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# INEQUALITIES FOR THE LAW OF LARGE NUMBERS

BY THOMAS G. KURTZ

*Universtiy of Wisconsin*

Let $X_1, X_2, X_3, \cdots$ be independent random variables and $a_1, a_2, a_3, \cdots$ positive real numbers. Define

$$F(t) = \sup_k P\{|X_k| > t\}$$

and

$$S_m = \sum_{k=1}^m a_k X_k .$$

Inequalities of the form

$$P\{\sup_m |S_m| > \delta\} \leqq C \sum_k \int_0^1 \varphi'(u) F(u/a_k)\, du$$

are given for a large class of functions $\varphi$, as well as inequalities of a somewhat different form that are appropriate for considering exponential convergence rates. Examples of how the inequalities can be used to prove rate theorems are also given.

**1. Introduction.** Let $X_1, X_2, X_3, \cdots$ be independent random variables with $E(X_k) = 0$. Let $S_0 = 0$ and

$$S_m = \sum_{k=1}^m a_k X_k , \qquad m \geqq 1 ,$$

where the $a_k$'s are positive real numbers. We are interested in giving bounds on $P\{\sup_{m \leqq n} |S_m| > \delta\}$ in terms of the sequence $\{a_k\}$ and

$$F(t) \equiv \sup_k P\{|X_k| > t) .$$

We will always assume $\lim_{t\to\infty} F(t) = 0$.

In Section 2, we obtain the basic bounds using inequalities from martingale theory; in Section 3 we combine these bounds with truncation; and in Section 4 we indicate how some particular results can be obtained using the inequalities.

The notation used for the most part follows that used by Hanson and Franck [1].

The best known result of the type we are interested in is Kolmogorov's Inequality which is essentially (2.3) with $\varphi(u) = u^2$. Related inequalities have been obtained using estimates of $E(|S_m|^r)$ for $r \geqq 1$. The results of Hanson and Franck and other authors (see the bibliographies in [1], [2], [3], [5]), which are stated in terms of the asymptotic behavior of sequences of sums, imply the existence of inequalities sharper than can be obtained in this manner. (For example, if $a_k = n^{-1}$ for $k \leqq n$ the moment inequalities, such as in [6], [7], give estimates on $P\{\sup_{m \leqq n} |S_m| > \delta\}$ that are $O(n^{-r/2})$ for $r > 2$ and $O(n^{-r+1})$ for $1 \leqq r \leqq 2$ while the asymptotic results suggest $o(n^{-r+1})$ for all $r \geqq 1$.) In fact the primary motivation of the work presented here was to obtain explicit inequalities equivalent to the asympotic results of Hanson and Franck in the sense that their results could be obtained from the inequalities. The results in Section 4 indicate that we have achieved that goal.

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At the time this paper was originally submitted for publication no method for obtaining exponential rate theorems using these techniques was apparent to the author. Professor I. Olkin subsequently pointed out the work of Hoeffding [3] which inspired a second look at exponential rates and led to the results in Section 5 which, while not direct applications of Lemma 2.2, are at least in the spirit of the rest of the paper.

**2. Basic inequalities.** Since the $S_m$ form a martingale we have

$$P\{\sup_{m \leqq n} |S_m| > \delta\} \leqq \frac{1}{\varphi(\delta)} E(\varphi(S_n)) \tag{2.1}$$

for every nonnegative, even, convex function $\varphi$. The problem is to estimate the expectation on the right.

(2.2) LEMMA. *Assume that $\varphi$ and $\varphi'$ are absolutely continuous. We consider two cases.*

(a) *Let $\varphi(0) = \varphi'(0)$ and $\varphi''$ be nonnegative and non-increasing on $(0, \infty)$. (For example, $\varphi(u) = |u|^\alpha$, $1 < \alpha \leqq 2$.) Then*

$$\begin{aligned} P\{\sup_m |S_m| > \delta\} &\leqq \frac{4}{\varphi(2\delta)} \textstyle\sum_k \int_0^\infty \varphi(a_k t)|dF(t)| \\ &= \frac{4}{\varphi(2\delta)} \textstyle\sum_k a_k \int_0^\infty \varphi'(a_k t)F(t)\,dt \\ &= \frac{4}{\varphi(2\delta)} \textstyle\sum_k \int_0^\infty \varphi'(u)F(u/a_k)\,du\,. \end{aligned} \tag{2.3}$$

(b) *Let $\varphi(0) = \varphi'(0) = 0$ and $\varphi''$ be nonnegative and bounded. (For example,*

$$\begin{aligned} \varphi(u) &= |u|^\alpha & |u| &\leqq 1 \\ &= \alpha(|u| - 1) + 1 & |u| &> 1\,, \end{aligned}$$

*for $\alpha \geqq 2$.) Then*

$$\begin{aligned} &P\{\sup_m |S_m| > \delta\} \\ &\quad\leqq \left(\textstyle\sum_k \int_0^\infty \varphi(a_k t)|dF(t)|\right) \textstyle\sum_{l=1}^{L} \frac{(\frac{1}{2}\|\varphi''\|(\sum_k a_k^2)\int_0^\infty t^2|dF(t)|)^{l-1}}{\prod_{m=0}^{l-1}\varphi(\delta/2^{m+1})} \\ &\qquad+ \frac{(\frac{1}{2}\|\varphi''\|(\sum_k a_k^2)\int_0^\infty t^2|dF(t)|)^{L}}{\prod_{m=0}^{L-1}\varphi(\delta/2^{m+1})}\,. \end{aligned} \tag{2.4}$$

*If $\sum_k a_k^2 \leqq 1$ then*

$$\begin{aligned} P\{\sup_m |S_m| > \delta\} &\leqq \left(\textstyle\sum_k \int_0^\infty \varphi(a_k t)|dF(t)|\right)M_1(\delta, \varphi, F, L) \\ &\qquad+ \left(\textstyle\sum_k a_k^2\right)^L M_2(\delta, \varphi, F, L) \\ &= \left(\textstyle\sum_k a_k \int_0^\infty \varphi'(a_k t)F(t)\,dt\right)M_1(\delta, \varphi, F, L) \\ &\qquad+ \left(\textstyle\sum_k a_k^2\right)^L M_2(\delta, \varphi, F, L)\,. \end{aligned} \tag{2.5}$$

PROOF.

$$\begin{aligned} E(\varphi(S_n)) &= \varphi(0) + \textstyle\sum_{m=1}^n E(\varphi(S_m) - \varphi(S_{m-1})) \\ &= \varphi(0) + \textstyle\sum_{m=1}^n E(E(\varphi(a_m X_m + S_{m-1}) - \varphi(S_{m-1}) \\ &\qquad - a_m X_m \varphi'(S_{m-1}) \mid S_{m-1}))\,, \end{aligned} \tag{2.6}$$

which reduces our problem to the problem of estimating

$$E(\varphi(a_m X_m + S_{m-1}) - \varphi(S_{m-1}) - a_m X_m \varphi'(S_{m-1}) \mid S_{m-1}) . \tag{2.7}$$

The basis for our estimate is the fact that

$$\begin{aligned} \varphi(u + z) - \varphi(z) - u\varphi'(z) &= \textstyle\int_0^u (\varphi'(v + z) - \varphi'(z))\, dv \\ &= \textstyle\int_0^u \int_0^v \varphi''(w + z)\, dw\, dv . \end{aligned} \tag{2.8}$$

Since $n$ is arbitrary and could in fact be $\infty$ if the sum on the right-hand side of (2.6) is convergent, we drop its use.

Under the assumption of part (a)

$$\begin{aligned} \varphi(u + z) - \varphi(z) - u\varphi'(z) &= \textstyle\int_0^u \int_0^v \varphi''(w + z)\, dw\, dv \\ &\leqq 2 \textstyle\int_0^{|u|} \int_0^{v/2} \varphi''(w)\, dw\, dv \\ &= 4\varphi\left(\frac{|u|}{2}\right) . \end{aligned}$$

In order to unify the final results we replace $\varphi(u)$ by $\varphi(2u)$ and (2.3) follows.

Under the assumptions of part (b) let

$$\begin{aligned} \varphi_\varepsilon(u) &= \varphi(|u| - \varepsilon) && |u| \geqq \varepsilon \\ &= 0 && |u| < \varepsilon . \end{aligned}$$

Then

$$\begin{aligned} \varphi_\varepsilon(u + z) - \varphi_\varepsilon(z) - u\varphi_\varepsilon'(z) &\leqq \varphi_\varepsilon(u + z) \leqq \varphi(u) && |z| \leqq \varepsilon \\ &\leqq \sup_x \varphi''(x) \frac{u^2}{2} \equiv ||\varphi''|| \frac{u^2}{2} && |z| > \varepsilon . \end{aligned}$$

Consequently, for $\varepsilon < \delta$

$$\begin{aligned} P\{\sup_m |S_m| > \delta\} \leqq \frac{1}{\varphi(\delta - \varepsilon)} \Big[ &(\textstyle\sum_k \int_0^\infty \varphi(a_k t)|dF(t)|) P\{\sup_m |S_m| \leqq \varepsilon\} \\ &+ \frac{||\varphi''||}{2} (\textstyle\sum_k a_k^2) \int_0^\infty t^2 |dF(t)| P\{\sup_m |S_m| > \varepsilon\} \Big] . \end{aligned} \tag{2.9}$$

Setting $\varepsilon = \delta/2$ and bounding $P\{\sup_m |S_m| \leqq \varepsilon\}$ by one, the inequality may be iterated to obtain (2.4).

REMARK. For $\varphi(u) = |u|^\alpha$, $1 < \alpha \leqq 2$ (2.3) becomes

$$P\{\sup_m |S_m| > \delta\} \leqq \frac{4}{(2\delta)^\alpha} (\textstyle\sum_k a_k^\alpha) \int_0^\infty t^\alpha |dF(t)| .$$

Taking

$$\begin{aligned} \varphi(u) &= u^2 && |u| \leqq 1 \\ &= 2|u| - 1 && |u| > 1 , \end{aligned}$$

(2.3) can be used to give a simple proof of the Weak Law of Large Numbers under the assumption that

$$\textstyle\int_0^\infty t|dF(t)| < \infty .$$

Since $\varphi'$ is non-decreasing, if we let $a = \max_k a_k$ (2.3) implies

(2.10) $$P\{\sup_m |S_m| > \delta\} \leqq \frac{4}{\varphi(2\delta)} (\sum_k a_k) \int_0^\infty \varphi'(at)F(t)\,dt\,.$$

**3. Truncation.**

(3.1) THEOREM. *Let* $\eta = \sup_k a_k \int_{1/a_k}^\infty F(t)\,dt$ *and* $e = \sum_k a_k \int_{1/a_k}^\infty F(t)\,dt$.

(a) *Suppose* $\varphi(0) = \varphi'(0) = 0$, $\varphi'' \geqq 0$, $\varphi''$ *is non-increasing on* $(0, \infty)$ *and* $\eta < 2\delta$. *Then*

$$P\{\sup_m |S_m| > \delta + e\} \leqq \left[\frac{4}{\varphi(2\delta - \eta)} + \frac{1}{\varphi(1)}\right] \sum_k \int_0^1 \varphi'(u)F(u/a_k)\,du\,.$$

(b) *Suppose* $\varphi(0) = \varphi'(0) = 0$, $\varphi'' \geqq 0$, *and* $\varphi''$ *is bounded. If there exist* $\varepsilon$, $L > 0$ *such that*

(3.2) $$\begin{gathered} \eta < \delta/2^L\,, \\ \varphi(\delta/2^L - \eta) > 0\,, \\ \sum_k \int_0^1 u^{1-\varepsilon}F(u/a_k)\,du \equiv M < \infty\,, \\ M^{L-1}u^{1+\varepsilon(L-1)} \leqq C\varphi'(u)\,, \end{gathered}$$ *and*

*then*

$$P\{\sup_m |S_m| > \delta + e\}$$
$$\leqq \left(\sum_k \int_0^1 \varphi'(u)F(u/a_k)\,du\right)\left[\frac{1}{\varphi(1)} + \sum_{l=1}^L \frac{(\|\varphi''\| \sum_k \int_0^1 uF(u/a_k)\,du)^{l-1}}{\prod_{m=0}^{l-1} \varphi(\delta/2^{m+1} - \eta)} + \frac{C\|\varphi''\|^L}{\prod_{m=0}^{L-1} \varphi(\delta/2^{m+1} - \eta)}\right].$$

REMARK. Let $a = \sup_k a_k$. We note that

$$e \leqq (\sum a_k) \int_{1/a}^\infty F(t)\,dt$$

and for $\alpha > 1$

$$\begin{aligned} e &\leqq (\sum_k a_k{}^\alpha) \sup_{y > 1/a} y^{\alpha-1} \int_y^\infty F(t)\,dt \\ &\leqq (\sum_k a_k{}^\alpha)(\sup_{x > 1/a} x^\alpha F(x)) \frac{1}{\alpha - 1}\,. \end{aligned}$$

If we drop the assumption of finite expectations, we can still say that

$$P\{|S_n| > \delta\} \leqq \frac{E(\varphi(S_n))}{\varphi(\delta)}$$

for every nonnegative even function $\varphi(u)$ that is non-decreasing for $u \geqq 0$. If in addition $\varphi'(u)$ is non-increasing for $u \geqq 0$ then

$$E(\varphi((S_n)) \leqq \sum_k E(\varphi(a_k X_k))\,.$$

If we assume $\varphi'(u) = 0$ for $u \geqq 1$ and $\varphi(\delta) > 0$ then

(3.3) $$P\{|S_n| > \delta\} \leqq \frac{1}{\varphi(\delta)} \sum_k \int_0^1 \varphi'(u)F(u/a_k)\,du\,.$$

This inequality may be used to obtain results similar to those due to Rohatgi [2].

PROOF. If we let $\hat{X}_k = (X_k \wedge (1/a_k)) \vee (-1/a_k)$ and define $\hat{S}_m = \sum_{k=1}^m a_k \hat{X}_k$, then

$$P\{\hat{S}_m \neq S_m\} \leqq \sum_{k=1}^m F(1/a_k)$$

and

$$|E(\hat{S}_m)| \leqq \sum_{k=1}^m a_k \int_{1/a_k}^\infty F(t)\, dt\, .$$

Consequently,

$$P\{\sup_m |S_m| > \delta + e\} \leqq P\{\sup_m |\hat{S}_m - E(\hat{S}_m)| > \delta\} + \sum_k F(1/a_k)\, . \tag{3.4}$$

We observe that since $F(t)$ is decreasing

$$\sum_k F(1/a_k) \leqq \frac{\sum_k \int_0^1 \varphi'(u)F(u/a_k)\, du}{\varphi(1)}\, . \tag{3.5}$$

Under the assumption of part (a) (2.3) implies

$$P\{\sup_m |\hat{S}_m - E(\hat{S}_m)| > \delta\} \leqq \frac{4}{\varphi(2\delta)} \sum_k a_k \int_0^{1/a_k} \varphi'(a_k t + \eta)F(t)\, dt\, . \tag{3.6}$$

If we replace $\varphi$ by

$$\begin{aligned} \psi(u) &= \varphi(|u| - \eta) && \text{for } |u| \geqq \eta \\ &= 0 && \text{for } |u| < \eta\, , \end{aligned}$$

then for $\eta < 2\delta$ (3.6) becomes

$$\begin{aligned} P\{\sup_m |\hat{S}_m - E(\hat{S}_m)| > \delta\} &\leqq \frac{4}{\varphi(2\delta - \eta)} \sum_k a_k \int_0^{1/a_k} \varphi'(a_k t)F(t)\, dt \\ &= \frac{4}{\varphi(2\delta - \eta)} \sum_k \int_0^1 \varphi'(u)F(u/a_k)\, du\, . \end{aligned} \tag{3.7}$$

Part (a) follows from (3.4), (3.5) and (3.7).

Under the assumptions of part (b), (2.4) implies

$$\begin{aligned} &P\{\sup_m |\hat{S}_m - E(\hat{S}_m)| > \delta\} \\ &\quad\leqq \left(\sum_k a_k \int_0^{1/a_k} \varphi'(a_k t)F(t)\, dt\right) \sum_{l=1}^L \frac{(\|\varphi''\| \sum_k a_k^2 \int_0^{1/a_k} tF(t)\, dt)^{l-1}}{\prod_{m=0}^{l-1} \varphi(\delta/2^{m+1} - \eta)} \\ &\qquad + \frac{(\|\varphi''\| \sum_k a_k^2 \int_0^{1/a_k} tF(t)\, dt)^L}{\prod_{m=0}^{L-1} \varphi(\delta/2^{m+1} - \eta)} \\ &\quad= \left(\sum_k \int_0^1 \varphi'(u)F(u/a_k)\, du\right) \sum_{l=1}^L \frac{(\|\varphi''\| \sum_k \int_0^1 uF(u/a_k)\, du)^{l-1}}{\prod_{m=0}^{l-1} \varphi(\delta/2^{m+1} - \eta)} \\ &\qquad + \frac{(\|\varphi''\| \sum_k \int_0^1 uF(u/a_k)\, du)^L}{\prod_{m=0}^{L-1} \varphi(\delta/2^{m+1} - \eta)}\, . \end{aligned} \tag{3.8}$$

Since

$$\sum_k \int_0^1 u^{1-\varepsilon}F(u/a_k)\, du = \sum_k a_k^{2-\varepsilon} \int_0^{1/a_k} t^{1-\varepsilon}F(t)\, dt \equiv M < \infty\, ,$$

Jensen's Inequality implies

$$\left(\sum_k \int_0^1 uF(u/a_k)\, du\right)^L \leqq M^{L-1} \sum_k \int_0^1 u^{1+\varepsilon(L-1)}F(u/a_k)\, du\, . \tag{3.9}$$

Consequently $M^{L-1}u^{1+\varepsilon(L-1)} \leqq C\varphi'(u)$ implies

(3.10) $$\left(\sum_k \int_0^1 uF(u/a_k)\,du\right)^L \leqq C \sum_k \int_0^1 \varphi'(u)F(u/a_k)\,du\ .$$

Part (b) follows from (3.4), (3.5), (3.8) and (3.10).

**4. Examples.** Suppose now that we have a sequence of collections of non-negative constants $\{a_{k,N}\}$ with $\lim_{N\to\infty} (\sup_k a_{k,N} \equiv a_N) = 0$,

$$S_m^{\ N} = \sum_{k=1}^m a_{k,N} X_k\ .$$

Many known theorems, including those of Hanson and Franck [1] (and Rohatgi [2]), can be obtained from the above inequalities by using various methods to bound

(4.1) $$\sum_k \int_0^1 \varphi'(u)F(u/a_k)\,du$$

by expressions in which the integral is factored out of the sum. For example (4.1) is bounded by

(4.2) $$\left(\sum_k a_k^{\ \alpha}\right) \int_0^1 \frac{\varphi'(u)}{u^\alpha} H_\alpha(u/a)\,du\ ,$$

where

$$H_\alpha(t) = \sup_{x \geqq t} x^\alpha F(x)\ .$$

(4.3) THEOREM. *Suppose* $\int_0^\infty F(t)\,dt < \infty$ *and* $H_\alpha(0) < \infty$. *In addition, for* $\alpha \geqq 2$ *suppose there is an* $\varepsilon > 0$ *such that*

(4.4) $$\limsup_{N\to\infty} \sum_k \int_0^1 u^{1-\varepsilon}F(u/a_{k,N})\,du \equiv M < \infty\ .$$

*Then* (a)

$$e_N \equiv \sum a_{k,N} \int_{1/a_{k,N}}^\infty F(t)\,dt \leqq O\left(\sum_k a_{k,N}^\alpha\right)$$

*and*

$$P\{\sup_m |S_m^{\ N}| > \delta\} \leqq O\left(\sum_k a_{k,N}^\alpha\right)\ ;$$

(b) *if* $\lim_{t\to\infty} H_\alpha(t) = 0$,

$$e_N \leqq o\left(\sum_k a_{k,N}^\alpha\right)$$

*and*

$$P\{\sup_m |S_m^{\ N}| > \delta\} \leqq o\left(\sum_k a_{k,N}^\alpha\right)\ ;$$

(c) *if* $\int_1^\infty t^{-1}H_\alpha(t)\,dt < \infty$,

$$\limsup_{N\to\infty} \sum_k a_{k,N}^\alpha < \infty\ ,$$

(*this is implied by* (4.4) *for* $\alpha \geqq 2$)

$$\eta(N) \sum_k a_{k,N}^\alpha \leqq N^{-1}\ ,$$

*and*

$$a_N \leqq N^{-\beta}\ , \qquad \textit{some} \quad \beta > 0\ ,$$

*then*

$$\sum_N \eta(N)P\{\sup_m |S_m^{\ N}| > \delta\} < \infty\ .$$

REMARK. This theorem corresponds to Theorems 1, 2 and 4 of [1].

PROOF. Let $\varphi(u) = u^{\alpha+\gamma}$ where $\gamma > 0$ and $\alpha + \gamma \leqq 2$ if $\alpha < 2$. Parts (a) and

(b) follow immediately from (4.2). To obtain part (c) we approximate $\sum_N N^{-1}H_\alpha(uN^\beta)$ by

$$\int_1^\infty \frac{1}{x} H_\alpha(ux^\beta)\, dx = \frac{1}{\beta} \int_u^\infty \frac{1}{t} H_\alpha(t)\, dt$$

and note $\int_0^1 (\varphi'(u)/u^\alpha)\beta^{-1} \int_u^\infty t^{-1}H_\alpha(t)\, dt\, du < \infty$.

Another important class of theorems corresponds to the assumptions

$$\begin{aligned} a_{k,N} &= 1/\gamma(N) & k &\leqq \psi(N) \\ &= 0 & k &> \psi(N) . \end{aligned}$$

The problem is to find conditions on $\gamma(N)$, $\psi(N)$ and $\eta(N)$ that imply

$$\sum \eta(N)P\{\sup_m |S_m{}^N| > \delta\} < \infty .$$

Using our inequalities this reduces to proving that

$$\sum_N \eta(N)\psi(N) \int_0^1 \varphi'(u)F(u\gamma(N))\, du < \infty .$$

Let $\lambda(N) = \eta(N)\psi(N)$ and assume we can approximate $\sum_N \lambda(N)F(u\gamma(N))$ by

$$\int_1^\infty \lambda(x)F(u\gamma(x))\, dx . \tag{4.5}$$

A change of variable in (4.5) gives

$$\frac{1}{u} \int_{u\gamma(1)}^\infty \lambda\left(\gamma^{-1}\left(\frac{t}{u}\right)\right) \gamma'^{-1}\left(\frac{t}{u}\right) F(t)\, dt . \tag{4.6}$$

Let

$$\Lambda(x) = \int_1^x \lambda(s)\, ds .$$

Then (4.6) can be rewritten as $\int_{u\gamma(1)}^\infty \Lambda(\gamma^{-1}(t/u))|dF(t)|$. Consequently, if

$$\int_0^\infty t^\alpha\, dF(t) < \infty ,$$

and $\varphi(u)$ is the same as in the proof of Theorem (4.3), then we can take

$$\Lambda(\gamma^{-1}(x)) \leqq Kx^\alpha ;$$

that is

$$\begin{aligned} \lambda(x) &\leqq K \frac{d}{dx} [\gamma(x)]^\alpha \\ &\leqq K\alpha[\gamma(x)]^{\alpha-1}\gamma'(x) . \end{aligned}$$

If $\alpha \geqq 2$, condition (3.2) introduces the additional requirement that $\psi(x)/\gamma(x)^{2-\varepsilon}$ be bounded for some $\varepsilon > 0$.

In particular, if we want $\eta(x) \equiv 1$ and $\psi(x) = \gamma(x)$, then for $\alpha = 1$, we can take $\gamma(x) = e^{kx}$, and for $\alpha = 1 + 1/\beta$, $\gamma(x) = x^\beta$. More generally if we want $\eta(x) = x^b$, $\psi(x) = x^c$ and $\gamma(x) = x^d$ then for $1 \leqq \alpha < 2$ we can take $b$, $c$ and $d$ satisfying $b + c \leqq \alpha d - 1$, and for $\alpha \geqq 2$, $b + c \leqq \alpha d - 1$ and $2d > c$.

Remark. An examination of the proofs will show that all we really need to assume is that $S_m = \sum_{k=1}^m a_k X_k$ is a martingale with respect to an increasing family of $\sigma$-algebras $\mathscr{F}_m$, i.e., that $S_m$ is $\mathscr{F}_m$-measurable and $E(X_m \mid \mathscr{F}_{m-1}) = 0$, and that

$$F(t) \geqq P\{|X_m| > t \mid \mathscr{F}_{m-1}\} \quad \text{a.s.}$$

In fact, we can even permit $a_m$ to be an $\mathscr{F}_{m-1}$-measurable random variable provided we take the expectation of the right-hand side of the inequalities, thus obtaining inequalities of the form

$$P\{\sup_m |S_m| > \delta\} \leqq C \sum_k \int_0^1 \varphi'(u)E(F(u/a_k))\, du \,.$$

**5. Exponential rates.** We prove the following lemma, analogous to Lemma 2.2.

(5.1) LEMMA. *Let* $\varphi(u) = e^u - 1 - u$, *and suppose* $\int_0^\infty \varphi(\alpha t)|dF(t)| < \infty$ *for some* $\alpha > 0$. *Then*

$$(5.2)\qquad \begin{aligned} P\{\sup_m |S_m| > \delta\} &\geqq 2\exp\{\inf_{\lambda>0}(\textstyle\sum_k \int_0^\infty \varphi(\lambda a_k t)|dF(t)| - \lambda\delta)\} \\ &= 2\exp\{\inf_{\lambda>0}(\textstyle\sum_k \lambda a_k \int_0^\infty F(t)\varphi'(\lambda a_k t)\, dt - \lambda\delta)\}\,. \end{aligned}$$

*Assuming* $\sum_k a_k = 1$ *and* $a = \max a_k$,

$$(5.3)\qquad \begin{aligned} P\{\sup_m |S_m| > \delta\} &\leqq 2\exp\left\{\frac{1}{a}\inf_s (\textstyle\int_0^\infty \varphi(st)|dF(t)| - s\delta)\right\}, \\ &= 2\exp\left\{\frac{1}{a}\inf_s s(\textstyle\int_0^\infty F(t)\varphi'(st)\, dt - \delta)\right\}. \end{aligned}$$

REMARK. Note that if $\eta(s) \equiv \int_0^\infty \varphi(st)|dF(t)|$ is finite for some $s$ then $\lim_{s\to 0}\eta(s)/s = 0$ and hence $\inf(\eta(s) - s\delta) < 0$.

PROOF. Let $\hat\varphi(x) = e^x + e^{-x}$. Using the fact that for $u > 0$, $\varphi(-u) \leqq \varphi(u)$ we have

$$(5.4)\qquad \begin{aligned} E(\hat\varphi(\lambda S_n)) &= \hat\varphi(0) + \textstyle\sum_{m=1}^n E(\hat\varphi(\lambda S_m) - \hat\varphi(\lambda S_{m-1})) \\ &= 2 + \textstyle\sum_{m=1}^n E(\exp[\lambda S_{m-1}]\varphi(\lambda a_m X_m) \\ &\qquad + \exp[-\lambda S_{m-1}]\varphi(-\lambda a_m X_m)) \\ &\leqq 2 + \textstyle\sum_{m=1}^n E(\varphi(\lambda a_m |X_m|))E(\hat\varphi(\lambda S_{m-1}))\,. \end{aligned}$$

Iterating the above inequality gives

$$(5.5)\qquad E(\hat\varphi(\lambda S_n)) \leqq 2\exp\{\textstyle\sum_{m=1}^n E(\varphi(\lambda a_m |X_m|))\}\,,$$

and hence

$$(5.6)\qquad \begin{aligned} P\{\sup_m |S_m| > \delta\} &\leqq \inf_\lambda \frac{E\hat\varphi(\lambda S_n))}{\hat\varphi(\lambda\delta)} \\ &\leqq \inf_{\lambda>0} \frac{2\exp\{\sum_{m=1}^n E(\varphi(\lambda a_m|X_m|))\}}{e^{\lambda\delta} + e^{-\lambda\delta}} \\ &\leqq 2\exp\{\inf_{\lambda>0}(\textstyle\sum_{m=1}^n E(\varphi(\lambda a_m |X_m|)) - \lambda\delta)\}\,. \end{aligned}$$

Inequality (5.2) follows from the fact that $\varphi(u)$ is increasing. To obtain (5.3), note that $\varphi(u)/u$ is increasing for positive $u$ and hence

$$\begin{aligned} \textstyle\sum_m \int_0^\infty \varphi(\lambda a_m t)|dF(t)| - \lambda\delta &= \lambda\left(\textstyle\sum_m a_m \int_0^\infty \frac{\varphi(\lambda a_m t)}{\lambda a_m}|dF(t)| - \delta\right) \\ &\leqq \lambda\left(\textstyle\int_0^\infty \frac{\varphi(\lambda a t)}{\lambda a}|dF(t)| - \delta\right). \end{aligned}$$

Letting $s = \lambda a$, (5.3) follows.

The following theorem is obtained in exactly the same way as Theorem (3.1).

(5.7) THEOREM. *Let* $q = \int_{1/a}^{\infty} F(t)\,dt$, $F_a(t) = F(t-q)$, *and* $e = \sum_k a_k \times \int_{1/a_k}^{\infty} F(t)\,dt$. *Then with* $\varphi(u) = e^u - 1 - u$

(5.8) $$P\{\sup_m |S_m| > \delta + e\} \leqq \sum_k F(1/a_k) + 2\exp\{\inf_{\lambda>0}(\sum_k \lambda \int_0^1 \varphi'(\lambda u)F_a(u/a_k)\,du - \lambda\delta)\}\,.$$

Lemma 5.1 can be used to prove the following:

(5.9) THEOREM. *Suppose* $F(t) \leqq \exp[-\lambda_0(t-T)]$ *for all* $t > T$. *Let* $a = \sup_k a_k$ *and* $A = \sum_k a_k^2/a$. *Then there exists a constant* $0 < \rho < 1$ *depending on* $\delta$, $T$, $\lambda_0$ *and* $A$ (*increasing as a function of* $A$) *such that*

$$P\{\sup_m |S_m| > \delta\} \leqq 2\rho^{1/a}\,.$$

REMARK. The constant $\rho$ can be taken to be

$$\rho = \exp\left\{\inf_s\left(A\frac{\lambda_0}{\lambda_0 - s}e^{sT} - s(AT + A/\lambda_0 + \delta) - A\right)\right\}.$$

This can be seen by writing the exponent on the right-hand side of (5.2) as

$$\frac{1}{a}\inf_{\lambda>0}\left(\sum_k \frac{a_k^2}{a}(\lambda a)^2 \int_0^\infty tF(t)\frac{\varphi'(\lambda a_k t)}{\lambda a_k t}\,dt - \lambda a\delta\right)$$

and applying the monotonicity of $\varphi'(u)/u$ to bound this by

$$\frac{1}{a}\inf_{\lambda>0}\left(\sum_k \frac{a_k^2}{a}\right)\lambda a \int_0^\infty F(t)\varphi'(\lambda a t)\,dt - \lambda a\delta)\,.$$

Substitute $s = \lambda a$ and $F(t) = 1 \wedge \exp[-\lambda_0(t-T)]$ in this expression and integrate.

We apply Theorem 5.7 to obtain

(5.10) THEOREM. *Let*

$$a_{k,N} = N^{-\beta} \qquad k \leqq N^\alpha$$
$$= 0 \qquad k > N^\alpha$$

*where* $\beta > \alpha$ *and* $\beta \geqq 1$ *or* $\beta = \alpha > 1$. *Suppose*

(5.11) $$\lim_{t\to\infty} \rho_0^{\,t} t^\alpha F(t^\beta) = 0\,.$$

*Then for* $1 < \rho < \rho_0$ *and* $\delta < 1$

$$\lim_{N\to\infty} \rho^{N\delta} P\{|\sum_k a_{k,N} X_k| > \delta\} = 0\,.$$

PROOF. Let $e_N = \sum_k a_{k,N} \int_{1/a_{k,N}}^{\infty} F(t)\,dt$ and replace $\delta$ in (5.8) by $(\delta - e_N)$. The first term on the right of (5.8) causes no difficulties. Let $b = \log \rho_0$ and let $\lambda = bN$. The exponent in the second term is bounded by

$$bNN^\alpha \int_0^1 (e^{bNu} - 1)F_a(uN^\beta)\,du - bN(\delta - e_N)\,.$$

We will be through if we can show

(5.12) $$\lim_{N\to\infty} N^\alpha \int_0^1 (e^{bNu} - 1)F_a(uN^\beta)\,du = 0\,.$$

A change of variable in (5.12) gives

$$N^\alpha \int_0^1 (\exp\{bNt^\beta\} - 1)F_a((tN)^\beta)\beta t^{\beta-1}\,dt$$
$$= \beta \int_0^1 t^{\beta-\alpha-1}(\exp\{-bN(t - t^\beta)\} - \exp\{-bNt\})t^\alpha N^\alpha \rho_0{}^{tN} F_a((tN)^\beta)\,dt\,.$$

Since (5.11) holds for $F$ it holds for $F_a$ uniformly for $a < 1$. If $\beta > \alpha$, $t^{\beta-\alpha-1}$ is integrable on [0, 1] and the theorem follows by the Lesbesgue Dominated Convergence Theorem.

If $\beta = \alpha > 1$, the integral is bounded by a constant times

$$\int_0^1 \frac{1}{t}(\exp\{bNt^\beta\} - 1)\exp\{-bNt\}\,dt\,.$$

The difficulty is close to zero so select $\varepsilon > 0$ such that $t - t^\beta$ is monotone on $[0, \varepsilon]$. Then

$$\int_0^\varepsilon \frac{1}{t}(\exp\{bNt^\beta\} - 1)\exp\{-bNt\}\,dt$$
$$\leqq \int_{[0,N^{-(1/\beta)}]} \frac{1}{t}\, bNt^\beta e^b \exp\{-bNt\}\,dt + \int_{[N^{-(1/\beta)},\varepsilon]} \frac{1}{t}\exp\{-bN(t - t^\beta)\}\,dt$$
$$\leqq (N^{-(1/\beta)})^{\beta-1} e^b \int_{[0,N^{-(1/\beta)}]} bN\exp\{-bNt\}\,dt$$
$$+ \int_{[N^{-(1/\beta)},\varepsilon]} \frac{1}{t}\exp\{-bN(t - t^\beta)\}\,dt$$
$$\leqq e^b N^{-(1-1/\beta)} + \int_{[N^{-(1/\beta)},\varepsilon]} \frac{1}{t}\exp\left\{-bN\left(N^{-(1/\beta)} - \frac{1}{N}\right)\right\}dt$$
$$= e^b(N^{-(1-1/\beta)} + \log(\varepsilon N^{1/\beta})\exp\{-bN^{1-1/\beta}\})\,.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706