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INEQUALITIES OF BERNSTEIN-JACKSON-TYPE AND THE DEGREE OF COMPACTNESS OF OPERATORS IN BANACH SPACES

By Bernd CARL

0. INTRODUCTION

Since the classical investigations of B. Riemann and D. Jackson one has been interested to know how certain properties like continuity, differentiability or integrability of 1-periodic functions influence the behaviour of Fourier coefficients and the "degree of approximation" by trigonometrical polynomials. Because every such function f corresponds to a convolution operator

$$x(t) \longrightarrow y(s) = \int_0^1 f(s-t)\, x(t)\, dt$$

in an appropriate function space whose approximation numbers may be estimated by the approximation (Bernstein) numbers of the function and whose eigenvalues coincide with the Fourier coefficients we have a special situation of a more general problem. Namely, we want to know how the "approximation" and the distribution of eigenvalues of an integral operator depends on the properties of its kernel. Much more generally the question arises whether there are characteristics for operators in Banach spaces implying a good approximation as well as a good behaviour of its eigenvalues. Obviously, the smoother the kernel the better the properties of an operator. This automatically leads to the problem of finding certain characteristics for the "degree of compactness" of an operator which guarantee a good approximation of the operator as well as a good behaviour of its eigenvalues. The entropy moduli (resp. entropy numbers) have turned out to be very natural characteristics simultaneously quantifying the "degree

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of compactness" of operators as well as the behaviour of its eigenvalues.

The starting point of the paper are finite dimensional covering problems and the degree of compactness of l_1 and l_∞ -factorizable operators. The main part of this paper is devoted to new relationships between entropy moduli (resp. entropy numbers) and Kolmogorov (resp. Gelfand and approximation) numbers for operators which may be interpreted as counterparts to the classical Bernstein-Jackson inequalities for functions. Via convolution operators we notice that these analogies are not only formal. Within the class of Banach spaces of type we establish new interesting statements on the equivalence between Kolmogorov, Gelfand, entropy and approximation numbers. We also discuss a quantification of results in the Riesz-Schauder-Theory in terms of entropy moduli and Gelfand numbers. Finally, we determine the largest distance between the "degree of approximation" and the "degree of compactness" of integral operators in $C[0,1]$ generated by smooth kernels. For illustrating the quantification of the Riesz-Schauder-Theory we treat some eigenvalue and compactness problems of nuclear operators and operators of Hille-Tamarkin-type.

Throughout this paper we use standard definitions and notations of Banach space theory. For the convenience of the reader we here collect some of them. In the following E, F, and G always denote (real or complex) Banach spaces. Given any Banach space E, we denote the closed unit ball by U_E, the dual Banach space by E'. If $x \in E$ and $a \in E'$, then $\langle x, a\rangle$ is the value of the functional a at the element x. Moreover, $\mathcal{L}(E,F)$ denotes the Banach space of all (bounded linear) operators S from E into F equipped with the usual norm. R denotes the real line and C the complex plane. We recall the definitions of some s-numbers. For every operator $S \in \mathcal{L}(E,F)$ the n^{th} Kolmogorov number $d_n(S)$, $n = 1, 2, \ldots,$ is defined by

$$d_n(S) := \inf_{N \subset F} \sup_{x \in U_E} \inf_{y \in N} \|Sx - y\|$$

$$= \inf_{N \subset F} \inf \{\delta > 0 : S(U_E) \subseteq \delta\, U_F + N\},$$

where N is an arbitrary subspace of F with dim $N < n$. Clearly, $d_n(S)$ is the infimum over all $(n-1)$-dimensional subspaces $N \subset F$ of inclination of the image $S(U_E)$ of the unit ball U_E from N. It expresses how "nicely" the image $S(U_E)$ can be approximated by $(n-1)$-dimensional subspaces of F. Using the canonical map Q_N^F from F onto F/N the Kolmogorov numbers may be described as

$$d_n(S) = \inf \{\| Q_N^F S \| : N \subset F,\ \dim N < n\}.$$

The dual concept to the Kolmogorov numbers has been suggested by Gelfand. The n^{th} Gelfand number of an operator $S \in \mathcal{L}(E,F)$ is defined by

$$c_n(S) := \inf \{\| S J_M^E \| : M \subset E,\ \operatorname{codim} M < n\},$$

where J_M^E denotes the natural embedding from M into E. Obviously, we have $c_n(S) = d_n(S')$, where S' is the dual operator of S. The concept corresponding to the well-known ϵ-entropy are the so-called entropy numbers introduced by B. Mitjagin and A. Pełczyński [22]. For $S \in \mathcal{L}(E,F)$ the n^{th} entropy number $\epsilon_n(S)$, $n = 1, 2, \ldots$, is defined to be the infimum of all $\epsilon > 0$ such that there are $y_1, \ldots, y_n \in F$ for which

$$S(U_E) \subseteq \bigcup_1^n (y_i + \epsilon\, U_F).$$

The analytic expression of this characteristic is

$$\epsilon_n(S) = \inf_{\{y_1, \ldots, y_n\} \subset F} \ \sup_{x \in U_E} \ \inf_{1 \leq i \leq n} \| Sx - y_i \|;$$

$\epsilon_n(S)$ means the best approximation of $S(U_E)$ by sets consisting of n elements. In the framework of operator ideals in Banach spaces it frequently will be helpful to switch over from ϵ_n to the dyadic entropy numbers $e_n(S) := \epsilon_{2^{n-1}}(S)$. For algebraic properties of the above characteristics we refer to ([23] (11), (12)). Finally, characteristics closely related to entropy numbers describing not only the "degree of compactness" but also the behaviour of eigenvalues of operators very well are the so-called entropy moduli. The n^{th} entropy modulus $g_n(S)$, $n = 1, 2, \ldots$, of an operator $S \in \mathcal{L}(E,F)$ is defined by

$$g_n(S) := \inf_{k=1,2,\ldots} k^{1/n} \epsilon_k(S).$$

Roughly speaking, the entropy moduli may be interpreted as the n^{th} root of the volume of an optimal covering of $S(U_E)$ by balls $y_i + \epsilon U_F$ considered in n-dimensional real Banach spaces. The reason for considering this characteristic is the following relationships between eigenvalues and entropy moduli for compact operators $S \in \mathfrak{L}(E, E)$ acting in complex Banach spaces [1], [5], [19]:

$$\Big(\prod_1^n |\lambda_i(S)|\Big)^{1/n} \leqslant g_{2n}(S), \qquad n = 1, 2, \ldots,$$

and

$$\Big(\prod_1^n |\lambda_i(S)|\Big)^{1/n} = \lim_{k \to \infty} g_{2n}^{1/k}(S^k), \quad n = 1, 2, \ldots .$$

Here the eigenvalues are ordered in non-increasing absolute values and counted according to their algebraic multiplicities. We mention some algebraic properties of the entropy moduli.

$$\| S \| = g_1(S) \geqslant g_2(S) \geqslant \ldots \geqslant 0 \quad \text{for} \quad S \in \mathfrak{L}(E, E),$$

$g_n(TS) \leqslant g_n(T)\, g_n(S)$ for $S \in \mathfrak{L}(E, F)$, $T \in \mathfrak{L}(F, G)$, $g_n(S) = 0$ if $S \in \mathfrak{L}(E, F)$ with rank $(S) < n$ and if E and F are real Banach spaces ($g_{2n-1}(S) = 0$ if E and F are complex Banach spaces), $g_n(I_n) = 1$, where I_n is the identity operator on an n-dimensional Banach space.

Finally, we introduce special Banach spaces. We say that a Banach space E is of (Rademacher) type p, $1 \leqslant p \leqslant 2$, if there is a constant $\tau > 0$ such that for every finite sequence $\{x_1, \ldots, x_n\} \subset E$ we have

$$\int_0^1 \Big\| \sum_1^n r_i(t)\, x_i \Big\|\, dt \leqslant \tau \Big(\sum_1^n \| x_i \|^p \Big)^{1/p},$$

where (r_i) denotes the sequence of Rademacher functions on the interval $[0, 1]$, i.e. $r_i(t) = \text{sign}\ (\sin 2^i \pi t)$. The Rademacher type p constant $\tau_p(E)$ is the smallest constant τ satisfying the above condition. For a Banach space we have the following characterization of type p [12], [21]:

For all independent E-valued random variables $z_1, \ldots, z_n$, $n = 1, 2, \ldots$, with finite p^{th} moment the inequality

$$\mathbf{E} \| \sum_{i=1}^{n} (z_i - \mathbf{E}\, z_i) \| \leqslant 4\, \tau_p(\mathrm{E}) \Big(\sum_{i=1}^{n} \mathbf{E} \| z_i \|^p \Big)^{1/p}$$

holds, where $\mathbf{E}$ is the mathematical expectation. For detailed information on the type of Banach spaces we refer to B. Maurey and G. Pisier [21] and to J. Hoffman-Jφrgensen [12]. Given $0 < p$, $u \leqslant \infty$ and a measure space (Ω, Σ, μ), the Lorentz space $\mathrm{L}_{p,u} := \mathrm{L}_{p,u}(\Omega, \Sigma, \mu)$ (with $\mathrm{L}_p := \mathrm{L}_{p,p}$) is the set of all $\Sigma -$ measurable (real or complex) valued functions f such that

$$\| f \|_{p,u} := \Big(\int_0^{\infty} f^*(t)^u\, t^{\frac{u}{p} - 1}\, dt \Big)^{1/u} < \infty .$$

Here f^* denotes the (equimeasurable) rearrangement of f (cf. [13]). For $\Omega = \{1, 2, \ldots\}$ (set of natural numbers) and $\mu =$ counting measure we get the Lorentz sequence space $l_{p,u}$ (with $l_p := l_{p,p}$) which may be re (quasi) normed by

$$\| (\xi_n) \|_{p,u} := \Big(\sum_{n=1}^{\infty} n^{\frac{u}{p} - 1}\, \xi_n^{*u} \Big)^{1/u} .$$

By l_p^n we denote the vector space of n-tuples equipped with $\| \cdot \|_p$ and by I_n we mean the identity operator between l_p^n spaces. We mention that the L_p spaces, $1 \leqslant p < \infty$, are of type min $(p\,; 2)$. The conjugate index to p, $1 \leqslant p \leqslant \infty$, is defined by $\frac{1}{p'} := 1 - \frac{1}{p}$.

There are several constants which enter into the estimates below. These constants are mostly denoted by the letters $\rho, \rho_0, \ldots$. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different constants in different parts of the paper. Let (a_n) and (b_n) be non-negative sequences. We write $a_n \lesssim b_n$ if there exists a constant $\rho > 0$ such that $a_n \leqslant \rho\, b_n$ for $n = 1, 2, \ldots$. The symbol $a_n \asymp b_n$ means $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

1. COVERING PROBLEMS AND THE DEGREE OF COMPACTNESS OF OPERATORS

The starting point of this paper are n-dimensional covering problems described by entropy numbers. Our interest is in getting good estimates for the radius of a given number of m-dimensional balls covering an n-dimensional set. It turns out that it is possible to obtain quite sharp general estimates which in many interesting examples give the best possible results. Especially, we get new estimates of entropy numbers of arbitrary operators from l_p^m into l_∞^m. It is not surprising therefore that sharp estimates on entropy have also various applications. In particular, we are in a position to characterize in terms of entropy numbers the degree of compactness of operators of the form DT where $D : l_\infty \longrightarrow l_\infty$ is a diagonal and T an arbitrary operator from a Hilbert space into l_∞.

Basic estimates.

We start our considerations with an improved version of a result going back to B. Maurey (cf. G. Pisier [25]).

LEMMA 1. – *Let* $n = 1, 2, \ldots$ *and* $S \in \mathcal{L}(\ell_1^n, E)$. *If* E *is a Banach space of type* p, *then in the real case*

$$\epsilon_{\binom{2n+k-1}{k}}(S) \leqslant 4\, \tau_p(E)\, k^{-1+1/p}\, \| S \|$$

for $k = 1, 2, \ldots$, *and in the complex case*

$$\epsilon_{\binom{4n+k-1}{k}}(S) \leqslant 4\sqrt{2}\, \tau_p(E)\, k^{-1+1/p}\, \| S \|$$

for $k = 1, 2, \ldots$

Proof. – First we assume that $l_1^n := l_1^n(\mathrm{R})$ is a real vector space. Putting $M_n := \{\pm Se_i : i = 1, \ldots, n\}$ we have

$$S(U_{l_1^n}) = \operatorname{conv} M_n,$$

where $e_1, \ldots, e_n$ denotes the canonical basis of l_1^n. For every $y \in S(U_{l_1^n})$ there is an E-valued random variable z on a suitable probability space (Ω, μ) with values in M_n such that $\mathbf{E} z = y$. Now we take k independent E-valued random variables $z_1, \ldots, z_k$ on (Ω^k, μ^k) with values in M_n such that $\mathbf{E} z_i = y$, $i = 1, \ldots, k$. Since E is of type p from $\|z_i\| \leqslant \|S\|$, $i = 1, \ldots, k$, we infer that

$$\mathbf{E} \left\| \sum_{i=1}^{k} z_i - ky \right\| = \mathbf{E} \left\| \sum_{i=1}^{k} (z_i - \mathbf{E} z_i) \right\|$$

$$\leqslant 4\, \tau_p(\mathrm{E}) \left(\sum_{i=1}^{k} \mathbf{E} \|z_i\|^p \right)^{1/p}$$

$$\leqslant 4\, \tau_p(\mathrm{E})\, k^{1/p} \|S\|.$$

Therefore

$$\mathbf{E} \left\| \frac{1}{k} \sum_{i=1}^{k} z_i - y \right\| \leqslant 4\, \tau_p(\mathrm{E})\, k^{-1+1/p} \|S\|.$$

Now there are $y_1, \ldots, y_k \in M_n$ so that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} y_i - y \right\| \leqslant 4\, \tau_p(\mathrm{E})\, k^{-1+1/p} \|S\|.$$

Because $\quad \operatorname{card} \left\{ \frac{1}{k} \sum_{i=1}^{k} y_i : y_i \in M_n \right\} \leqslant \binom{2n+k-1}{k}$

we obtain

$$\epsilon_{\binom{2n+k-1}{k}}(S) \leqslant 4\, \tau_p(\mathrm{E})\, k^{-1+1/p} \|S\|$$

for $k = 1, 2, \ldots$.

Finally, let $l_1^n := l_1^n(\mathbf{C})$ be complex. There is an isomorphism $J : l_1^n(\mathbf{C}) \longrightarrow l_1^{2n}(\mathbf{R})$ such that $\|J\|\,\|J^{-1}\| \leqslant \sqrt{2}$.

Thus

$$\epsilon_{\binom{4n+k-1}{k}}(S) \leqslant \|J\| \, \epsilon_{\binom{4n+k-1}{k}}(SJ^{-1})$$

$$\leqslant 4\, \tau_p(E)\, k^{-1+1/p} \, \|J\| \, \|SJ^{-1}\|$$

$$\leqslant 4\sqrt{2}\, \tau_p(E)\, k^{-1+1/p} \, \|S\|$$

for $k = 1, 2, \ldots$.

□

The next statement is a consequence of a basic inequality in the theory of stochastic processes obtained by Sudakov (cf. [6]).

LEMMA 2. – *Let* $n = 1, 2, \ldots,$ *and* $S \in \mathcal{L}(E, l_2^n)$. *If* E' *is a Banach space of type p, then*

$$(\log k)^{1/2} \, \varepsilon_{k+1}(S) \leqslant \rho(p)\, \tau_p(E') \left(\sum_1^n \|S' e_i\|^p \right)^{1/p}$$

for $k = 1, 2, \ldots,$ *where*

$$\rho(p) \leqslant \begin{cases} 4\sqrt{\pi} \left(\dfrac{\Gamma\left(\dfrac{1+p}{2}\right)}{\Gamma\left(\dfrac{1}{2}\right)} \right)^{1/p} & \textit{(real case)} \\[4ex] 4\sqrt{\pi}\, 2^{1/p} \left(\dfrac{\Gamma\left(\dfrac{1+p}{2}\right)}{\Gamma\left(\dfrac{1}{2}\right)} \right)^{1/p} & \textit{(complex case)} \end{cases}$$

Proof. – The result of Sudakov already announced states that for a Gaussian process $\theta_1, \ldots, \theta_m$, the inequality

$$(\log m)^{1/2} \inf_{1 \leqslant i \neq j \leqslant m} (\mathbf{E}\, |\theta_i - \theta_j|^2)^{1/2} \leqslant \sqrt{2\pi}\, \mathbf{E} \sup_{1 \leqslant i \leqslant m} |\theta_i|$$

holds. Let $S \in \mathcal{L}(E, l_2^n(R))$ and fix k. Then we may choose elements $a_1, \ldots, a_k \in U_E$ such that

$$\epsilon_{k+1}(S) \leqslant \inf_{1 \leqslant i \neq j \leqslant k} \|Sa_i - Sa_j\| .$$

Furthermore, let μ be the Gaussian measure on the real line with the characteristic function $\exp(-\eta^2)$ and let μ^n denote the n-fold product of μ and consider the Gaussian random variables $\theta_i(x) := \langle x, Sa_i \rangle$, $i = 1, 2, \ldots, k$, on (R^n, μ^n). Observe that the coordinate functionals $\langle x, e_i \rangle$, where e_i, $i = 1, \ldots, n$, is the canonical basis in l_2^n, are independent random variables on the probability space (R^n, μ^n) with the characteristic function $\exp(-\eta^2)$. By [23, p. 28] we have

$$\mathrm{E}\,|\theta_i - \theta_j|^2 = \int_{R^n} |\langle x, Sa_i - Sa_j \rangle|^2 \, d\mu^n(x) = 2\,\|Sa_i - Sa_j\|^2 .$$

Since the Rademacher functions attain only the values $+1$ and -1 it follows from the symmetry of the random variables $\langle x, e_i \rangle$, $i = 1, \ldots, n$, that for every $t \in [0, 1]$

$$\sum_1^n \langle x, e_i \rangle S' e_i \quad \text{and} \quad \sum_1^n r_i(t) \langle x, e_i \rangle S' e_i$$

possess the same distribution. Consequently,

$$\int_{R^n} \Big\| \sum_1^n \langle x, e_i \rangle S' e_i \Big\| \, d\mu^n(x) = \int_{R^n} \Big\| \sum_1^n r_i(t) \langle x, e_i \rangle S' e_i \Big\| \, d\mu^n(x).$$

Thus

$$\begin{aligned} \int_{R^n} \Big\| \sum_1^n \langle x, e_i \rangle S' e_i \Big\| \, d\mu^n(x) &= \int_{R^n} \int_0^1 \Big\| \sum_1^n r_i(t) \langle x, e_i \rangle S' e_i \Big\| \, dt \, d\mu^n(x) \\ &\leqslant \tau_p(E') \int_{R^n} \Big(\sum_1^n |\langle x, e_i \rangle|^p \, \|S' e_i\|^p \Big)^{1/p} d\mu^n(x) \\ &\leqslant \tau_p(E') \Big(\sum_1^n \int_{R^n} |\langle x, e_i \rangle|^p \, d\mu^n(x) \, \|S' e_i\|^p \Big)^{1/p} \\ &\leqslant \tau_p(E') \Big(\int_{R^n} |\langle x, e_1 \rangle|^p \, d\mu^n(x) \Big)^{1/p} \Big(\sum_1^n \|S' e_i\|^p \Big)^{1/p} . \end{aligned}$$

Notice that (cf. [23] p. 289)

$$\left(\int_{\mathbf{R}^n} |\langle x, e_1 \rangle|^p \, d\,\mu^n(x)\right)^{1/p} = \rho_0(p) = 2\left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/p}.$$

Putting all together we obtain the desired assertion

$$(\log k)^{1/2}\, \epsilon_{k+1}(\mathrm{S}) \leqslant 2\sqrt{\pi}\, \rho_0(p)\, \tau_p(\mathrm{E}')\left(\sum_1^n \|\mathrm{S}' e_i\|^p\right)^{1/p}$$

in the real case. The complex case may be treated similarly.

Remark. – From a deep comparison theorem for p-stable and Gaussian processes of M. Marcus and G. Pisier [20] one may derive the following statement cf. [16]. Let $n = 1, 2, \ldots,$ and $\mathrm{S} \in \mathcal{L}(\mathrm{E}, l_p^n)$. If E' is a Banach space of type q and $1 < q < p < 2$, then

$$\epsilon_{k+1}(\mathrm{S}) \leqslant \rho(p, q)\, \tau_q(\mathrm{E}')\, (\log k)^{-1+1/p} \left(\sum_1^n \|\mathrm{S}' e_i\|^q\right)^{1/q}$$

for $k = 1, 2, \ldots$.

Operators in $\mathcal{L}(l_1^n, \mathrm{E})$.

In the sequel we establish sharp estimates for entropy numbers of operators from l_1^n into Banach spaces of type p.

PROPOSITION 1. – *Let* $n = 1, 2, \ldots$ *and* $\mathrm{S} \in \mathcal{L}(l_1^n, \mathrm{E})$. *If* E *is a Banach space of type* p, *then*

$$e_k(\mathrm{S}) \leqslant \rho\, \tau_p(\mathrm{E})\, f(k, n, p)\, \|\mathrm{S}\|$$

for $k = 1, 2, \ldots,$ *where*

$$f(k, n, p) = 2^{-\max(k/n\,;\,1)} \min\left\{1\,;\left[\max\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\,;\frac{1}{n}\right)\right]^{1-1/p}\right\}$$

and $\rho \leqslant 2^6\, 40^{1-1/p} \leqslant 448$.

Proof. – From Lemma 1 we obtain

$$\epsilon_{\binom{2n+k-1}{k}}(S) \leqslant 4\,\tau_p(E)\,k^{-1+1/p}\,\|S\| \quad \text{for} \quad k = 1, 2, \ldots, n$$

in the real case. Passing from k to $\dfrac{k}{20\log\left(\dfrac{n}{k}+1\right)}$ we get for the dyadic entropy numbers

$$e_k(S) \leqslant 4 \cdot 20^{1-1/p}\,\tau_p(E)\left(\frac{\log\left(\dfrac{n}{k}+1\right)}{k}\right)^{1-1/p} \|S\|$$

for $\log(n+1) \leqslant k \leqslant n$. Obviously, $e_k(S) \leqslant \|S\|$ for $1 \leqslant k \leqslant \log(n+1)$. Furthermore, if $k \geqslant n+1$, then

$$\begin{aligned} e_k(S) &\leqslant e_n(S)\,e_{k-n}(I_n : l_1^n \longrightarrow l_1^n) \leqslant 4 \cdot 2^{-(k-n)/n}\,e_n(S) \\ &\leqslant 8 \cdot 2^{-k/n}\,e_n(S) \end{aligned}$$

(cf. [23 (12)]). Choosing, for each k and n, the best estimate we obtain the desired assertion with $\rho_0 \leqslant 32 \cdot 20^{1-1/p}$ in the real case. Finally, let $l_1^n = l_1^n(C)$ be complex. There is an isomorphism $J : l_1^n(C) \longrightarrow l_1^{2n}(R)$ such that $\|J\|\,\|J^{-1}\| \leqslant \sqrt{2}$. Thus

$$\begin{aligned} e_k(S : l_1^n(C) \longrightarrow E) &\leqslant \|J\|\,e_k(SJ^{-1} : l_1^{2n}(R) \longrightarrow E) \\ &\leqslant \rho_0\,\|J\|\,f(k, 2n, p)\,\|SJ^{-1}\| \\ &\leqslant \rho_0\,2^{1/2}\,2^{1-1/p}\,f(k, n, p)\,\|S\| \end{aligned}$$

in the complex case, which yields the desired estimate.

□

Now we turn to the well-known type p constant of L_p spaces.

LEMMA 3. – *Let* $L_p := L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p < \infty$. *Then*

$$\tau_{\min(p;2)}(L_p) \leqslant K_p \leqslant \sqrt{p},$$

where K_p *denotes the constant in Khintchin's inequality.*

Proof. – If $x_1, \ldots, x_n \in L_p(\Omega, \Sigma, \mu)$, Then by Khintchin's inequality (cf. [23] (E.5)) we get

$$\left(\int_0^1 \left\| \sum_1^n r_i(t)\, x_i \right\| dt\right)^p \leqslant \int_0^1 \left\| \sum_1^n r_i(t)\, x_i \right\|^p dt$$

$$\leqslant \int_0^1 \int_\Omega \left| \sum_1^n r_i(t)\, x_i(\omega) \right|^p d\mu(\omega)\, dt$$

$$\leqslant \int_\Omega \int_0^1 \left| \sum_1^n r_i(t)\, x_i(\omega) \right|^p dt\, d\mu(\omega)$$

$$\leqslant K_p^p \int_\Omega \left(\sum_{i=1}^n |x_i(\omega)|^2 \right)^{p/2} d\mu(\omega).$$

From

$$\int_\Omega \left(\sum_1^n |x_i(\omega)|^2 \right)^{p/2} d\mu(\omega) \leqslant \sum_1^n \|x_i\|_p^p \quad \text{for} \quad 1 \leqslant p \leqslant 2$$

and

$$\int_\Omega \left(\sum_1^n |x_i(\omega)|^2 \right)^{p/2} d\mu(\omega) \leqslant \left(\sum_1^n \|x_i\|_p^2 \right)^{p/2} \quad \text{for} \quad 2 \leqslant p < \infty$$

we obtain the desired estimate

$$\int_0^1 \left\| \sum_1^n r_i(t)\, x_i \right\| dt \leqslant K_p \left(\sum_1^n \|x_i\|_p^{\min(p;2)} \right)^{1/\min(p;2)},$$

where
$$K_p = \begin{cases} 1 & \text{for } 1 \leqslant p \leqslant 2 \\[2ex] 2^{1/2} \left(\dfrac{\Gamma\left(\dfrac{1+p}{2}\right)}{\Gamma\left(\dfrac{1}{2}\right)} \right)^{1/p} & \text{for } 2 \leqslant p < \infty \end{cases}$$

cf. Haagerup [10]. Hence

$$\tau_{\min(p;2)}(L_p) \leqslant K_p \leqslant \sqrt{p} \qquad \text{for } 1 \leqslant p < \infty.$$

□

Combining Proposition 1 and Lemma 3 we establish the following statement, of which use will be made in later arguments.

PROPOSITION 2. – *Let* $n = 1, 2, \ldots,$ *and* $S \in \mathcal{L}(l_1^n, L_p)$, $1 \leqslant p < \infty$. *Then*

$$e_k(S) \leqslant \rho \sqrt{p} f(k, n, \min(p; 2)) \| S \| \quad \text{for} \quad k = 1, 2, \ldots,$$

where f *is the function in Proposition 1.*

In order to show that the estimate in Proposition 2 is the best possible we need a result due to C. Schütt [29].

LEMMA 4. – *Let* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $n = 1, 2, \ldots$. *Then there are absolute constants* $\rho_0, \rho_1 > 0$ *such that*

$$\rho_0 g(k, n, q, p) \leqslant e_k(I_n : l_q^n \longrightarrow l_p^n) \leqslant \rho_1 g(k, n, q, p)$$

for $k = 1, 2, \ldots,$ *where*

$$g(k, n, q, p) = 2^{-\max(k/n; 1)} \min\left(1; \left(\max\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}; \frac{1}{n}\right)\right)^{1/q - 1/p}\right).$$

SUPPLEMENT 1. – *Let* $1 \leqslant p < \infty$ *and* $n = 1, 2, \ldots$. *Then there are operators* $A_n \in \mathcal{L}(l_1^n, l_p^n)$ *such that*

$$\rho_0 f(k, n, \min(p; 2)) \leqslant e_k(A_n) \leqslant \rho_1 \sqrt{p} f(k, n, \min(p; 2))$$

for $k = 1, 2, \ldots,$ *where* f *is the function in Proposition 1.*

Proof. – In the case $1 \leqslant p \leqslant 2$ the assertion follows from Lemma 4. For $2 \leqslant p < \infty$ we take the Walsh-matrices

$$A_n := n^{-1/2} (e^{2\pi i kl/n})_{k,l = 1, \ldots, n}.$$

Obviously, $A_n^* A_n = I_n$ and therefore $\| A_n : l_2^n \longrightarrow l_2^n \| = 1$. Applying Lemma 4 we get

$$\begin{aligned}
\rho_0 f(k, n, 2) &\leqslant e_k(I_n : l_1^n \longrightarrow l_2^n) \leqslant e_k(A_n^* A_n : l_1^n \longrightarrow l_2^n) \\
&\leqslant \| A_n^* : l_p^n \longrightarrow l_2^n \| \, e_k(A_n : l_1^n \longrightarrow l_p^n) \\
&\leqslant n^{1/2 - 1/p} e_k(A_n : l_1^n \longrightarrow l_p^n).
\end{aligned}$$

Thus

$$\rho_0 \, n^{1/p-1/2} f(k,n,2) \leqslant e_k(\mathrm{A}_n : l_1^n \longrightarrow l_p^n)$$

for $k,n = 1,2,\ldots$. On the other hand we obtain from Proposition 2

$$e_k(\mathrm{A}_n : l_1^n \longrightarrow l_p^n) \leqslant \rho\sqrt{p}\, f(k,n,2)\, \| \mathrm{A}_n : l_1^n \longrightarrow l_p^n \|$$
$$\leqslant \rho\sqrt{p}\, n^{1/p-1/2} f(k,n,2)$$

for $k,n = 1,2,\ldots$. This completes the proof.

□

The previous results will be used to get also quite sharp estimates of the entropy of operators from l_1^n into l_∞^m .

PROPOSITION 3. – *Let* $n,m = 1,2,\ldots,$ *and* $\mathrm{S} \in \mathcal{L}(l_1^n, l_\infty^m)$. *Then* $e_k(\mathrm{S}) \leqslant \rho\, h(k,n,m) \| \mathrm{S} \|$ *for* $k = 1,2,\ldots,$ *where*

$$h(k,n,m) = 2^{-\max(k/n,k/m,1)} \min\left(1\,;\max\left(1\,;\log^{1/2}\left(\frac{m}{k}+1\right)\right)\right) *$$
$$* \min\left(1\,;\left(\max\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\,;\frac{1}{n}\right)\right)^{1/2}\right).$$

Proof. – By Proposition 2 and Lemma 4 we check, for $2 \leqslant p < \infty$, the estimate

$$e_{2k}(\mathrm{S} : l_1^n \longrightarrow l_\infty^m)$$
$$\leqslant e_k(\mathrm{I}_m : l_p^m \longrightarrow l_\infty^m)\, e_k(\mathrm{S} : l_1^n \longrightarrow l_p^m)$$
$$\leqslant \rho_0\sqrt{p}\, g(k,m,p,\infty)\, f(k,n,2)\, \| \mathrm{S} : l_1^n \longrightarrow l_p^m \|$$
$$\leqslant \rho_0\sqrt{p}\, g(k,m,p,\infty)\, m^{1/p} f(k,n,2)\, \| \mathrm{S} : l_1^n \longrightarrow l_\infty^m \| .$$

Putting $p = 2\max\left(1\,;\log\left(\frac{m}{k}+1\right)\right)$ we see that

$$g(k,m,p,\infty)\, m^{1/p} \leqslant \rho_1\, 2^{-\max(k/m\,;1)} .$$

Thus

$$e_{2k}(\mathrm{S}) \leqslant \rho_2\, 2^{-2\max(k/n\,;\,k/m\,;\,1)} \max\left(1\,;\log\left(\frac{m}{k}+1\right)\right) *$$

$$* \min\left(1\,;\left(\max\left(\frac{\log\left(\frac{n}{k}+1\right)}{k}\,;\frac{1}{n}\right)\right)^{1/2}\right) \|\,\mathrm{S}\,\|\,.$$

Finally, passing from $2k$ to k in the preceding estimate and comparing this with

$$e_k(\mathrm{S}) \leqslant 8 \cdot 2^{-\max(k/n\,;\,k/m\,;\,1)}\ \|\,\mathrm{S}\,\|$$

we get the desired assertion by choosing, for each k, n and m, the best estimate.

□

Operators in $\mathcal{L}(\mathrm{E}, l_\infty^m)$.

In the "dual" situation $\mathrm{S} \in \mathcal{L}(\mathrm{E}, l_\infty^m)$ where E' is a Banach space of type p we may only give in general upper estimates of entropy numbers by combining Lemma 2 and Lemma 4. However for operators from l_p^n into l_∞^m we get quite sharp estimates of entropy numbers. For this purpose we need a striking and deep Embedding-Theorem recently proved by Johnson and Schechtman [14] (cf. also Pisier [26]) in the case $1 < p < 2$ and which in the case $p = 2$ already goes back to Figiel, Lindenstrauss and Milman [7] (cf. also Kashin [15] and Szarek [30]).

LEMMA 5 (Embedding-Theorem). – *Let* $1 < p \leqslant 2$. *For each* $\epsilon > 0$, *there is a constant* $c(p, \epsilon) > 0$ *such that* l_1^m *contains, for each* m, *a subspace* $(1+\epsilon)$*-isomorphic to* l_p^k *with* $k \geqslant c\,(p, \epsilon)\, m$.

PROPOSITION 4. – *Let* $n, m = 1, 2, \ldots,$ *and* $\mathrm{S} \in \mathcal{L}(l_p^n, l_\infty^m)$, $1 < p \leqslant 2$. *Then*

$$e_k(\mathrm{S}) \leqslant \rho\,(p)\; h\,(k, n, m)\ \|\,\mathrm{S}\,\|$$

for $k = 1, 2, \ldots,$ *where* $h\,(k, n, m)$ *is the function of Proposition 3.*

Proof. – By Lemma 5 there is a constant $c(p) > 0$ such that, for each n, there exists an n-dimensional subspace $E_n \subset l_1^{[n/c(p)]}$ and an isomorphism $A : l_p^n \longrightarrow E_n$ such that $\|A\| \, \|A^{-1}\| \leqslant 2$. Consider the diagram

$$\begin{array}{ccccccc} l_p^n & \xrightarrow{\ A\ } & E_n & \xrightarrow{\ A^{-1}\ } & l_p^n & \xrightarrow{\ \ S\ \ } & l_\infty^m \\ & & & \searrow^{J} & & \nearrow_{T} & \\ & & & & l_1^{\left[\frac{n}{c(p)}\right]} & & \end{array}$$

Since l_∞^m has the extension property by the Hahn-Banach-Theorem there is an extension $T: l_1^{[n/c(p)]} \longrightarrow l_\infty^m$ of SA^{-1} such that $\|T\| = \|SA^{-1}\|$.

Applying Proposition 3 we check

$$\begin{aligned} e_k(S) &\leqslant e_k(SA^{-1}A) \leqslant \|A\| \, e_k(SA^{-1}) \\ &\leqslant \|A\| \, e_k(TJ) \leqslant \|A\| \, \|J\| \, e_k(T) \\ &\leqslant \rho_0 \, h\left(k, \left[\frac{n}{c(p)}\right], m\right) \|A\| \, \|T\| \\ &\leqslant \rho_0 \, h\left(k, \left[\frac{n}{c(p)}\right], m\right) \|A\| \, \|A^{-1}\| \, \|S\| \\ &\leqslant \rho(p) \, h(k, n, m) \, \|S\| \end{aligned}$$

which yields the desired assertion.

□

In particular, for $n = m$, we have the following statement

PROPOSITION 5. – *Let* $n = 1, 2, \ldots,$ *and* $S \in \mathcal{L}(l_p^n, l_\infty^n)$, $1 < p \leqslant 2$. *Then*

$$e_k(S) \leqslant \rho(p) \, h(k, n) \, \|S\|$$

for $k = 1, 2, \ldots,$ *where*

$$h(k,n) = 2^{-\max(k/n\,;\,1)} \min\left(1\,;\max\left(\frac{\log\left(\frac{n}{k}+1\right)}{k^{1/2}}\,;\frac{1}{n^{1/2}}\right)\right).$$

By taking Walsh-matrices one may show similarly as in the proof to Supplement 1 that the previous estimate is almost optimal.

SUPPLEMENT 2. – *For* $n = 1, 2, \ldots,$ *there exist* $A_n \in \mathcal{L}(l_p^n, l_\infty^m)$, $1 < p \leqslant 2$, *such that*

$$\rho_0 \frac{h(k,n)}{\max\left(1\,;\log^{1/2}\left(\frac{n}{k}+1\right)\right)} \leqslant e_k(A_n) \leqslant \rho_1(p)\, h(k,n)$$

for $k = 1, 2, \ldots$.

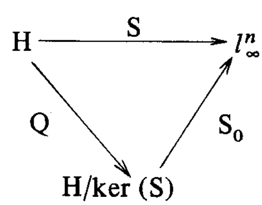
Finally, the previous results may be applied for characterizing the degree of compactness of operators from a Hilbert space H into l_∞^n .

PROPOSITION 6. – *Let* $n = 1, 2, \ldots,$ *and* $S \in \mathcal{L}(H, l_\infty^n)$. *Then*

$$e_k(S) \leqslant \rho\, h(k,n)\, \|S\|$$

for $k = 1, 2, \ldots,$ *where* $h(k,n)$ *is the function in Proposition 5.*

Proof. – For given $S \in \mathcal{L}(H, l_\infty^n)$ we consider the factorization



where Q is the quotient map to the quotient space H/ker (S). Obviously, H/ker (S) is again a Hilbert space with the dimension less than or equal to n and $\|S\| = \|S_0\|$. Moreover, there exists an isomorphism $A: \; H/\ker(S) \longrightarrow l_2^m$, $m := \dim(H/\ker(S)) \leqslant n$, such that $\|A\|\,\|A^{-1}\| = 1$.

By Proposition 5 we have

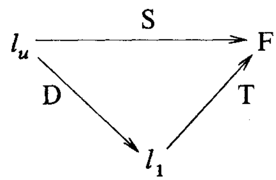
$$\begin{aligned} e_k(\mathrm{S}) &\leqslant e_k(\mathrm{S}_0\, \mathrm{A}^{-1}\, \mathrm{AQ}) \leqslant \| \mathrm{AQ} \|\, e_k(\mathrm{S}_0\, \mathrm{A}^{-1}) \\ &\leqslant h(k, n)\, \| \mathrm{A} \|\, \| \mathrm{S}_0\, \mathrm{A}^{-1} \| \\ &\leqslant h(k, n)\, \| \mathrm{S} \| . \end{aligned}$$

□

Degree of compactness of operators.

The preceding results may be applied to determine the degree of compactness of several operators. We give some important sharp results on the entropy of factorizable operators through l_1 and l_∞. The first statement has already been treated in [2]. It is a consequence of Lemma 1 and of the technique to the proof of Theorem 1 in [2].

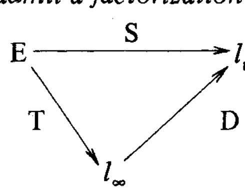
THEOREM 1. – *Let* F *be a Banach space of type* q *and let* $\mathrm{S} \in \mathfrak{L}(l_u, \mathrm{F})$ *admit a factorization through* l_1,



where $\mathrm{D}(\xi_i) := (\delta_i\, \xi_i)$, $(\delta_i) \in l_{r,t}$, *and* $\mathrm{T} \in \mathfrak{L}(l_1, \mathrm{F})$. *If* $1 \leqslant u \leqslant \infty$, $0 < t \leqslant \infty$, $r < u'$, *and* $\dfrac{1}{s} = \dfrac{1}{r} + \dfrac{1}{u} - \dfrac{1}{q}$, *then* $(e_n(\mathrm{S})) \in l_{s,t}$.

The "dual" situation to Theorem 1 has been considered by T. Kühn [16].

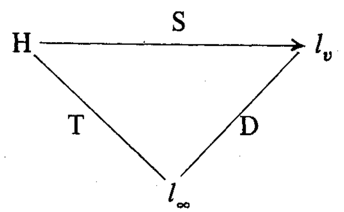
THEOREM 2. – *Let* E *be a Banach space such that* E′ *is of type* p *and let* $\mathrm{S} \in \mathfrak{L}(\mathrm{E}, l_v)$ *admit a factorization through* l_∞,



with a diagonal operator $D(\xi_l) := (\delta_l \xi_l)$, $(\delta_l) \in l_{r,t}$, *and* $T \in \mathcal{L}(E, l_\infty)$. If $1 \leqslant v \leqslant \infty$, $0 < t \leqslant \infty$, $r < \min(p, v)$ *and* $\frac{1}{s} = \frac{1}{r} + 1 - \frac{1}{p} - \frac{1}{v}$, *then* $(e_n(S)) \in l_{s,t}$.

However, in the above statement there is the condition $r < \min(p, v)$, we do not know whether in general it can be replaced by the weaker condition $r < v$. In the special case of Hilbert spaces H we may give an answer in the affirmative. Indeed, by Proposition 6 and with the technique to the proof of Theorem 1 in [2] we may establish the following statement.

THEOREM 3. – *Let* H *be a Hilbert space and let* $S \in \mathcal{L}(H, l_v)$ *admit a factorization through* l_∞,



with a diagonal operator $D(\xi_l) := (\delta_l \xi_l)$, $(\delta_l) \in l_{r,t}$, *and* $T \in \mathcal{L}(H, l_\infty)$. *If* $1 \leqslant v \leqslant \infty$, $0 < t \leqslant \infty$, $r < v$, *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{2} - \frac{1}{v}$, *then* $(e_n(S)) \in l_{s,t}$.

Remark. – The conclusions in the preceding theorems cannot be improved. Indeed, under the conditions of the preceding theorems, for $t_0 < t$, there exist operators such that $(e_n(S)) \notin l_{s,t_0}$ (cf. [2]).

2. INEQUALITIES OF BERNSTEIN-JACKSON-TYPE

In this section we shall establish new inequalities between entropy moduli and Kolmogorov (resp. Gelfand) numbers for operators which may be interpreted as counterparts to the classical

Bernstein and Jackson inequalities for functions. It turns out that the corresponding analogies are not only formal.

The main part is devoted to inequalities of Jackson-type. As a consequence of the inequalities of Bernstein and Jackson-type we obtain interesting statements on the equivalence of Kolmogorov, Gelfand, entropy numbers as well as entropy moduli. In particular, we show for operators $S \in \mathcal{L}(E, F)$, where the Banach spaces E and F' are of type 2, the equivalence

$$c_n(S) \asymp n^{-\alpha} \text{ iff } d_n(S) \asymp n^{-\alpha} \text{ iff } e_n(S) \asymp n^{-\alpha} \text{ iff } g_n(S) \asymp n^{-\alpha}$$

for $0 < \alpha < \infty$.

Inequalities of Bernstein-type.

The following inequality of Bernstein-type between dyadic entropy numbers and Kolmogorov (resp. Gelfand) numbers has already been proved in [1].

THEOREM 4. – *Let* $s \in \{c, d\}$ *and* $0 < \alpha < \infty$. *Then for all Banach spaces* E *and* F *and all* $S \in \mathcal{L}(E, F)$ *the inequality*

$$\sup_{1 \leq k \leq n} k^\alpha e_k(S) \leq \rho(\alpha) \sup_{1 \leq k \leq n} k^\alpha s_k(S), \quad n = 1, 2, \ldots,$$

is valid.

Inequalities of Jackson-type.

Before stating the inequalities of Jackson-type let us prove two lemmas. The first one may be found implicitly in [24].

LEMMA 6. – *Let* $S \in \mathcal{L}(E, F)$ *and* $\epsilon > 0$. *Then the following assertions are valid :*

(i) *There exist* $x_1, x_2, \ldots \in E$ *and* $b_1, b_2, \ldots \in F'$ *such that* $\|x_i\| \leq 1$, $\|b_k\| \leq 1$, $\langle Sx_i, b_k\rangle = 0$ *for* $i < k$, *and* $(1 + \epsilon) |\langle Sx_k, b_k\rangle| \geq d_k(S)$, $k = 1, 2, \ldots$.

(ii) *There exist* $x_1, x_2, \ldots \in B$ *and* $b_1, b_2, \ldots \in F'$ *such that* $\|x_i\| \leq 1$, $\|b_k\| \leq 1$, $\langle Sx_i, b_k\rangle = 0$ *for* $i > k$, *and* $(1 + \epsilon) |\langle Sx_k, b_k\rangle| \geq c_k(S)$, $k = 1, 2, \ldots$.

Proof. – We show (i) by induction, the second assertion may be treated similarly. Suppose that x_i and b_k for $i < n$ and $k < n$ have already been found with $\|x_i\| \leqslant 1$, $\|b_k\| \leqslant 1$, $\langle Sx_i, b_k \rangle = 0$ for $i < k$, and $(1 + \epsilon) |\langle Sx_k, b_k \rangle| \geqslant d_k(S)$, $k = 1, 2, \ldots, n - 1$.

Then we put $N := \operatorname{span}\{Sx_i : i < n\}$. Since $\dim N < n$, we may find $x_n \in E$ such that $\|x_n\| \leqslant 1$ and

$$(1 + \epsilon) \|Q_N^F \, Sx_n\| \geqslant \|Q_N^F \, S\| \geqslant d_n(S).$$

Moreover, there is $b_n^0 \in (F/N)'$ with $\|b_n^0\| \leqslant 1$ and

$$|\langle Q_N^F \, Sx_n, b_n^0 \rangle| = \|Q_N^F \, Sx_n\|.$$

If $b_n := (Q_N^F)' \, b_n^0$, then

$(1 + \epsilon) |\langle Sx_n, b_n \rangle| \geqslant d_n(S)$ and $\langle Sx_i, b_n \rangle = 0$ for $i < n$.

□

In order to formulate the next lemma we need the following quantity. If $S \in \mathcal{L}(E, F)$, then $\gamma_n(S)$, $n = 1, 2, \ldots,$ is defined by

$\gamma_n(S) := \sup\{g_n(BSX) : \|X : l_1^n \longrightarrow E\| \leqslant 1, \|B : F \longrightarrow l_\infty^n\| \leqslant 1\}.$

LEMMA 7. – *Let* $s \in \{c, d\}$ *and* $S \in \mathcal{L}(E, F)$. *If* E *and* F *are real Banach spaces, then*

$$\left(\prod_1^n s_k(S)\right)^{1/n} \leqslant n \, \gamma_n(S), \quad n = 1, 2, \ldots .$$

If E *and* F *are complex Banach spaces, then*

$$\left(\prod_1^n s_k(S)\right)^{1/n} \leqslant n \, \gamma_{2n}(S), \quad n = 1, 2, \ldots .$$

Proof. – We show the assertion for $s = d$, the statement for $s = c$ may be treated similarly. Let E and F be real Banach spaces and $\delta > 0$. By Lemma 6 there exist $x_1, x_2, \ldots \in E$ and $b_1, b_2, \ldots \in F'$ such that $\|x_i\| \leqslant 1$, $\|b_k\| \leqslant 1$, $\langle Sx_i, b_k \rangle = 0$ for $i < k$, and $(1 + \delta) |\langle Sx_k, b_k \rangle| \geqslant d_k(S)$. Hence the matrix $(\langle Sx_i, b_k \rangle)$ has superdiagonal form. Setting

$$X_n := \sum_1^n e_i \otimes x_i \quad \text{and} \quad B_n := \sum_1^n b_i \otimes e_i$$

where e_i denotes the canonical basis of the n-dimensional real vector space R^n, we get operators $X_n \in \mathcal{L}(l_1^n, E)$ and $B_n \in \mathcal{L}(F, l_\infty^n)$ with $\|X_n\| = \sup_{1 \leq i \leq n} \|x_i\| \leq 1$ and $\|B_n\| = \sup_{1 \leq i \leq n} \|b_i\| \leq 1$.
Obviously, the operator $I_n B_n SX_n \in \mathcal{L}(l_1^n, l_1^n)$ where $I_n \in \mathcal{L}(l_\infty^n, l_1^n)$ denotes the identity operator is generated by the triangle matrix $(\langle Sx_i, b_k \rangle)$, $1 \leq i, k \leq n$. Consequently,

$$\prod_1^n d_k(S) \leq (1+\delta)^n \prod_1^n |\langle Sx_k, b_k \rangle| \leq (1+\delta)^n |\det(I_n B_n SX_n)|.$$

Furthermore, let μ be the Lebesgue measure on l_1^n. Then

$$I_n B_n SX_n (U_{l_1^n}) \subseteq \bigcup_1^k \{y_i + \epsilon U_{l_1^n}\}$$

implies

$$\mu(I_n B_n SX_n(U_{l_1^n})) \leq \sum_1^k \mu(y_i + \epsilon U_{l_1^n}) \leq k\, \epsilon^n \mu(U_{l_1^n}).$$

On the other hand we have

$$\mu(I_n B_n SX_n(U_{l_1^n})) = |\det(I_n B_n SX_n)|\, \mu(U_{l_1^n}).$$

Combining the preceding two relations we may conclude that

$$|\det(I_n B_n SX_n)| \leq k\, \epsilon^n.$$

This yields

$$\begin{aligned}(\prod_1^n d_k(S))^{1/n} &\leq (1+\delta)\, g_n(I_n B_n SX_n) \\ &\leq (1+\delta)\, \|I_n\|\, g_n(B_n SX_n) \\ &\leq (1+\delta)\, n\, g_n(B_n SX_n)\end{aligned}$$

with $\|X_n : l_1^n \longrightarrow E\| \leq 1$ and $\|B_n : F \longrightarrow l_\infty^n\| \leq 1$.
Hence

$$(\prod_1^n d_k(S))^{1/n} \leq (1+\delta)\, n\, \gamma_n(S).$$

Finally, $\delta \longrightarrow 0$ completes the proof in the real case. The complex counterpart of this inequality may be treated similarly.

□

We now are able to prove the basic inequality of Jackson-type.

THEOREM 5. – *Let* $s \in \{c, d\}$ *and* $S \in \mathcal{L}(E, F)$. *If* E *is a Banach space of type* p *and* F' *a Banach space of type* q, *then in the real case the inequality*

$$(\prod_1^n s_k(S))^{1/n} \leqslant \rho\, \tau_p(E)\, \tau_q(F')\, n^{-1+1/p+1/q}\, g_n(S),\quad n = 1, 2, \ldots,$$

is valid, where $\rho \leqslant 768\, e^2 \sqrt{\pi} \left(\dfrac{\Gamma\left(\frac{1+q}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{1/q} \leqslant 10^4$ *and in the complex case the inequality*

$$(\prod_1^n s_k(S))^{1/n} \leqslant \rho\, \tau_p(E)\, \tau_q(F')\, n^{-1+1/p+1/q}\, g_{2n}(S),\quad n = 1, 2, \ldots,$$

is valid, where $\rho \leqslant 3\,840\, e^2 \sqrt{2\pi} \left(2 \dfrac{\Gamma\left(\frac{1+q}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{1/q} \leqslant 10^6$.

Proof. – We turn to the real case. Lemma 1 implies

$$g_n(X : l_1^n \longrightarrow E) \leqslant \binom{3n}{n}^{1/n} \epsilon_{\binom{3n}{n}}(X : l_1^n \longrightarrow E) \leqslant 3e\, \epsilon_{\binom{3n}{n}}(X)$$

$$\leqslant 12e\, \tau_p(E)\, n^{-1+1/p}\, \| X : l_1^n \longrightarrow E \|.$$

By Lemma 2 we obtain

$$g_n(B : F \longrightarrow l_2^n) \leqslant (2^{n+1})^{1/n}\, \epsilon_{2^{n+1}}(B : F \longrightarrow l_2^n)$$

$$\leqslant 4\, \rho(q)\, \tau_q(F')\, n^{1/q-1/2}\, \| B : F \longrightarrow l_\infty^n \|,$$

where $\rho(q)$ denotes the constant in Lemma 2.

On the other hand by C. Schütt ([29] proof to Theorem 1) we have

$$\epsilon_{2k\binom{n+k}{n}}(I_n : l_1^n \longrightarrow l_\infty^n) \leqslant k^{-1}.$$

Thus

$$g_n(\mathrm{I}_n : l_1^n \longrightarrow l_\infty^n) \leqslant \left(2^n \binom{2n}{n}\right)^{1/n} \epsilon_{2^n \binom{2n}{n}} (\mathrm{I}_n : l_1^n \longrightarrow l_\infty^n)$$

$$\leqslant 4e\, n^{-1} .$$

Therefore

$$g_n(\mathrm{I}_n : l_2^n \longrightarrow l_\infty^n) \leqslant g_n(\mathrm{I}_n : l_1^n \longrightarrow l_\infty^n) \, \| \mathrm{I}_n : l_2^n \longrightarrow l_1^n \|$$

$$\leqslant 4e\, n^{-1/2} .$$

Hence

$$g_n(\mathrm{B} : \mathrm{F} \longrightarrow l_\infty^n) \leqslant g_n(\mathrm{I}_n : l_2^n \longrightarrow l_\infty^n)\, g_n(\mathrm{B} : \mathrm{F} \longrightarrow l_2^n)$$

$$\leqslant 16e\, \rho(q)\, \tau_q(\mathrm{F}')\, n^{-1+1/q} \, \| \mathrm{B} : \mathrm{F} \longrightarrow l_\infty^n \| .$$

Combining the previous estimates we get

$$g_n(\mathrm{BSX}) \leqslant g_n(\mathrm{B})\, g_n(\mathrm{S})\, g_n(\mathrm{X})$$

$$\leqslant \rho\, \tau_p(\mathrm{E})\, \tau_q(\mathrm{F}')\, n^{-2+1/p+1/q}\, g_n(\mathrm{S})\, \|\mathrm{X}\| \, \|\mathrm{B}\| ,$$

where $\rho \leqslant 768\, e^2 \sqrt{\pi}\, \rho(q)$. Finally, Lemma 7 yields the desired inequality in the real case.

The complex counterpart may be treated similarly.

□

The previous inequalities are optimal in the sense of the following statement.

SUPPLEMENT 3. – *Let* $s \in \{c, d\}$ *and* $1 < p, q \leqslant 2$. *If* $0 \leqslant \alpha < -1 + \dfrac{1}{p} + \dfrac{1}{q}$ *and* $\rho > 0$, *then an inequality of the form* $(\prod_1^n s_k(\mathrm{S}))^{1/n} \leqslant \rho\, \tau_p(\mathrm{E})\, \tau_q(\mathrm{F}')\, n^\alpha\, g_n(\mathrm{S})$, $n = 1, 2, \ldots,$ *cannot hold for all* $\mathrm{S} \in \mathfrak{L}(\mathrm{E}, \mathrm{F})$ *all Banach spaces* E *of type* p *and all Banach spaces* F' *of type* q.

Proof. – We show the assertion for $s = d$. Define canonical operators $\mathrm{J}_m \in \mathfrak{L}(l_p^m, l_p)$ and $\mathrm{Q}_m \in \mathfrak{L}(l_{q'}, l_{q'}^m)$ by

$$\mathrm{J}_m(\xi_1, \ldots, \xi_m) := (\xi_1, \ldots, \xi_m, 0, 0, \ldots)$$

and

$$Q_m(\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots) := (\xi_1, \ldots, \xi_m).$$

Clearly, $\| J_m \| = \| Q_m \| = 1$. We define the operators $S_m \in \mathfrak{L}(l_p, l_{q'})$ by $S_m := Q_m I_m J_m$, where $I_m \in \mathfrak{L}(l_p^m, l_{q'}^m)$. A result of Gluskin [8] [9] yields

$$\rho_0(p, q) \leqslant d_n(I_{[nq'/2]}) = d_n(S_{[nq'/2]}) \quad \text{for} \quad n = 1, 2, \ldots,$$

and by [29] we have

$$g_n(S_{[nq'/2]}) = g_n(I_{[nq'/2]}) \leqslant 2e_n(I_{[nq'/2]})$$
$$\leqslant \rho_1(p, q) \left(\frac{\log(n+1)}{n}\right)^{-1+1/p+1/q}$$

for $n = 1, 2, \ldots$. By Lemma 3 we have $\tau_p(l_p) = 1$ for $1 \leqslant p \leqslant 2$ and $\tau_q((l_{q'})') = 1$ for $1 < q \leqslant 2$. Assume that the inequality

$$d_n(S_{[nq'/2]}) \leqslant \left(\prod_1^n d_k(S_{[nq'/2]})\right)^{1/n} \leqslant \rho\, n^\alpha g_n(S_{[nq'/2]})$$

holds with some $\rho > 0$ and $n = 1, 2, \ldots$. Then we get

$$\rho_0(p, q) \leqslant \rho\, \rho_1(p, q) \left(\frac{\log(n+1)}{n}\right)^{-1+1/p+1/q}$$

for $n = 1, 2, \ldots$. This estimate implies for $n = 1, 2, \ldots$ that $\alpha \geqslant -1 + 1/p + 1/q$.

□

COROLLARY 1. – *Let* $s \in \{c, d\}$ *and* $S \in \mathfrak{L}(E, F)$. *If* E *and* F *are arbitrary Banach spaces, then in the real case the inequality*

$$\left(\prod_1^n s_k(S)\right)^{1/n} \leqslant n\, g_n(S) \quad \textit{for} \quad n = 1, 2, \ldots,$$

is valid and in the complex case the inequality

$$\left(\prod_1^n s_k(S)\right)^{1/n} \leqslant n\, g_{2n}(S) \quad \textit{for} \quad n = 1, 2, \ldots,$$

is valid.

Proof. – The assertion follows immediately from $\gamma_n(S) \leqslant g_n(S)$ and Lemma 7.

□

The inequalities in Corollary 1 are optimal in the sense of the following statement:

SUPPLEMENT 4. – *Let* $s \in \{c, d\}$. *If* $0 \leqslant \alpha < 1$, *and* $\rho > 0$, *then an inequality of the form*

$$\left(\prod_1^n s_k(S)\right)^{1/n} \leqslant \rho\, n^\alpha\, g_n(S) \quad \textit{for} \quad n = 1, 2, \ldots,$$

cannot hold for all $S \in \mathfrak{L}(E,)$ *and all Banach spaces* E *and* F.

Proof. – Assume that such an inequality holds with some $\rho > 0$ and $n = 1, 2, \ldots$. Then the argument in the proof to Supplement 3 guarantees with $E = l_p$, $F = l_{q'}$, $1 < p, q \leqslant 2$, that $\alpha > -1 + 1/p + 1/q$. Letting $p \longrightarrow 1$, $q \longrightarrow 1$, we obtain $\alpha \geqslant 1$.

□

Equivalence of Bernstein and Jackson inequalities.

In the sequel we shall determine the Banach spaces in the class of type where we get equivalence of Bernstein and Jackson inequalities. Especially, we obtain some interesting statements on the duality of Kolmogorov and entropy numbers.

THEOREM 6. – *Let* $S \in \mathfrak{L}(E, F)$ *and* $0 < \alpha < \infty$. *If the Banach spaces* E *and* F' *are of type 2, then*

$$\sup_{1 \leqslant k \leqslant n} k^\alpha g_k(S) \asymp \sup_{1 \leqslant k \leqslant n} k^\alpha e_k(S) \asymp \sup_{1 \leqslant k \leqslant n} k^\alpha c_k(S) \asymp \sup_{1 \leqslant k \leqslant n} k^\alpha d_k(S).$$

Proof. – For $s \in \{c, d\}$ we see from Theorem 5 that

$$s_k(S) \leqslant \rho\, \tau_2(E)\, \tau_2(F')\, g_k(S) \leqslant 2\rho\, \tau_2(E)\, \tau_2(F')\, e_k(S).$$

Combining this estimate with Theorem 4 we obtain the desired assertion.

THEOREM 7. – *Let* $s \in \{c, d, e, g\}$, $0 < \alpha < \infty$, *and* $S \in \mathcal{L}(E, F)$. *If the Banach spaces* E *and* F′ *are of type 2, then*

$$\sup_{1 \leq k \leq n} k^{\alpha} s_k(S) \asymp \sup_{1 \leq k \leq n} k^{\alpha} s_k(S').$$

Proof. – Because $c_k(S) = d_k(S')$ and since E″ is of type 2 we have

$$\begin{aligned}\sup_{1 \leq k \leq n} k^{\alpha} e_k(S) &\leq \rho(\alpha) \sup_{1 \leq k \leq n} k^{\alpha} c_k(S) \\ &\leq \rho(\alpha) \sup_{1 \leq k \leq n} k^{\alpha} d_k(S') \\ &\leq \rho\, \rho(\alpha)\, \tau_2(E'')\, \tau_2(F') \sup_{1 \leq k \leq n} k^{\alpha} e_k(S')\end{aligned}$$

and

$$\begin{aligned}\sup_{1 \leq k \leq n} k^{\alpha} e_k(S') &\leq \rho(\alpha) \sup_{1 \leq k \leq n} k^{\alpha} d_k(S') \\ &\leq \rho(\alpha) \sup_{1 \leq k \leq n} k^{\alpha} c_k(S) \\ &\leq \rho\, \rho(\alpha)\, \tau_2(E)\, \tau_2(F') \sup_{1 \leq k \leq n} k^{\alpha} e_k(S)\end{aligned}$$

by Theorem 4 and Theorem 5. Putting together these estimates we obtain the assertion for $s = e$. Finally, if $s \in \{c, d, g\}$, Theorem 6 yields

$$\begin{aligned}\sup_{1 \leq k \leq n} k^{\alpha} s_k(S) \asymp \sup_{1 \leq k \leq n} k^{\alpha} e_k(S) \asymp \sup_{1 \leq k \leq n} k^{\alpha} e_k(S') \\ \asymp \sup_{1 \leq k \leq n} k^{\alpha} s_k(S').\end{aligned}$$

□

There are several immediate consequences of the preceding theorems. The first one is the following statement.

COROLLARY 2. – *Let* $0 < \alpha < \infty$ *and* $S \in \mathcal{L}(E, F)$. *If* E *and* F′ *are Banach spaces of type 2, then*

$$c_n(S) \asymp n^{-\alpha} \text{ iff } d_n(S) \asymp n^{-\alpha} \text{ iff } e_n(S) \asymp n^{-\alpha} \text{ iff } g_n(S) \asymp n^{-\alpha}.$$

Proof. – We show that $c_n(S) \asymp n^{-\alpha}$ implies $d_n(S) \asymp n^{-\alpha}$. From $\rho_0 n^{-\alpha} \leqslant c_n(S) \leqslant \rho_1 n^{-\alpha}$ we obtain by Theorem 6 $n^\alpha d_n(S) \leqslant \rho_2 \sup_{1 \leqslant k \leqslant n} k^\alpha c_k(S) \leqslant \rho_3$, and thus $d_n(S) \leqslant \rho_3 n^{-\alpha}$. Now we turn to the converse estimate. Applying again Theorem 6 we see that

$$\begin{aligned}\rho_0 (mn)^\alpha &\leqslant \sup_{1 \leqslant k \leqslant mn} k^{2\alpha} c_k(S) \leqslant \rho_4 \sup_{1 \leqslant k \leqslant mn} k^{2\alpha} d_k(S) \\ &\leqslant \rho_4 \sup_{1 \leqslant k \leqslant n} k^{2\alpha} d_k(S) + \rho_4 \sup_{n \leqslant k \leqslant mn} k^{2\alpha} d_k(S) \\ &\leqslant \rho_5 \sup_{1 \leqslant k \leqslant n} k^{2\alpha} c_k(S) + \rho_4 (mn)^{2\alpha} d_n(S) \\ &\leqslant \rho_6 n^\alpha + \rho_4 (mn)^{2\alpha} d_n(S).\end{aligned}$$

This implies

$$d_n(S) \geqslant \left(\frac{\rho_0 m^\alpha - \rho_6}{\rho_4 m^{2\alpha}}\right) n^{-\alpha}.$$

Choosing $m = \left[\left(\frac{1+\rho_6}{\rho_0}\right)^{1/\alpha}\right]$ we check $d_n(S) \geqslant \rho_7 n^{-\alpha}$. The remaining implications may be verified similarly.

□

Employing the same arguments as in the proof of the preceding corollary and using Theorem 7 we may also establish the following statement.

COROLLARY 3. – *Let* $s \in \{c, d, e, g\}$, $0 < \alpha < \infty$, *and* $S \in \mathfrak{L}(E, F)$. *If* E *and* F′ *are of type 2, then*

$$s_n(S) \asymp n^{-\alpha} \iff s_n(S') \asymp n^{-\alpha}.$$

Now we turn to operators of $l_{p,u}$-type. Let $s \in \{c, d, e, g\}$ and $0 < p, u \leqslant \infty$. An operator $S \in \mathfrak{L}(E, F)$ is said to be of $l_{p,u}$-type, $S \in \mathfrak{L}^{(s)}_{p,u}(E, F)$, iff $(s_n(S)) \in l_{p,u}$. Put

$$L^{(s)}_{p,u}(S) := \|(s_n(S))\|_{p,u}, \quad S \in \mathfrak{L}^{(s)}_{p,u}(E, F).$$

The class of these operators form a quasinormed operator ideal $[\mathfrak{L}^{(s)}_{p,u}; L^{(s)}_{p,u}]$ for $s \in \{c, d, e\}$ (cf. [23] (14)). The next statement may be easily checked via Theorem 6 and the following inequality of Hardy (cf. [24]). For $x = (\xi_k)$ we put

$$\|x\|_{r,u,v} := \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{1/r-1/u-1/v} \sup_{1 \leq k \leq n} k^{1/v} \xi_k^* \right)^u \right)^{1/u} & \text{if } 0 < u < \infty \\ \sup_n n^{1/r-1/v} \sup_{1 \leq k \leq n} k^{1/v} \xi_k^* & \text{if } u = \infty, \end{cases}$$

where $0 < v < \min(r; u)$. Then Hardy's inequality states that

$$\|x\|_{r,u} \leq \|x\|_{r,u,v} \leq \rho(r,u,v) \|x\|_{r,u} \quad \text{for} \quad x \in l_{r,u}.$$

THEOREM 8. – *Let* $0 < r, u \leq \infty$ *and let* E *and* F′ *be spaces of type 2. Then*

$$\mathcal{L}^{(c)}_{r,u}(\mathrm{E},\mathrm{F}) = \mathcal{L}^{(d)}_{r,u}(\mathrm{E},\mathrm{F}) = \mathcal{L}^{(e)}_{r,u}(\mathrm{E},\mathrm{F}) = \mathcal{L}^{(g)}_{r,u}(\mathrm{E},\mathrm{F}).$$

In the case of spaces $\mathrm{L}_p(\Omega, \Sigma, \mu)$ we even have the following interesting result.

THEOREM 9. – *Let* $0 < r < \infty$, $0 < u \leq \infty$, *and* $1 < p, p < \infty$. *Then the equality* $\mathcal{L}^{(c)}_{r,u}(\mathrm{L}_p, \mathrm{L}_q) = \mathcal{L}^{(d)}_{r,u}(\mathrm{L}_p, \mathrm{L}_q)$ *is valid for all* L_p *and* L_q *spaces if and only if* $1 < q \leq 2 \leq p < \infty$.

Proof. – Observe that L_p and $(\mathrm{L}_q)'$ are of type 2 by Lemma 3 if $1 < q \leq 2 \leq p < \infty$. Hence Theorem 8 implies $\mathcal{L}^{(c)}_{r,u}(\mathrm{L}_p, \mathrm{L}_q) = \mathcal{L}^{(d)}_{r,u}(\mathrm{L}_p, \mathrm{L}_q)$. It remains to show the converse implication. For this purpose we put $\mathrm{L}_p := \mathrm{L}_p[0,1]$ and $\mathrm{L}_q := l_q$. It is well-known that l_2 is a complemented subspace of $\mathrm{L}_p[0,1]$, $1 < p < \infty$, (cf. [17]). Therefore there exists a projection P from $\mathrm{L}_p[0,1]$ onto l_2. Using the operators $\mathrm{J}_n \in \mathcal{L}(l_q^n, l_q)$, $\mathrm{Q}_n \in \mathcal{L}(l_2, l_2^n)$ and $\mathrm{I}_n \in \mathcal{L}(l_2^n, l_q^n)$ in the proof to Supplement 3 we define $\mathrm{S}_n \in \mathcal{L}(\mathrm{L}_p[0,1], l_q)$ by $\mathrm{S}_n := \mathrm{J}_n \mathrm{I}_n \mathrm{Q}_n \mathrm{P}$. Clearly, $c_k(\mathrm{S}_n) \asymp c_k(\mathrm{I}_n)$, $d_k(\mathrm{S}_n) \asymp d_k(\mathrm{I}_n)$. In [8, 9] Gluskin has shown that

$$d_k(\mathrm{I}_n : l_2^n \longrightarrow l_q^n) \asymp \min(1; k^{-1/2} n^{1/q}), \ k = 1, 2, \ldots, n,$$

and by [23 (11)] we have

$$c_k(\mathrm{I}_n : l_2^n \longrightarrow l_q^n) \asymp 1, \quad k = 1, \ldots, \frac{n}{2}.$$

A straightforward computation yields

$$L^{(c)}_{r,u}(S_n) \asymp L^{(c)}_{r,u}(I_n) \asymp n^{1/r}$$

and

$$L^{(d)}_{r,u}(S_n) \asymp L^{(d)}_{r,u}(I_n) \asymp \begin{cases} n^{\max(1/r+1/q-1/2\,;\,2/qr)}, & r \neq 2 \\ n^{1/q}(\log(n+1))^{1/u}, & r = 2. \end{cases}$$

Assume now $\mathfrak{L}^{(c)}_{r,u}(L_p[0,1], l_q) = \mathfrak{L}^{(d)}_{r,u}(L_p[0,1], l_q)$ for $1 < p < \infty$ and $2 < q < \infty$. The closed graph theorem guarantees the existence of constants $\rho_0 > 0$ and $\rho_1 > 0$ such that

$$\rho_0 L^{(c)}_{r,u}(S) \leqslant L^{(d)}_{r,u}(S) \leqslant \rho_1 L^{(c)}_{r,u}(S)$$

for all $S \in \mathfrak{L}^{(c)}_{r,u}(L_p[0,1], l_q)$. Taking $S = S_n$, $n = 1, 2\ldots$, this inequality gives a contradiction. Because $c_n(S) = d_n(S')$ the assumptions $1 < p < 2$ and $1 < q < \infty$ also yield a contradiction. Hence the proof is complete.

□

Finally, we give an additional result concerning approximation numbers. The n^{th} approximation number of an operator $S \in \mathfrak{L}(E, F)$ is defined by

$$a_n(S) := \inf \{\| S - A \| : \text{rank}(A) < n\}.$$

Since $c_n(S) = a_n(S)$ for $S \in \mathfrak{L}(H, F)$ and $d_n(S) = a_n(S)$ for $S \in \mathfrak{L}(E, H)$ (cf. [23] (11)), where H denotes a Hilbert space we obtain from Theorem 8 the following statement

COROLLARY 4. – *Let* $0 < r, u \leqslant \infty$ *and let* E *and* F' *be Banach spaces of type 2. Then*

$$\mathfrak{L}^{(a)}_{r,u}(E, H) = \mathfrak{L}^{(g)}_{r,u}(E, H) \quad \textit{and} \quad \mathfrak{L}^{(a)}_{r,u}(H, F) = \mathfrak{L}^{(g)}_{r,u}(H, F).$$

Classical Bernstein – Jackson inequalities.

The inequalities in this section concerning operators in Banach spaces may be interpreted as counterparts to the classical Bernstein and Jackson inequalities. It turns out that corresponding analogies exist between

(i) entropy moduli (resp. entropy numbers) of operators and the modulus of continuity of functions

(ii) Kolmogorov (resp. Gelfand or approximation) numbers of operators and Bernstein numbers of functions.

Let $L_p^*[0,1]$, $1 \leqslant p < \infty$, and $L_\infty^*[0,1] := C[0,1]$ for $p = \infty$ denote the spaces of p-summable and continuous 1-periodic functions, respectively. If $f \in L_p^*[0,1]$, then the modulus of continuity is defined by

$$\omega^{(p)}(f,\delta) := \sup_{0<|h|\leqslant\delta} \left(\int_0^1 |f(x+h) - f(x)|^p \, dx\right)^{1/p} .$$

The n^{th} Bernstein number $E_n^{(p)}(f)$, $f \in L_p^*[0,1]$, is defined by

$$E_n^{(p)}(f) := \inf \|f - t\|_p , \quad n = 0,1,2,\ldots,$$

where the infimum is taken over all trigonometrical polynomials t with degree $(t) < n$, $n = 1,2,\ldots$, and where $t \equiv 0$ iff degree $(t) < 0$. Obviously, $E_0^{(p)}(f) = \|f\|_p$.

Bernstein inequalities

The Bernstein inequality for functions $f \in L_p^*[0,1]$ says (cf. e.g., [18], [19])

$$\omega^{(p)}\left(f,\frac{1}{n}\right) \leqslant \frac{\rho}{n} \sum_1^n E_k^{(p)}(f) \quad \text{for} \quad n = 1,2,\ldots .$$

Theorem 4 implies, for $s \in \{c,d,a\}$, an analogous inequality for operators:

$$g_n(S) \leqslant 2e_n(S) \leqslant \frac{\rho}{n} \sum_1^n s_k(S) \quad \text{for} \quad n = 1,2,\ldots .$$

Jackson inequalities

The Jackson inequality for functions $f \in L_p^*[0,1]$ says (cf. e.g., [18], [31])

$$E_n^{(p)}(f) \leqslant \rho\, \omega^{(p)}\left(f,\frac{1}{n}\right) \quad \text{for} \quad n = 1,2,\ldots .$$

If S acts between Hilbert spaces, we have an analogous inequality for operators, and $s \in \{c, d, a\}$,

$$s_n(S) \leqslant \rho\, g_n(S) \leqslant 2\, \rho\, e_n(S) \quad \text{for} \quad n = 1, 2, \ldots .$$

Notice that the analogies established above are not only formal. Indeed, if S_f is a convolution operator generated by a function f, then there are relationships between Kolmogorov (resp. Gelfand or approximation) numbers of S_f and the Bernstein numbers of f as well as relationships between the entropy moduli (resp. entropy numbers) of S_f and the modulus of continuity of f. For this purpose let $f \in L_p^*[0,1]$, $1 \leqslant p \leqslant \infty$, then the convolution operator is defined by

$$S_f g := \int_0^1 f(s-t)\, g(t)\, dt.$$

The operator S_f may be considered as a map from $L_{p'}^*[0,1]$ into $C^*[0,1]$ (cf. [32]). By [1] we have the following statement: Let $s \in \{c, d, a\}$ and $f \in L_p^*[0,1]$, $1 \leqslant p \leqslant \infty$. Then for $S_f \in \mathfrak{L}(L_{p'}^*[0,1], C^*[0,1])$ the inequalities

$$s_1(S_f) = \| S_f \| \leqslant \| f \|_p = E_0^{(p)}(f),$$

$$s_{2n}(S_f) \leqslant E_n^{(p)}(f) \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$g_n(S_f) \leqslant 2 e_n(S_f) \leqslant \rho^{(r)} \left(n^{-1/r} \| f \|_p + \omega^{(p)} \left(f, \frac{1}{n} \right) \right)$$

for $0 < r < \infty$, $n = 1, 2, \ldots$, are valid.

3. RIESZ – SCHAUDER THEORY

The results proved in the previous sections allow us to quantify the following qualitative statements already going back to F. Riesz [27] and J. Schauder [28]:

– The sequence of eigenvalues of a compact operator tends to zero.

– An operator is compact if and only if its dual operator is compact.

Since we are also interested in eigenvalues all Banach spaces under considerations are assumed to be complex. So we may establish the following quantification of the Riesz-Schauder-Theory :

(i) Let $S \in \mathcal{L}(E, F)$ be a compact operator. Then

$$\Big(\prod_1^n |\lambda_i(S)|\Big)^{1/n} \leqslant g_{2n}(S) \quad \text{and} \quad \Big(\prod_1^{2n} |\lambda_i(S)|\Big)^{1/2n} \leqslant \rho \, \Big(\prod_1^n c_k(S)\Big)^{1/n}$$

for $n = 1, 2, \ldots,$ (cf. [5], [24]).

(ii) Let $S \in \mathcal{L}(E, F)$ and let E and F' be Banach spaces of type 2. Then

$$g_n(S) \asymp n^{-\alpha} \quad \text{if and only if} \quad g_n(S') \asymp n^{-\alpha}$$

and

$$c_n(S) \asymp n^{-\alpha} \quad \text{if and only if} \quad c_n(S') \asymp n^{-\alpha} .$$

It should be mentioned that in general the statement in (ii) concerning Gelfand numbers is false if E or F' are not of type 2. Indeed, for the diagonal operator $D_\alpha(\xi_n) := (n^{-\alpha} \xi_n)$ we have (cf. [15])

$$c_n(D_\alpha : l_1 \longrightarrow l_2) \asymp n^{-\alpha - 1/2} \quad \text{and} \quad c_n(D'_\alpha : l_2 \longrightarrow l_\infty) \asymp n^{-\alpha} .$$

Consequently, we have in general not the same "degree of compactness" for an operator and its dual operator in terms of Gelfand numbers. Up to this time we have not been able to give a complete answer to the following "duality problem of entropy moduli".

PROBLEM. – *Let* $S \in \mathcal{L}(E, F)$ *and let* E *and* F *be arbitrary Banach spaces. Does*

$$g_n(S) \asymp n^{-\alpha} \quad \text{imply} \quad g_n(S') \asymp n^{-\alpha} ,$$

where $0 < \alpha < \infty$ *and* $n = 1, 2, \ldots$?

We now turn to an interesting example of an integral operator in $C[0, 1]$ generated by a smooth kernel. We show that there exists a smooth kernel such that the distance of the "degree of compactness" in terms of Gelfand numbers and of the "degree of compactness" in terms of entropy moduli of the corresponding integral operator is large. Since the Gelfand and the approximation numbers coincide for operators acting in $C[0,1]$ we have also the same distance between the "degree of approximation" in terms of approximation numbers and the "degree of compactness" in terms of entropy moduli. We prepare two lemmas.

LEMMA 8. – *Let* $m = 1, 2, \dots,$ *and* $S \in \mathcal{L}(l_\infty^m, F)$. *Then*

$$\left(\prod_1^n c_k(S)\right)^{1/n} \leqslant \rho \left(n \log\left(\frac{m}{n} + 1\right)\right)^{1/2} g_n(S)$$

for $n = 1, 2, \dots, m$.

Proof. – From Lemma 3 we get $\tau_2(l_p^m) \leqslant \sqrt{p}$, $2 \leqslant p < \infty$, and $\tau_1(F') = 1$. Thus by Theorem 5 we have

$$\left(\prod_1^n c_k(S: l_p^m \longrightarrow F)\right)^{1/n} \leqslant \rho_0 \sqrt{p}\, n^{1/2} g_n(S: l_p^m \longrightarrow F).$$

A result of C. Schütt [29] yields

$$g_n(I_m : l_p^m \longrightarrow l_\infty^m) \leqslant \rho_1 \left(\frac{\log\left(\frac{m}{n} + 1\right)}{n}\right)^{1/p}.$$

Combining the preceding estimates with

$$c_k(S : l_\infty^m \longrightarrow F) \leqslant m^{1/p} c_k(S : l_p^m \longrightarrow F)$$

we get

$$\left(\prod_1^n c_k(S : l_\infty^m \longrightarrow F)\right)^{1/n}$$

$$\leqslant \rho_2 \sqrt{p} \left(\frac{m}{n}\right)^{1/p} \log^{1/p}\left(\frac{m}{n} + 1\right) n^{1/2} \cdot g_n(S : l_\infty^m \longrightarrow F).$$

Putting $p = \log\left(\frac{m}{n} + 1\right)$ we obtain the desired estimate.

□

LEMMA 9. – *For each* $m = 1, 2, \dots,$ *there exists a* $m \times m$ *matrix* $A = (\epsilon_{ij})$, $\epsilon_{ij} = \pm 1$, *such that simultaneously the following estimates are satisfied:*

$$\rho_0\, m \leqslant a_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m) \leqslant m$$

and

$$\rho_1 \frac{m}{n^{1/2} \log^{1/2}\left(\frac{m}{n} + 1\right)} \leqslant g_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m)$$

$$\leqslant 2e_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m) \leqslant \rho_2 \frac{m}{n^{1/2}} \log\left(\frac{m}{n} + 1\right)$$

for $n = 1, 2, \ldots, \left[\rho_3 \frac{m}{\log m}\right]$, *where* ρ_0, ρ_1, ρ_2, ρ_3 *are positive constants.*

Proof. – The first assertion is a recent result of S. Heinrich [11]. The estimate from below of the second assertion is a consequence of Lemma 8 with $\mathrm{F} = l_\infty^m$ and

$$c_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m) = a_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m).$$

The estimate from above may be checked via

$$e_n(\mathrm{A} : l_\infty^m \longrightarrow l_\infty^m) \leqslant m^{1/2}\, e_n(\mathrm{A} : l_2^m \longrightarrow l_\infty^m)$$

and Lemma 5.

□

Let $\alpha = 0, 1, 2, \ldots$, we define by $\mathrm{C}^\alpha[0,1]^2$ the space of all kernels $\mathrm{K}(s,t)$ having continuous derivatives up to the order α in both variables. By

$$x(t) \longrightarrow y(s) = \int_0^1 \mathrm{K}(s,t)\, x(t)\, dt$$

we define an operator $\mathrm{S_K} \in \mathcal{L}(\mathrm{C}[0,1], \mathrm{C}[0,1])$ admitting a factorization

$$\begin{array}{ccc} \mathrm{C}[0,1] & \xrightarrow{\ \mathrm{S_K}\ } & \mathrm{C}[0,1] \\ \mathrm{J} \downarrow & & \uparrow \mathrm{J} \\ \mathrm{L}_1[0,1] & \xrightarrow{\ \mathrm{S_K^0}\ } & \mathrm{C}^\alpha[0,1] \end{array}$$

with embedding operators J and a bounded operator $\mathrm{S_K^0}$. We are prepared to establish the following interesting statement.

THEOREM 10. – (i) *Let* $K \in C^{\alpha}[0,1]^2$, $\alpha = 1, 2, \dots$. *Then for* $S_K \in \mathcal{L}(C[0,1], C[0,1])$ *it holds*

$$a_n(S_K) \leqslant \rho_0\, n^{-\alpha} \quad \text{and} \quad g_n(S_K) \leqslant \rho_1\, n^{-\alpha-1/2}$$

for $n = 1, 2, \dots$.

(ii) *For any* $\epsilon > 0$ *there exist a* $K \in C^{\alpha}[0,1]^2$ *such that*

$$\rho_0(\epsilon)\, n^{-\alpha-\epsilon} \leqslant a_n(S_K) \leqslant \rho_1\, n^{-\alpha}$$

and

$$\rho_2(\epsilon)\, n^{-\alpha-1/2-\epsilon} \leqslant g_n(S_K) \leqslant \rho_3\, n^{-\alpha-1/2}$$

are valid for $n = 1, 2, \dots$.

Proof. – The first assertion of (i) follows from the factorization of S_K and $a_n(J : C^{\alpha}[0,1] \longrightarrow C[0,1]) \lesssim n^{-\alpha}$. On the other hand from the factorization

$$\begin{array}{ccc}
C[0,1] & \xrightarrow{\quad S_K \quad} & C[0,1] \\
J \downarrow & & \uparrow \\
L_2[0,1] & & J \\
J \downarrow & & | \\
L_1[0,1] & \xrightarrow{\quad S_K^0 \quad} & C^{\alpha}[0,1]
\end{array}$$

we conclude via Theorem 3 and a spline technique argument similar to those in [4] that $g_n(JS_K^0 J : L_2[0,1] \longrightarrow C[0,1]) \lesssim n^{-\alpha-1/2}$. This implies the second assertion of (i).

Finally, we only sketch (ii). For each $k = 0, 1, 2, \dots$, we choose $2^k \times 2^k$ matrices $A_{2^k} = (\epsilon_{ij}^{(k)})$, $\epsilon_{ij}^{(k)} = \pm 1$, satisfying the assertion in Lemma 9. By

$$A_{2^k} = (\epsilon_{ij}^{(k)}) \longrightarrow K_{2^k}(s,t) = \sum_{i,j=1}^{2^k} \epsilon_{ij}^{(k)}\, \varphi_i^{(k)}(s)\, \varphi_j^{(k)}(t),$$

we assign every such matrix a kernel where $\varphi_i^{(k)}$ are suitable infinitely differentiable non-negative functions with

$\operatorname{supp}(\varphi_i^{(k)}) \subset \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right]$. The desired kernel is a blockwise sum of multiples of the $K'_{2^n}s$. We omit the precise construction and refer to Heinrich [11].

□

This example shows that the difference of the order of convergence between approximation (resp. Gelfand) numbers and entropy moduli of integral operators in $C[0,1]$ generated by smooth kernels is 1/2. At the end of the paper we illustrate the usefulness of the inequality between eigenvalues and entropy moduli. We consider two examples of eigenvalue problems already known in the literature (cf. [2], [3], [16]). They are immediate consequences of Theorem 1 and 2 and of $|\lambda_n(S)| \leqslant g_{2n}(S)$.

Operators of Hille-Tamarkin type

An operator $S(\xi_i) := (\delta_{ij}\xi_j)$ with the condition

$$\sum_j \Big(\sum_i |\delta_{ij}|^p\Big)^{q/p} < \infty, \quad 1 \leqslant p, q < \infty,$$

is called an operator of Hille-Tamarkin type. We may establish the following statement: Let

$$1 \leqslant p, q < \infty, \qquad \frac{1}{q}+\frac{1}{p} > 1, \qquad \text{and} \qquad \frac{1}{s} = \frac{1}{p}+\frac{1}{q} - \max\left(\frac{1}{2};\frac{1}{p}\right).$$

Then $S \in \mathcal{L}(l_p, l_p)$ and

$$(g_n(S)) \in l_{s,q} \qquad \text{and} \qquad (\lambda_n(S)) \in l_{s,q}.$$

The result is optimal in the following sense. Under the above conditions for $q_0 < q$, there exists an operator S_0 such that $(g_n(S_0)) \in l_{s,q}$ and $(\lambda_n(S)) \notin l_{s,q_0}$.

Nuclear operators

An operator $S \in \mathcal{L}(E, F)$ is said to be *r*-nuclear (in the sense of Grothendieck), $0 < r < 1$, if there are sequences $(a_i) \subset U_{E'}$, $(y_i) \subset U_F$ and $(\delta_i) \in l_r$ such that

$$Sx = \sum_1^\infty \delta_i \langle x, a_i\rangle y_i \quad \text{for} \quad x \in E.$$

We may establish the following statement.

If $S \in \mathcal{L}(L_p, L_p)$, $1 < p < \infty$, is an r-nuclear operator, $0 < r < 1$, then $(g_n(S)) \in l_{s,r}$ and $(\lambda_n(S)) \in l_{s,r}$, where $\frac{1}{s} = \frac{1}{r} - \left|\frac{1}{2} - \frac{1}{p}\right|$. The result is optimal. Under the above conditions, for $r_0 < r$, there exists an r-nuclear operator $S \in \mathcal{L}(L_p, L_p)$ such that $(g_n(S)) \in l_{s,r}$ and $(\lambda_n(S)) \notin l_{s,r_0}$.

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