Mathematical
Inequalities
& Applications
Volume 18, Number 4 (2015), 1427–1442 doi:10.7153/mia-18-111

# INEQUALITIES FOR THE BETA FUNCTION

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*Abstract.* Let $g(x) := (e/x)^x\Gamma(x+1)$ and $F(x,y) := g(x)g(y)/g(x+y)$. Let $D^{(k)}_{x,y}$ be the $k$th differential in Taylor's expansion of $\log F(x,y)$. We prove that when $(x,y) \in \mathbb{R}^2_+$ one has $(-1)^{k-1}D^{(k)}_{x,y}(X,Y) > 0$ for every $X,Y \in \mathbb{R}_+$, and that when $k$ is even the conclusion holds for every $X,Y \in \mathbb{R}$ with $(X,Y) \neq (0,0)$. This implies that Taylor's polynomials for $\log F$ provide upper and lower bounds for $\log F$ according to the parity of their degree. The formula connecting the Beta function to the Gamma function shows that the bounds for $F$ are actually bounds for Beta.

*Mathematics subject classification* (2010): Primary 26D07; Secondary 26D99.
*Keywords and phrases*: Euler Beta function, completely monotone function, explicit bounds.

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