

INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN ℓ_p^{n*}

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Abstract

Let $B_p^n = \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$ be the unit ball in ℓ_p^n . We prove the inequalities for the volume of the B_p^n :

$$V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}$$

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+p}} V_{B_p^n} \leq V_{B_p^{n+1}}$$

for all $n \geq 1$ and $p \geq 1$, where $V_{B_p^n}$ denotes the volumes of B_p^n . Furthermore, we obtain the upper and lower bounds of $V_{B_p^{n+1}}/V_{B_p^n}$ and $V_{B_p^{n+1}}/V_{B_p^n}$. Our results are generalizations for inequalities in \mathbb{R}^n proved and refined by G.D. Anderson et al., K.H. Borgwardt, D.A.Klajn and G.-C. Rota and H. Alzer.

1. Introduction

Let $B_p^n = \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$ be the unit ball in ℓ_p^n , and $V_{B_p^n}$ denotes the volume of unit ball in ℓ_p^n . Then $V_{B_2^n}$ means the volume of unit ball in \mathbb{R}^n . In past several years, there have been many works about the inequalities for $V_{B_2^n}$. According to the results of G.D.Anderson, M.K.Vamanamurthy and M.Vuorinen in [3] and of D.A.Klajn and G.-C. Rota in [7], we have

$$V_{B_2^{n+1}}^{\frac{1}{n+1}} < V_{B_2^n}^{\frac{1}{n}}, (n = 1, 2, \dots). \quad (1.1)$$

Another inequality regarding the upper and lower bounds for the ratio of $V_{B_2^{n+1}}/V_{B_2^n}$ was obtained by Brogwardt in [5]:

$$\sqrt{\frac{2\pi}{n+2}} \leq \frac{V_{B_2^{n+1}}}{V_{B_2^n}} \leq \sqrt{\frac{2\pi}{n+1}}, (n = 1, 2, \dots), \quad (1.2)$$

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which leads to

$$1 < \frac{V_{B_2^n}^2}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < \sqrt{\frac{n+2}{n}}, (n = 2, 3, \dots). \quad (1.3)$$

The next inequality about $V_{B_2^n}$ is proved by H. Alzer in [2], and he pointed out that

$$1 < \frac{V_{B_2^n}^2}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < 1 + \frac{1}{n}, (n = 2, 3, \dots). \quad (1.4)$$

This inequality can also be deduced from the results in [3]. However, the right-hand side inequality of (1.4) is weaker than that of (1.3) as $\sqrt{\frac{n+2}{n}} \leq 1 + \frac{1}{n}$.

Inequalities (1.1), (1.2) and (1.4) have been refined by Horst Alzer in [2]. His results are:

$$\frac{2}{\sqrt{\pi}}V_{B_2^{n+1}}^{\frac{n}{n+1}} \leq V_{B_2^n} < \sqrt{e}V_{B_2^{n+1}}^{\frac{n}{n+1}}, (n = 1, 2, \dots); \quad (1.5)$$

$$\sqrt{\frac{2\pi}{n + \frac{8}{\pi} - 1}} \leq \frac{V_{B_2^{n+1}}}{V_{B_2^n}} < \sqrt{\frac{2\pi}{n + \frac{3}{2}}}, (n = 1, 2, \dots); \quad (1.6)$$

$$\left(1 + \frac{1}{n}\right)^{2 - \frac{\log \pi}{\log 2}} \leq \frac{V_{B_2^n}^2}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}, (n = 2, 3, \dots). \quad (1.7)$$

On the other hand, there are also lots of results about the volume of B_p^n , such as in M.Meyer and A.Pajor [8], M.Schmuckenschlager [10], Jesus Bastero etc [4] and Peng Gao [6]. From these results and those inequalities for the volumes of unit ball in \mathbb{R}^n , it is natural to ask whether there exist similar inequalities for the volumes of unit ball in ℓ_p^n ? In this paper, we give the answer to this question by proving Theorem 1 and 2 and Corollary 1, which are similar to (1.1), (1.2) and (1.3). Moreover, we prove Theorem 3 and 4, whose results are similar to (1.5) and (1.6). Our results are:

$$V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}, (n = 1, 2, \dots);$$

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+p}} V_{B_p^n} \leq V_{B_p^{n+1}}, (n = 1, 2, \dots);$$

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+p}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} \leq 2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+1}}, (n = p, p+1, \dots);$$

$$\sqrt[p]{\frac{n+1}{p+n-1}} \leq \frac{V_{B_p^n}^2}{V_{B_p^{n-1}}V_{B_p^{n+1}}} \leq \sqrt[p]{\frac{n+p}{n}}, (n = p, p+1, \dots);$$

$$aV_{B_p^{n+1}}^{\frac{n}{n+1}} \leq V_{B_p^n} < bV_{B_p^{n+1}}^{\frac{n}{n+1}}, (n = p-1, p, \dots);$$

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+A}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} < 2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+B}}, (n = 1, 2, \dots),$$

where $a = \frac{p-1}{\Gamma(\frac{p-1}{p}+1)} \sqrt{\Gamma(2)}$, $b = \sqrt[p]{e}$, $A = p \left(\frac{\Gamma(\frac{2}{p}+1)}{\Gamma(\frac{1}{p}+1)} \right)^p - 1$ and $B = \frac{p+1}{2}$.

2. Volume of the unit ball in ℓ_p^n and some inequalities of $\Gamma(x)$ and $\Psi(x)$

Before we start our proof, it is necessary for us to introduce the formula of the volumes of unit ball in ℓ_p^n spaces and some properties of gamma function and psi function (the logarithmic derivative of the gamma function).

Lemma 1. *Let $B_p^n = \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$, then*

$$V_{B_p^n} = \frac{(2\Gamma(\frac{1}{p} + 1))^n}{\Gamma(\frac{n}{p} + 1)}, \quad (2.1)$$

where $V_{B_p^n}$ is the volume of unit ball in ℓ_p^n .

The proof of Lemma 1 can be found in [12].

Lemma 2. *For all $x > 0$ we have*

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \log \sqrt{2\pi} + \frac{1}{12x} + O\left(\frac{1}{x^3}\right), \quad (2.2)$$

$$\log \Gamma(x) > \left(x - \frac{1}{2}\right) \log x - x + \log \sqrt{2\pi}, \quad (2.3)$$

and

$$x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), (x \rightarrow \infty). \quad (2.4)$$

Lemma 2 is provided by Horst Alzer in [2], Another paper of him [1] gives us the proof of (2.3), which is also found in [11], and the proofs of (2.2) and (2.4) can be given in [9].

Lemma 3. *For $x > 0$, let*

$$\Psi(x) = \frac{d \log(\Gamma(x))}{dx} = \frac{1}{\Gamma(x)} \frac{d\Gamma(x)}{dx}.$$

We also have the integral representations

$$\Psi(x) = -C + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \quad (2.5)$$

where $C = \text{Euler's constant}$,

$$\Psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}} dt, \quad (2.6)$$

and the asymptotic formula

$$\Psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \quad (2.7)$$

Then,

$$\Psi(x) < \log x - \frac{1}{2x}. \quad (2.8)$$

Lemma 3 is also mentioned in [2] and [11]. The integral representation and asymptotic formula of $\Psi(x)$ is given in [9]. Actually, (2.6) follows from (2.5) by differentiation, and the proof of (2.8) is proved in [1.3], which can be deduced from (2.7) easily.

Lemma 4. *Let $n \geq 0$ be an integer and let $x > 0$ and $s \in (0, 1)$ be real numbers. Then*

$$A_n(s; x) < \Psi(x+1) - \Psi(x+s), \quad (2.9)$$

where

$$A_n(s; x) = (1-s) \left(\frac{1}{x+s+n} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+s)} \right). \quad (2.10)$$

We can find the proof of Lemma 4 in [1]. Horst Alzer proved this Lemma by Jensen's inequality,

$$h(su + (1-s)v) < sh(u) + (1-s)h(v), \quad (u, v > 0; u \neq v; 0 < s < 1).$$

3. Inequalities for $V_{B_p^n}$

Theorem 1. *For all integers $n \geq 1$, we have*

$$V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}. \quad (3.1)$$

Proof. We define for positive real numbers x

$$f(x) = \frac{2\Gamma(\frac{1}{p} + 1)}{\left(\Gamma(\frac{x}{p} + 1)\right)^{\frac{1}{x}}}.$$

Differentiation yields

$$\frac{df(x)}{dx} = \frac{2\Gamma(\frac{1}{p} + 1)}{x^2 \left(\Gamma(\frac{x}{p} + 1)\right)^{\frac{1}{x}}} \left(\log \Gamma(\frac{x}{p} + 1) - \frac{x}{p} \Psi(\frac{x}{p} + 1) \right).$$

Then, we define for $y > 1$

$$g(y) = \log \Gamma(y) - (y-1)\Psi(y).$$

Differentiation yields

$$\frac{dg(y)}{dy} = -(y-1) \frac{d\Psi(y)}{dy}.$$

By (2.6), we know

$$\frac{d\Psi(y)}{dy} = \int_0^\infty e^{-yt} \frac{t}{1-e^{-t}} dt > 0. \quad (3.2)$$

According to (3.2), $\frac{dg(y)}{dy} \leq 0$ for $y \geq 1$. Thus, for $y > 1$

$$g(y) < g(1) = 0,$$

which implies

$$\frac{df(x)}{dx} < 0.$$

Hence, we obtain that $V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}$.

Theorem 2. *For all integers $n \geq 1$, we have*

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+p}} V_{B_p^n} \leq V_{B_p^{n+1}}. \quad (3.3)$$

Proof. We define for positive real numbers x

$$f(x) = 2\Gamma\left(\frac{1}{p} + 1\right) \frac{\Gamma\left(\frac{x}{p} + 1\right)}{\Gamma\left(\frac{x+1}{p} + 1\right)}.$$

Differentiation yields

$$\frac{df(x)}{dx} = 2\Gamma\left(\frac{1}{p} + 1\right) \frac{\Gamma\left(\frac{x}{p} + 1\right)}{\Gamma\left(\frac{x+1}{p} + 1\right)} \left(\Psi\left(\frac{x}{p} + 1\right) - \Psi\left(\frac{x+1}{p} + 1\right) \right) < 0.$$

As $\Psi(x)$ is an increasing function by (3.5). Hence, we obtain

$$\left(2\Gamma\left(\frac{1}{p} + 1\right)\right)^p \frac{p}{n+p} = \prod_n^{n+p-1} f(i) \leq f^p(n).$$

Hence, the theorem is proved. It may be noted that the equality sign holds, if and only if $p = 1$.

Corollary 1. *For all integers $n \geq p$, we have*

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+p}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} \leq 2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+1}}, \quad (3.4)$$

and

$$\sqrt[p]{\frac{n+1}{p+n-1}} \leq \frac{V_{B_p^n}^2}{V_{B_p^{n-1}}V_{B_p^{n+1}}} \leq \sqrt[p]{\frac{n+p}{n}}. \quad (3.5)$$

Proof. The above leads to (3.3). Proceeding precisely in the same way as what we have done in the previous proof, we obtain that, for all $n \geq p$,

$$\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^p \frac{p}{n+p} = \prod_n^{n+p-1} f(i) \leq f^p(n) \leq \prod_{n-p+1}^n f(i) = \left(2\Gamma\left(\frac{1}{p}+1\right)\right)^p \frac{p}{n+1}. \quad (3.6)$$

Then, applying (3.3), we can obtain (3.4) easily. All equality sign hold, if and only if $p = 1$.

4. Bounds of $V_{B_p^n}$

Theorem 3. *For all integers $n \geq p - 1$, we have*

$$aV_{B_p^{n+1}}^{\frac{n}{n+1}} \leq V_{B_p^n} < bV_{B_p^{n+1}}^{\frac{n}{n+1}}, \quad (4.1)$$

with $a = \frac{p-1}{\Gamma(\frac{p-1}{p}+1)}\sqrt[p]{\Gamma(2)}$ and $b = \sqrt[p]{e}$.

Proof. First, we define the sequence

$$\begin{aligned} x_n &= \log V_{B_p^n} - \frac{n}{n+1} \log V_{B_p^{n+1}} \\ &= \frac{n}{n+1} \log \Gamma\left(\frac{n+1}{p}+1\right) - \log \Gamma\left(\frac{n}{p}+1\right), \quad (n = p-1, p, \dots), \end{aligned}$$

and for positive real number x , let

$$f(x) = \frac{x}{x+\frac{1}{p}} \log \Gamma\left(x+\frac{1}{p}+1\right) - \log \Gamma(x+1),$$

then,

$$p\left(x+\frac{1}{p}\right)^2 \frac{df(x)}{dx} = \log \Gamma\left(x+\frac{1}{p}+1\right) + px\left(x+\frac{1}{p}\right)\Psi\left(x+\frac{1}{p}+1\right) - p\left(x+\frac{1}{p}\right)^2\Psi(x+1).$$

We define for $y = x + 1 + \frac{1}{p} \geq 2$

$$g(y) = \log \Gamma(y) + (py - p - 1)(y - 1)\Psi(y) - p(y - 1)^2\Psi\left(y - \frac{1}{p}\right).$$

Applying (2.3),(2.8) and Lemma 4, we consider that

$$\begin{aligned} g(y) &\geq \log \sqrt{2\pi} + \frac{1}{2} + \frac{1}{2} \log y - y \\ &\quad - \frac{1}{2y} + (y-1)^2 \left(\frac{1}{y - \frac{1}{p} + 2} + \frac{1}{y(y - \frac{1}{p})} + \frac{1}{(y+1)(y+1 - \frac{1}{p})} \right) \\ &\geq \log \sqrt{2\pi} + \frac{1}{2} + \frac{1}{2} \log y - y - \frac{1}{2y} + (y-1)^2 \left(\frac{1}{y+2} + \frac{1}{y^2} + \frac{1}{(y+1)^2} \right). \end{aligned}$$

A simple calculation reveals for $y \geq 2$,

$$(y-1)^2 \left(\frac{1}{y+2} + \frac{1}{y^2} \right) + \frac{1}{(y+1)^2} - y - \frac{1}{2y} \geq -2,$$

which means

$$g(y) \geq \log \sqrt{2\pi} + \frac{1}{2} + \frac{1}{2} \log 2 - 2 > 0$$

Thus, $\frac{df(y)}{dy} > 0$, so that $x_n (n = 1, 2, \dots)$ is strictly increasing. Applying (2.2),

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left(\frac{2n+p}{2p} \log \frac{n+p+1}{n+p} - \frac{1}{2(n+1)} \log(n+p+1) + O\left(\frac{1}{n}\right) \right) \\ &= \frac{1}{p}. \end{aligned}$$

Hence, for all $n \geq p-1$

$$\frac{\frac{p-1}{p} \sqrt{\Gamma(2)}}{\Gamma(\frac{p-1}{p} + 1)} V_{B_p^{n+1}} \leq V_{B_p^n} < \sqrt[p]{e} V_{B_p^{n+1}}.$$

Theorem 4. For all integers $n \geq 1$, we have

$$2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+A}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} < 2\Gamma\left(\frac{1}{p} + 1\right) \sqrt[p]{\frac{p}{n+B}} \quad (4.2)$$

with $A = p \left(\frac{\Gamma(\frac{2}{p} + 1)}{\Gamma(\frac{1}{p} + 1)} \right)^p - 1$ and $B = \frac{p+1}{2}$.

Proof. Double-inequality (4.2) is equivalent to

$$B < ph \left(\frac{n}{p} \right) \leq A,$$

where

$$h(x) = \left(\frac{\Gamma(x+1 + \frac{1}{p})}{\Gamma(x+1)} \right)^p - x, \quad (x > 0).$$

Define $r = p \left(\frac{\Gamma(x+1+\frac{1}{p})}{\Gamma(x+1)} \right)^p \left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1) \right)$ and $L(r, s) = \frac{r-s}{\log r - \log s}$. Let $s = 1$. Differentiation yields

$$\begin{aligned} \frac{1}{L(r, s)} \frac{dh(x)}{dx} &= p \log \Gamma(x+1+\frac{1}{p}) - p \log \Gamma(x+1) \\ &\quad + \log \left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1) \right) + \log p. \end{aligned}$$

Define $q(x) = \frac{1}{L(r, s)} \frac{dh(x)}{dx}$, and from (2.5) and (2.6), we obtain

$$\begin{aligned} \left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1) \right) \frac{dq(x)}{dx} &= \frac{d\Psi(x+1+\frac{1}{p})}{dx} - \frac{d\Psi(x+1)}{dx} \\ &\quad + p \left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1) \right)^2 \\ &= -\int_0^\infty e^{-xt} t \delta(t) dt + p \left(\int_0^\infty e^{-xt} \delta(t) dt \right)^2, \end{aligned}$$

where

$$\delta(t) = \frac{-e^{-(1+\frac{1}{p})t} + e^{-t}}{1 - e^{-t}}.$$

Applying the convolution theorem for Laplace transforms, we get

$$\left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1) \right) \frac{dq(x)}{dx} = \int_0^\infty e^{-xt} \int_0^t (p\delta(s)\delta(t-s) - \delta(t)) ds dt.$$

Let $0 < s < t$, we have

$$\begin{aligned} &p\delta(s)\delta(t-s) - \delta(t) \\ &= \frac{p(1 - e^{-\frac{s}{p}})(1 - e^{-\frac{t-s}{p}})(1 - e^{-t}) - (1 - e^{-\frac{t}{p}})(1 - e^{-s})(1 - e^{-(t-s)})}{(e^s - 1)(e^{t-s} - 1)(e^t - 1)} \\ &= \frac{(1 - e^{-\frac{s}{p}})(1 - e^{-\frac{t-s}{p}})(1 - e^{-\frac{t}{p}})(p \sum_{i=0}^{p-1} e^{-\frac{i}{p}t} - \sum_{i=0}^{p-1} e^{-\frac{i}{p}s} \sum_{i=0}^{p-1} e^{-\frac{i}{p}(t-s)})}{(e^s - 1)(e^{t-s} - 1)(e^t - 1)} \\ &> 0. \end{aligned}$$

Thus, for $x > 0$, $\frac{dq(x)}{dx} > 0$.

Applying (2.4) and (2.7), we get

$$\begin{aligned} \lim_{z \rightarrow \infty} e^{q(z)} &= \lim_{z \rightarrow \infty} p \left(\frac{\Gamma(z+1+\frac{1}{p})}{\Gamma(z+1)} z^{-\frac{1}{p}} \right)^p z \left(\Psi(z+1+\frac{1}{p}) - \Psi(z+1) \right) \\ &= 1, \end{aligned}$$

which means $q(x) < 0$.

We conclude that $h(x)$ is a decreasing function. Hence, for $n \geq 1$

$$p \lim_{n \rightarrow \infty} h\left(\frac{n}{p}\right) < ph\left(\frac{n}{p}\right) \leq ph\left(\frac{1}{p}\right) = p \left(\frac{\Gamma\left(\frac{2}{p} + 1\right)}{\Gamma\left(\frac{1}{p} + 1\right)} \right)^p - 1.$$

From (2.4),

$$\lim_{n \rightarrow \infty} h(n) = \frac{p+1}{2p}.$$

This is the end of the proof.

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