\hbar(\frac{1}{2})\neq 0$ and $q\in(0,1)$. Additionally, let $a,b\in I$, $a<b$.*

(i) If f is symmetric about $\frac{a+x}{2}$, $x\in(a,b)$, then we have the following inequality for left q-h-integrals:

$$\begin{aligned}\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+x}{2}\right)&\leq\frac{1-q}{(1-q)(x-a)+qh_1}\int_a^x f(t)\,{}_{h_1}d_qt \\ &\leq f(x)\int_0^1\hbar(t)\,{}_h d_qt+f(a)\int_0^1\hbar(1-t)_h d_qt,\end{aligned} \tag{23}$$

where $h_1=(x-a)h$.

(ii) If f is symmetric about $\frac{x+b}{2}$, $x\in(a,b)$, then we have the following inequality for right q-h-integrals:

$$\begin{aligned}\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{x+b}{2}\right)&\leq\frac{1-q}{(1-q)(b-x)+qh_2}\int_x^b f(t)\,{}_{h_2}d_qt \\ &\leq f(b)\int_0^1\hbar(t)_h d_qt+f(x)\int_0^1\hbar(1-t)_h d_qt,\end{aligned} \tag{24}$$

where $h_2=(b-x)h$.

Proof. We prove (i) and (ii) as follows:

(i) By using the $\hbar$-convexity of f, the following inequality is yielded:

$$\frac{1}{\hbar(\frac{1}{2})}f\left(\frac{a+x}{2}\right)\leq f(ta+(1-t)x)+f(tx+(1-t)a),\ t\in[0,1].$$

Taking q-h-integral on both sides, we have

$$\frac{1}{\hbar(\frac{1}{2})}f\left(\frac{a+x}{2}\right)\leq\frac{1-q}{(1-q)+qh}\left(\int_0^1 f(x-t(x-a))_h d_qt+\int_0^1 f(a+t(x-a))_h d_qt\right). \tag{25}$$

It is given that $f(a+x-z)=f(z)$ for all $z\in(a,x)$. Therefore, inequality (25) takes the following form:

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+x}{2}\right)\leq\frac{1-q}{(1-q)+qh}\int_0^1 f(a+t(x-a))_h d_qt. \tag{26}$$

From the definition of the left q-h-integral, we have

$$\begin{aligned} &\frac{(1-q)+qh}{(1-q)(x-a)+qh_1}\int_a^x f(t)\,{}_{h_1}d_qt \\ &=((1-q)+qh)\sum_{k=0}^{\infty}q^kf(q^ka+(1-q^k)x+kq^kh_1)=\int_0^1 f(a+(x-a)t)_hd_qt. \end{aligned} \tag{27}$$

Now, by using the $\hbar$-convexity of f, the last term of (27) can be estimated as follows:

$$\int_0^1 f(a+t(x-a))\,{}_hd_qt \le f(x)\int_0^1 \hbar(t)_hd_qt+f(a)\int_0^1\hbar(1-t)_hd_qt.$$

Hence, from (27), we obtain the following inequality:

$$\frac{(1-q)+qh}{(1-q)(x-a)+qh_1}\int_a^x f(t)\,{}_{h_1}d_qt \le f(x)\int_0^1 \hbar(t)_hd_qt+f(a)\int_0^1\hbar(1-t)_hd_qt. \tag{28}$$

Inequalities (26), (27) and (28) constitute the required inequality (23).

(ii) Again, by using the $\hbar$-convexity of f, one can have the following inequality:

$$\frac{1}{\hbar(\frac{1}{2})}f\left(\frac{x+b}{2}\right) \le f(tx+(1-t)b)+f(tb+(1-t)x),\ t\in[0,1].$$

Taking the q-h-integral on both sides, we have

$$\frac{1}{\hbar(\frac{1}{2})}f\left(\frac{x+b}{2}\right) \le \frac{1-q}{(1-q)+qh}\left(\int_0^1 f(b-t(b-x))_hd_qt+\int_0^1 f(x+t(b-a))_hd_qt\right). \tag{29}$$

It is given that $f(x+b-z)=f(z)$ for all $z\in(x,b)$. Therefore, inequality (29) takes the following form:

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{x+b}{2}\right) \le \frac{1-q}{(1-q)+qh}\int_0^1 f(a+t(x-a))_hd_qt. \tag{30}$$

From the definition of the right q-h-integral, we have

$$\begin{aligned} &\frac{(1-q)+qh}{(1-q)(b-x)+qh_2}\int_x^b f(t)\,{}_{h_2}d_qt \\ &=((1-q)+qh)\sum_{k=0}^{\infty}q^kf(q^kx+(1-q^k)b+kq^kh_2)=\int_0^1 f(x+(b-x)t)_hd_qt. \end{aligned} \tag{31}$$

Now, by using the $\hbar$-convexity of f, the last term of (31) can be estimated as follows:

$$\int_0^1 f(x+t(b-x))\,{}_hd_qt \le f(b)\int_0^1 \hbar(t)_hd_qt+f(x)\int_0^1\hbar(1-t)_hd_qt.$$

Hence, from (31), we obtain the following inequality:

$$\frac{(1-q)+qh}{(1-q)(x-a)+qh_2}\int_x^b f(t)\,{}_{h_2}d_qt \le f(x)\int_0^1 \hbar(t)_hd_qt+f(a)\int_0^1\hbar(1-t)_hd_qt. \tag{32}$$

Inequalities (30), (31) and (32) constitute the required inequality (24). $\square$

Corollary 3. *By setting $h=0$, in (23) and (24), the following inequalities hold for left and right q-integrals, respectively:*

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+x}{2}\right) \leq \frac{1}{x-a}\int_a^x f(t)\,d_qt \leq f(x)\int_0^1 \hbar(t)\,d_qt + f(a)\int_0^1 \hbar(1-t)d_qt,$$

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{x+b}{2}\right) \leq \frac{1}{b-x}\int_x^b f(t)\,d_qt \leq f(b)\int_0^1 \hbar(t)d_qt + f(x)\int_0^1 \hbar(1-t)_h d_qt.$$

Theorem 11. *Under the assumptions of above Theorem 10, one can obtain the following inequality:*

$$\begin{aligned}\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \frac{1-q}{(1-q)(b-a)+qh_3}\int_a^b f(t)\,{}_{h_3}d_qt \\ &\leq f(b)\int_0^1 \hbar(t)\,{}_h d_qt + f(a)\int_0^1 \hbar(1-t)_h d_qt,\end{aligned} \tag{33}$$

where $h_3 = (b-a)h$.

Proof. By setting $x = b$ in (23) or $x = a$ in (24), the required inequality (33) can be obtained. □

Corollary 4. *By setting* $h = 0$ *in* (33)*, the following inequality is obtained for q-integrals:*

$$\frac{1}{2\hbar(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)d_qt \leq f(b)\int_0^1 \hbar(t)_h d_qt + f(a)\int_0^1 \hbar(1-t)_h d_qt. \tag{34}$$

Theorem 12. *Let* $f : [0, b_1] \to \mathbb{R}$ *be an m-convex function and* $q \in (0,1)$*. Additionally, let* $a, b \in [0, b_1]$*,* $a < b$*.*
(i) If $f\left(\frac{a+x-z}{m}\right) = f(z)$*,* $z \in (a, x)$*, then we have*

$$\begin{aligned}f\left(\frac{a+x}{2}\right) &\leq \frac{(1-q)(1+m)}{2((1-q)(x-a)+qh_1)}\int_a^x f(x)_{h_1}d_qx \\ &\leq \frac{(1-q)+qh}{2(1-q)}\left(f(x)\left(\frac{q}{1+q}+(1-q)hS\right)+mf\left(\frac{a}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right),\end{aligned} \tag{35}$$

(ii) If $f\left(\frac{x+b-z}{m}\right) = f(z)$*,* $z \in (x, b)$*, then we have*

$$\begin{aligned}f\left(\frac{x+b}{2}\right) &\leq \frac{(1-q)(1+m)}{2((1-q)(b-x)+qh_2)}\int_x^b f(x)_{h_2}d_qx \\ &\leq \frac{(1-q)+qh}{2(1-q)}\left(f(b)\left(\frac{q}{1+q}+(1-q)hS\right)+mf\left(\frac{x}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right).\end{aligned} \tag{36}$$

Proof. Using m-convexity of f, we have the following inequality:

$$f\left(\frac{a+x}{2}\right) \leq \frac{1}{2}(f(a+t(x-a)) + mf\left(\frac{x-t(x-a)}{m}\right).$$

Using the given condition $f\left(\frac{a+x-z}{m}\right) = f(z)$, $z \in (a, x)$ and taking the left q-h-integral on both sides, we have

$$2f\left(\frac{a+x}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)+qh}\int_0^1 f(a+t(x-a))_h d_qx. \tag{37}$$

By using the m-convexity of f, we obtain

$$\int_0^1 f(a+t(x-a))_h d_q t \le f(x)\int_0^1 t\,{}_h d_q t + mf\left(\frac{a}{m}\right)\int_0^1 (1-t)\,{}_h d_q t. \tag{38}$$

From (37), (27) and (38), we obtain the following inequality:

$$2f\left(\frac{a+x}{2}\right) \le \frac{(1-q)(1+m)}{(1-q)(x-a)+qh_1}\int_a^x f(x)_{h_1} d_q x \le f(x)\int_0^1 t\,{}_h d_q t + mf\left(\frac{a}{m}\right)\int_0^1 (1-t)_h d_q t. \tag{39}$$

Similarly, from (37) and the definition of a right q-h-integral, one can obtain the following inequality:

$$2f\left(\frac{x+b}{2}\right) \le \frac{(1-q)(1+m)}{(1-q)(b-x)+qh_2}\int_x^b f(x)_{h_2} d_q x \le f(b)\int_0^1 t_h d_q t + mf\left(\frac{x}{m}\right)\int_0^1 (1-t)_h d_q t. \tag{40}$$

By definition, we have

$$\int_0^1 t\,{}_h d_q t = \frac{(1-q)+qh}{1-q}\left(\frac{q}{1+q}+(1-q)hS\right) \tag{41}$$

and

$$\int_0^1 (1-t)\,{}_h d_q t = \frac{(1-q)+qh}{1-q}\left(\frac{1}{1+q}-(1-q)hS\right). \tag{42}$$

By using (41), (42) in the inequalities (39) and (40), the required inequalities (35) and (36) are obtained. □

Corollary 5. *By setting $h=0$, in (35) and (36), the following inequalities hold for left and right q-integrals, respectively:*

$$f\left(\frac{a+x}{2}\right) \le \frac{1+m}{2(x-a)}\int_a^x f(x) d_q x \le \frac{1}{2(1+q)}\left(qf(x)+mf\left(\frac{a}{m}\right)\right), \tag{43}$$

$$f\left(\frac{x+b}{2}\right) \le \frac{1+m}{2(b-x)}\int_x^b f(x) d_q x \le \frac{1}{2(1+q)}\left(qf(b)+mf\left(\frac{x}{m}\right)\right). \tag{44}$$

Theorem 13. *Under the assumptions of Theorem 12, the following inequality holds for q-h-integrals:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{(1-q)(1+m)}{2((1-q)(b-a)+qh_3)}\int_a^b f(x)_{h_3} d_q x \\ &\le \frac{(1-q)+qh}{2(1-q)}\left(f(b)\left(\frac{q}{1+q}+(1-q)hS\right)+mf\left(\frac{a}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right). \end{aligned} \tag{45}$$

Proof. By setting $x=b$ in (35) or $x=a$ in (36), the required inequality (45) can be obtained. □

Corollary 6. *By setting $h=0$ in (45), the following inequality for q-integrals holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1+m}{2(b-a)}\int_a^b f(x) d_q x \le \frac{1}{2(1+q)}\left(qf(b)+mf\left(\frac{a}{m}\right)\right). \tag{46}$$

3. Conclusions

We have presented new integral inequalities for q-h-integrals, which hold implicitly for q- and h-integrals at the same time. Some particular cases are obtained for q-integrals.

These inequalities have been established for convex, symmetric $\hbar$-convex and m-convex functions satisfying a symmetric-like condition. Several implications are deducible from the main results. The idea of this paper is applicable in studying difference and differential equations as well as generalizing inequalities, which hold for ordinary derivatives and integrals. In future work, we are interested in establishing Ostrowski and Opial-type inequalities for q-h-integrals.

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References

1. Lazarević, M.P. *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*; Wseas Press: Attica, Greece, 2012.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier: New York, NY, USA; London, UK, 2006.
3. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *282*, 1–19. [CrossRef]
4. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Basel, Switzerland, 2012.
5. Kac, V.; Cheung, P. *Quantum Calculus, Universitext*; Springer: New York, NY, USA, 2002.
6. Alp, N.; Sarikaya, M.Z.; Kunt, M.; Iscan, I. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex function. *J. King Saud Univ.* **2018**, *30*, 193–203. [CrossRef]
7. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
8. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Quantum integral inequalities for convex function. *J. Math. Inequal.* **2015**, *9*, 781–793. [CrossRef]
9. Ogunmez, H.; Ozkan, U. Fractional quantum integral inequalities. *J. Inequal. Appl.* **2011**, *2011*, 787939. [CrossRef]
10. Butt, S.I.; Budak, H.; Nonlaopon, K. New quantum Mercer estimates of Simpson–Newton-like inequalities via convexity. *Symmetry* **2022**, *14*, 1935. [CrossRef]
11. Prabseang, J.; Nonlaopon, K.; Tariboon, J. Quantum Hermite–Hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal.* **2019**, *13*, 675–686. [CrossRef]
12. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.-M. Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 7. [CrossRef]
13. Bermudo, S.; Kórus, P.; Valdés, J.E.N. On q-Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
14. Neang, P.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Fractional (p,q)-calculus on finite intervals and some integral inequalities. *Symmetry* **2021**, *13*, 504. [CrossRef]
15. Vivas-Cortez, M.; Ali, M.A.; Budak, H.; Kalsoom, H.; Agarwal, P. Some new Hermite-Hadamard and related inequalities for convex functions via (p,q)-Integral. *Entropy* **2021**, *23*, 828. [CrossRef] [PubMed]
16. Varosanec, S. On $\hbar$-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
17. Toader, G.H. Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization*; Univ. Cluj-Napoca: Cluj Napoca, Romania, 1984; pp. 329–338.
18. Sarikaya, M.Z.; Saglam, A.; Yildirim, H. On some Hadamard type inequalities for $\hbar$-convex functions. *J. Math. Inequal.* **2008**, *2*, 335–345. [CrossRef]
19. Farid, G.; Afzal, Z. Further on quantum-plank derivatives and integrals in composite forms. *Open J. Math. Anal.* **2022**, *6*, 130–138.
20. Liu, Y.; Farid, G.; Abuzaid, D.; Nonlaopon, K. On q-Hermite-Hadamard inequalities via q-h-integrals. *Symmetry* 2022, 14, 2648. [CrossRef]