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Inequalities of Hermite-Hadamard-like Type for the Functions whose Second Derivatives in Absolute Value are Convex and Concave

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Abstract

In this article, new estimates on generalization of Hermite-Hadamard-like type inequalities for functions whose second derivatives in absolute value at certain powers are convex and concave are established.

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1 Introduction

Recall that a function $f : I \subseteq R \to R$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$, and f is said to be concave on I if the inequality (1) holds in reversed direction.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f : I \subseteq R \to R$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then the following double inequalities hold:

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{f(a)+f(b)}{2}. \tag{2}$$

Both inequalities hold in the reversed direction if f is concave.

It was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality (2) was nowhere mentioned in the mathematical literature untill 1893. In [1], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (2) was proved by Hadamard in 1893. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis. That is why, the inequality (2) was known as Hermite-Hadamard inequality. We note that Hermite-Hadamard's inequality may be regarded as a refinements of the concept of convexity and it follows easily from Jensen's inequality. This inequality (2) has been received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [2]-[17].

In recent paper[14], Tseng et. al established the following result which gives a refinement of (2):

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\le \frac{f\big(\frac{3a+b}{4}\big)+f\big(\frac{a+3b}{4}\big)}{2} \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\le \frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big] \le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3}$$

where $f : [a, b] \to R$ is a convex function.

In[9], Latif established some new Hadamard-type inequalities for whose derivatives in absolute values are convex:

Theorem 1.1. *Let $f : I \subseteq R \to R$ be a differentiable function define on the interior I^0 of an interval I in R such that $f' \in L([a, b])$, where $a, b \in I^0$ with $a < b$. If $\mid f' \mid^q$ is convex on $[a, b]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\begin{aligned} &\Big|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\mathrm{d}u\Big| \\ &\le \Big(\frac{1}{4}\Big)\Big[\frac{(x-a)^2}{b-a}\Big\{\Big(\frac{5\mid f'(x)\mid^q+\mid f'(a)\mid^q}{6}\Big)^{\frac{1}{q}} \\ &\qquad +\Big(\frac{\mid f'(x)\mid^q+5\mid f'(a)\mid^q}{6}\Big)^{\frac{1}{q}}\Big\} \\ &\quad +\frac{(b-x)^2}{b-a}\Big\{\Big(\frac{5\mid f'(x)\mid^q+\mid f'(b)\mid^q}{6}\Big)^{\frac{1}{q}} \\ &\qquad +\Big(\frac{\mid f'(x)\mid^q+5\mid f'(b)\mid^q}{6}\Big)^{\frac{1}{q}}\Big\}\Big] \end{aligned} \tag{4}$$

for all $x \in [a, b]$.

Here, if we choose $x = \frac{a+b}{2}$ in (4), we have some Hermite-Hadamard inequalities which gives an estimate between $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ and $\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]$ for functions whose derivatives in absolute value are convex.

In this article, a new general identity for continuously twice differentiable functions is established. By making use of this equality, author has obtained new estimates on generalization of Hermite-Hadamard-like type inequalities for functions whose second derivatives in absolute value at certain powers are convex and concave.

2 Main results

In this section, for the simplicity of the notation, let

$$\begin{aligned} & I_f(x,a,b) \\ & \equiv \frac{2}{b-a}\int_a^b f(u)\mathrm{d}u - f(x) - \frac{(b-x)f(b)+(x-a)f(a)}{b-a} \\ & \quad + \frac{(x-a)^2(f'(x)-f'(a))+(b-x)^2(f'(b)-f'(x))}{4(b-a)} \end{aligned}$$

for any $x \in [a,b]$.

In order to prove our main results, we need the following lemma:

Lemma 1. *Let $f : I \subseteq R \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. Then, for any $x \in [a,b]$ we have the following identity:*

$$\begin{aligned} & I_f(x,a,b) \\ & = \frac{(x-a)^3}{8(b-a)}\int_0^1 t^2\Big\{f''\big(\frac{1+t}{2}x + \frac{1-t}{2}a\big) \\ & \qquad + f''\big(\frac{1-t}{2}x + \frac{1+t}{2}a\big)\Big\}\mathrm{d}t \\ & \quad + \frac{(b-x)^3}{8(b-a)}\int_0^1 t^2\Big\{f''\big(\frac{1+t}{2}x + \frac{1-t}{2}b\big) \\ & \qquad + f''\big(\frac{1-t}{2}x + \frac{1+t}{2}b\big)\Big\}\mathrm{d}t. \end{aligned}$$

Proof. By integration by parts and making use of the substitution $u =$

$\frac{1+t}{2}x + \frac{1-t}{2}a$, we can state

$$\begin{aligned}(i)\quad &\frac{(x-a)^3}{8(b-a)}\int_0^1 t^2 f''\Big(\frac{1+t}{2}x + \frac{1-t}{2}a\Big)\mathrm{d}t \\ &= \frac{(x-a)^2}{4(b-a)}f'(x) - \frac{x-a}{b-a}f(x) + \frac{2}{b-a}\int_{\frac{a+x}{2}}^{x} f(u)\mathrm{d}u. \qquad (5)\end{aligned}$$

By similar way we get

$$\begin{aligned}(ii)\quad &\frac{(x-a)^3}{8(b-a)}\int_0^1 t^2 f''\Big(\frac{1-t}{2}x + \frac{1+t}{2}a\Big)\mathrm{d}t \\ &= -\frac{(x-a)^2}{4(b-a)}f'(a) - \frac{x-a}{b-a}f(a) + \frac{2}{b-a}\int_{a}^{\frac{a+x}{2}} f(u)\mathrm{d}u, \qquad (6)\\ (iii)\quad &\frac{(b-x)^3}{8(b-a)}\int_0^1 t^2 f''\Big(\frac{1+t}{2}x + \frac{1-t}{2}b\Big)\mathrm{d}t \\ &= -\frac{(b-x)^2}{4(b-a)}f'(x) - \frac{b-x}{b-a}f(x) + \frac{2}{b-a}\int_{x}^{\frac{x+b}{2}} f(u)\mathrm{d}u, \qquad (7)\\ (iv)\quad &\frac{(b-x)^3}{8(b-a)}\int_0^1 t^2 f''\Big(\frac{1-t}{2}x + \frac{1+t}{2}b\Big)\mathrm{d}t \\ &= \frac{(b-x)^2}{4(b-a)}f'(b) - \frac{b-x}{b-a}f(b) + \frac{2}{b-a}\int_{\frac{x+b}{2}}^{b} f(u)\mathrm{d}u. \qquad (8)\end{aligned}$$

By the equalities (5)-(8), we get the desired result.

Theorem 2.1. *Let $f : I \subset [0, \infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a, b]$, where $a, b \in I^0$ with $a < b$. If $| f'' |$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}\Big|I_f(x,a,b)\Big| \le\ &\frac{(x-a)^3}{24(b-a)}\Big\{\big|f''(a)\big| + \big|f''(x)\big|\Big\} \\ &+ \frac{(b-x)^3}{24(b-a)}\Big\{\big|f''(x)\big| + \big|f''(b)\big|\Big\}. \qquad (9)\end{aligned}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{(x-a)^3}{8(b-a)}\int_0^1 t^2\Big\{\Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}a)\Big| \\
&\qquad +\Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}a)\Big|\Big\}\mathrm{d}t \\
&\quad +\frac{(b-x)^3}{8(b-a)}\int_0^1 t^2\Big\{\Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}b)\Big| \\
&\qquad +\Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}b)\Big|\Big\}\mathrm{d}t.
\end{aligned}$$

Since $\mid f'' \mid$ is convex on $[a,b]$, we have

$$\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{(x-a)^3}{8(b-a)}\Big[\int_0^1 t^2\Big\{\frac{1+t}{2}\Big|f''(x)\Big|+\frac{1-t}{2}\Big|f''(a)\Big|\Big\}\mathrm{d}t \\
&\qquad +\int_0^1 t^2\Big\{\frac{1-t}{2}\Big|f''(x)\Big|+\frac{1+t}{2}\Big|f''(a)\Big|\Big\}\mathrm{d}t\Big] \\
&\quad +\frac{(b-x)^3}{8(b-a)}\Big[\int_0^1 t^2\Big\{\frac{1+t}{2}\Big|f''(x)\Big|+\frac{1-t}{2}\Big|f''(b)\Big|\Big\}\mathrm{d}t \\
&\qquad +\int_0^1 t^2\Big\{\frac{1-t}{2}\Big|f''(x)\Big|+\frac{1+t}{2}\Big|f''(b)\Big|\Big\}\mathrm{d}t\Big] \\
&= \frac{(x-a)^3}{8(b-a)}\Big[\Big\{\frac{7}{24}\Big|f''(x)\Big|+\frac{1}{24}\Big|f''(a)\Big|\Big\} \\
&\qquad +\Big\{\frac{1}{24}\Big|f''(x)\Big|+\frac{7}{24}\Big|f''(a)\Big|\Big\}\Big] \\
&\quad +\frac{(b-x)^3}{8(b-a)}\Big[\Big\{\frac{7}{24}\Big|f''(x)\Big|+\frac{1}{24}\Big|f''(b)\Big|\Big\} \\
&\qquad +\Big\{\frac{1}{24}\Big|f''(x)\Big|+\frac{7}{24}\Big|f''(b)\Big|\Big\}\Big],
\end{aligned}$$

which completes the proof.

Corollary 2.1. *Under the conditions of Theorem 2.1, if we choose* $x=\frac{a+b}{2}$ *in the inequality (9), by the inequalities (2) and (3) we have*

$$\begin{aligned}
&\Big|I_f(\frac{a+b}{2},a,b)\Big| \\
&\leq \frac{(b-a)^2}{192}\Big\{\big|f''(a)\big|+2\big|f''(\frac{a+b}{2})\big|+\big|f''(b)\big|\Big\} \\
&\leq \frac{(b-a)^2}{96}\Big\{\big|f''(a)\big|+\big|f''(b)\big|\Big\}.
\end{aligned}$$

Remark 2.1. By letting $x=a$ (or $x=b$) in Theorem 2.1, we have:

$$\begin{aligned}&\Big|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u-\frac{f(a)+f(b)}{2}+\frac{(b-a)}{8}\big(f'(b)-f'(a)\big)\Big|\\&\leq\frac{(b-a)^2}{48}\Big\{\big|f''(a)\big|+\big|f''(b)\big|\Big\}.\end{aligned}$$

Theorem 2.2. *Let $f:I\subset[0,\infty)\to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f^{''}\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|\ f''\ |^q$ is convex on $[a,b]$ for some fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$ the following inequality holds:*

$$\begin{aligned}&\Big|I_f(x,a,b)\Big|\\&\leq\frac{1}{8}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^3}{b-a}\Big\{\Big(\frac{\big|f''(a)\big|^q+3\big|f''(x)\big|^q}{4}\Big)^{\frac{1}{q}}\\&\qquad+\Big(\frac{3\big|f''(a)\big|^q+\big|f''(x)\big|^q}{4}\Big)^{\frac{1}{q}}\Big\}\\&\quad+\frac{(b-x)^3}{b-a}\Big\{\Big(\frac{\big|f''(x)\big|^q+3\big|f''(b)\big|^q}{4}\Big)^{\frac{1}{q}}\\&\qquad+\Big(\frac{3\big|f''(x)\big|^q+\big|f''(b)\big|^q}{4}\Big)^{\frac{1}{q}}\Big\}\Big].\end{aligned}\tag{10}$$

Proof. From Lemma 1 and using the well-known Hölder inequality, we have

$$\begin{aligned}&\Big|I_f(x,a,b)\Big|\\&\leq\frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''\big(\frac{1+t}{2}x+\frac{1-t}{2}a\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&+\frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}a\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&+\frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''\big(\frac{1+t}{2}x+\frac{1-t}{2}b\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&+\frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}b\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&=\frac{1}{8}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big[\Big\{\frac{(x-a)^3}{(b-a)}\Big(\int_0^1\Big|f''\big(\frac{1+t}{2}x+\frac{1-t}{2}a\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&+\frac{(x-a)^3}{(b-a)}\Big(\int_0^1\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}a\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big\}\\&+\Big\{\frac{(b-x)^3}{(b-a)}\Big(\int_0^1\Big|f''\big(\frac{1+t}{2}x+\frac{1-t}{2}b\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&+\frac{(b-x)^3}{(b-a)}\Big(\int_0^1\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}b\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big\}\Big].\end{aligned}\tag{11}$$

Since $\mid f'' \mid^q$ is convex on $[a,b]$, by using (2) we have

$$(i) \int_0^1 \Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}a)\Big|^q \mathrm{d}t \le \frac{\big|f''(a)\big|^q + 3\big|f''(x)\big|^q}{4}, \tag{12}$$

and

$$(ii) \int_0^1 \Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}a)\Big|^q \mathrm{d}t \le \frac{3\big|f''(a)\big|^q + \big|f''(x)\big|^q}{4}, \tag{13}$$

$$(iii) \int_0^1 \Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}b)\Big|^q \mathrm{d}t \le \frac{3\big|f''(x)\big|^q + \big|f''(b)\big|^q}{4}, \tag{14}$$

$$(iv) \int_0^1 \Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}b)\Big|^q \mathrm{d}t \le \frac{\big|f''(x)\big|^q + 3\big|f''(b)\big|^q}{4}. \tag{15}$$

By substituting (12)-(15) in (11), we get the desired result.

Corollary 2.2. *Under the conditions of Theorem 2.2, if we choose $x = \frac{a+b}{2}$ in the inequality (10), by the inequalities (2) and (3) we have*

$$\begin{aligned}
&\Big|\frac{2}{b-a}\int_a^b f(u)\mathrm{d}u - f(\frac{a+b}{2}) - \frac{f(a)+f(b)}{2} + \frac{(b-a)}{16}\big\{f'(b)-f'(a)\big\}\Big| \\
&\le \Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}} \frac{(b-a)^2}{64} \\
&\quad \times \Big[\Big(\frac{\big|f''(a)\big|^q + 3\big|f''(\frac{a+b}{2})\big|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3\big|f''(a)\big|^q + \big|f''(\frac{a+b}{2})\big|^q}{4}\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{\big|f''(\frac{a+b}{2})\big|^q + 3\big|f''(b)\big|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3\big|f''(\frac{a+b}{2})\big|^q + \big|f''(b)\big|^q}{4}\Big)^{\frac{1}{q}}\Big] \\
&\le \Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}} \Big(\frac{1}{8}\Big)^{\frac{1}{q}} \Big(1 + 3^{\frac{1}{q}} + 5^{\frac{1}{q}} + 7^{\frac{1}{q}}\Big) \frac{(b-a)^2}{64}\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\}.
\end{aligned}$$

Proof. By using the fact that

$$\Sigma_{k=1}^n (u_k + v_k)^s \le \Sigma_{k=1}^n (u_k)^s + \Sigma_{k=1}^n (v_k)^s \tag{16}$$

for $u_k, v_k \ge 0, 1 \le k \le n$ and $0 \le s < 1$, we get

$$\begin{aligned}
&\Big(\big|f''(a)\big|^q + 3\big|f''(\frac{a+b}{2})\big|^q\Big)^{\frac{1}{q}} + \Big(3\big|f''(a)\big|^q + \big|f''(\frac{a+b}{2})\big|^q\Big)^{\frac{1}{q}} \\
&+ \Big(\big|f''(\frac{a+b}{2})\big|^q + 3\big|f''(b)\big|^q\Big)^{\frac{1}{q}} + \Big(3\big|f''(\frac{a+b}{2})\big|^q + \big|f''(b)\big|^q\Big)^{\frac{1}{q}} \\
&\le \Big(\frac{1}{2}\Big)^{\frac{1}{q}} \Big(1 + 3^{\frac{1}{q}} + 5^{\frac{1}{q}} + 7^{\frac{1}{q}}\Big)\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\},
\end{aligned}$$

which implies that the second inequality holds.

Theorem 2.3. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a,b \in I^0$ with $a < b$. If $| f'' |^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, then for any $x \in [a,b]$ the following inequality holds:*

$$
\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{1}{24}\Big[\frac{(x-a)^3}{b-a}\Big\{\Big(\frac{\big|f''(a)\big|^q + 7\big|f''(x)\big|^q}{8}\Big)^{\frac{1}{q}} + \Big(\frac{7\big|f''(a)\big|^q + \big|f''(x)\big|^q}{8}\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \frac{(b-x)^3}{b-a}\Big\{\Big(\frac{7\big|f''(x)\big|^q + \big|f''(b)\big|^q}{8}\Big)^{\frac{1}{q}} + \Big(\frac{\big|f''(x)\big|^q + 7\big|f''(b)\big|^q}{8}\Big)^{\frac{1}{q}}\Big\}\Big].
\end{aligned}
$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$
\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^2 \mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^2\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^2 \mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^2\Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^2 \mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^2\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^2 \mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^2\Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&= \frac{1}{8}\Big(\frac{1}{3}\Big)^{1-\frac{1}{q}}\Big[\frac{(x-a)^3}{(b-a)}\Big\{\Big(\int_0^1 t^2\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \Big(\int_0^1 t^2\Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \frac{(b-x)^3}{(b-a)}\Big\{\Big(\int_0^1 t^2\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \Big(\int_0^1 t^2\Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big\}\Big]. \qquad (17)
\end{aligned}
$$

Since $| f'' |^q$ is convex on $[a,b]$, by using (2) we have

$$
(i) \int_0^1 t^2\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}a)\Big|^q \mathrm{d}t \leq \frac{\big|f''(a)\big|^q + 7\big|f''(x)\big|^q}{24}, \qquad (18)
$$

and

$$(ii)\int_0^1 t^2\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}a\big)\Big|^q \mathrm{d}t \le \frac{7\big|f''(a)\big|^q+\big|f''(x)\big|^q}{24}, \qquad (19)$$

$$(iii)\int_0^1 t^2\Big|f''\big(\frac{1+t}{2}x+\frac{1-t}{2}b\big)\Big|^q \mathrm{d}t \le \frac{7\big|f''(x)\big|^q+\big|f''(b)\big|^q}{24}, \qquad (20)$$

$$(iv)\int_0^1 t^2\Big|f''\big(\frac{1-t}{2}x+\frac{1+t}{2}b\big)\Big|^q \mathrm{d}t \le \frac{\big|f''(x)\big|^q+7\big|f''(b)\big|^q}{24}. \qquad (21)$$

By substituting (18)-(21) in (17), we get the desired result.

Corollary 2.3. *Under the conditions of Theorem 2.3, if we choose* $x=\frac{a+b}{2}$*, by the inequalities (2),(3) and (16) we have*

$$\begin{aligned}
&\Big|\frac{2}{b-a}\int_a^b f(u)\mathrm{d}u - f\big(\frac{a+b}{2}\big) - \frac{f(a)+f(b)}{2} \\
&\qquad + \frac{(b-a)}{16}\{f'(b)-f'(a)\}\Big| \\
&\le \Big(\frac{1}{8}\Big)^{\frac{1}{q}}\frac{(b-a)^2}{192} \\
&\quad \times \Big[\Big(\big|f''(a)\big|^q+7\big|f''\big(\frac{a+b}{2}\big)\big|^q\Big)^{\frac{1}{q}} + \Big(7\big|f''(a)\big|^q+\big|f''\big(\frac{a+b}{2}\big)\big|^q\Big)^{\frac{1}{q}} \\
&\quad + \Big(7\big|f''\big(\frac{a+b}{2}\big)\big|^q+\big|f''(b)\big|^q\Big)^{\frac{1}{q}} + \Big(\big|f''\big(\frac{a+b}{2}\big)\big|^q+7\big|f''(b)\big|^q\Big)^{\frac{1}{q}}\Big] \\
&\le \Big(\frac{1}{16}\Big)^{\frac{1}{q}}\Big(1+7^{\frac{1}{q}}+9^{\frac{1}{q}}+15^{\frac{1}{q}}\Big)\frac{(b-a)^2}{192}\Big\{\big|f''(a)\big|+\big|f''(b)\big|\Big\}.
\end{aligned}$$

Theorem 2.4. *Let* $f : I \subset [0,\infty) \to R$ *be a twice differentiable function on the interior* I^0 *of an interval* I *such that* $f'' \in L_1[a,b]$*, where* $a,b \in I^0$ *with* $a<b$*. If* $|\, f''\,|^q$ *is concave on* $[a,b]$ *for some fixed* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then for any* $x\in[a,b]$ *the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\le \frac{1}{8}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^3}{b-a}\Big\{\Big|f''\big(\frac{a+3x}{4}\big)\Big|+\Big|f''\big(\frac{3a+x}{4}\big)\Big|\Big\} \\
&\quad + \frac{(b-x)^3}{b-a}\Big\{\Big|f''\big(\frac{x+3b}{4}\big)\Big|+\Big|f''\big(\frac{3x+b}{4}\big)\Big|\Big\}\Big].
\end{aligned}$$

Proof. From Lemma 1 and using the well-known Hölder inequality for $q>1$

with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\le \frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(x-a)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^3}{8(b-a)}\Big(\int_0^1 t^{2p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&= \frac{1}{8}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big[\frac{(x-a)^3}{(b-a)}\Big\{\Big(\int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}a)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big\} \\
&\quad + \Big\{\frac{(b-x)^3}{(b-a)}\Big\{\Big(\int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}b)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big\}\Big].
\end{aligned} \tag{22}
$$

Since $\mid f'' \mid^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to obtain:

$$
(i) \int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}a)\Big|^q \mathrm{d}t \le \Big|f''(\frac{a+3x}{4})\Big|^q, \tag{23}
$$

and

$$
(ii) \int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}a)\Big|^q \mathrm{d}t \le \Big|f''(\frac{3a+x}{4})\Big|^q, \tag{24}
$$

$$
(iii) \int_0^1 \Big|f''(\frac{1+t}{2}x+\frac{1-t}{2}b)\Big|^q \mathrm{d}t \le \Big|f''(\frac{3x+b}{4})\Big|^q, \tag{25}
$$

$$
(iv) \int_0^1 \Big|f''(\frac{1-t}{2}x+\frac{1+t}{2}b)\Big|^q \mathrm{d}t \le \Big|f''(\frac{x+3b}{4})\Big|^q. \tag{26}
$$

By substituting (23)-(26) in (22), we get the desired result.

Corollary 2.4. *Under the conditions of Theorem 2.4, if we choose* $x=\frac{a+b}{2}$*,* *then we have*

$$\begin{aligned}
&\Big|\frac{2}{b-a}\int_a^b f(u)\mathrm{d}u - f(\frac{a+b}{2}) - \frac{f(a)+f(b)}{2} \\
&\qquad + \frac{(b-a)}{16}\{f'(b)-f'(a)\}\Big| \\
&\leq \Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\frac{(b-a)^2}{64}\Big[\Big|f''(\frac{5a+3b}{8})\Big| + \Big|f''(\frac{7a+b}{8})\Big|\Big\} \\
&\quad + \Big|f''(\frac{a+7b}{8})\Big| + \Big|f''(\frac{3a+6b}{8})\Big|\Big].
\end{aligned}$$

Theorem 2.5. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on the interior I^0 of an interval I such that $f'' \in L_1[a,b]$, where $a,b \in I^0$ with $a < b$. If $| f'' |^q$ is concave on $[a,b]$ for some fixed $q \geq 1$, then for any $x \in [a,b]$ the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{1}{24}\Big[\frac{(x-a)^3}{b-a}\Big\{\Big|f''(\frac{a+7x}{8})\Big| + \Big|f''(\frac{7a+x}{4})\Big|\Big\} \\
&\quad + \frac{(b-x)^3}{b-a}\Big\{\Big|f''(\frac{7x+b}{8})\Big| + \Big|f''(\frac{x+7b}{4})\Big|\Big\}\Big].
\end{aligned} \tag{27}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\Big|I_f(x,a,b)\Big| \\
&\leq \frac{(x-a)^3}{8(b-a)}\int_0^1 t^2\Big\{\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}a)\Big| \\
&\qquad + \Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}a)\Big|\Big\}\mathrm{d}t \\
&\quad + \frac{(b-x)^3}{8(b-a)}\int_0^1 t^2\Big\{\Big|f''(\frac{1+t}{2}x + \frac{1-t}{2}b)\Big| \\
&\qquad + \Big|f''(\frac{1-t}{2}x + \frac{1+t}{2}b)\Big|\Big\}\mathrm{d}t.
\end{aligned} \tag{28}$$

Since $| f'' |$ is concave on $[a,b]$, we have

$$\begin{aligned}
&(i)\int_0^1 t^2\Big\{\frac{1+t}{2}\Big|f''(x)\Big| + \frac{1-t}{2}\Big|f''(a)\Big|\Big\}\mathrm{d}t \\
&\leq \Big(\int_0^1 t^2\mathrm{d}t\Big)\Big|f''\Big(\frac{\int_0^1 t^2\big(\frac{1+t}{2}x + \frac{1-t}{2}a\big)\mathrm{d}t}{\int_0^1 t^2\mathrm{d}t}\Big)\Big| \\
&= \frac{1}{3}\Big|f''(\frac{a+7x}{8})\Big|,
\end{aligned} \tag{29}$$

$$(ii) \int_0^1 t^2\Big\{\frac{1-t}{2}\Big|f''(x)\Big| + \frac{1+t}{2}\Big|f''(a)\Big|\Big\}\mathrm{d}t \le \frac{1}{3}\Big|f''(\frac{7a+x}{8})\Big|, \tag{30}$$

$$(iii) \int_0^1 t^2\Big\{\frac{1+t}{2}\Big|f''(x)\Big| + \frac{1-t}{2}\Big|f''(b)\Big|\Big\}\mathrm{d}t \le \frac{1}{3}\Big|f''(\frac{7x+b}{8})\Big|, \tag{31}$$

$$(iv) \int_0^1 t^2\Big\{\frac{1-t}{2}\Big|f''(x)\Big| + \frac{1+t}{2}\Big|f''(b)\Big|\Big\}\mathrm{d}t \le \frac{1}{3}\Big|f''(\frac{x+7b}{8})\Big|. \tag{32}$$

By substituting (29)-(32) in (28), we get the desired result.

Corollary 2.5. *Under the conditions of Theorem 2.5, if we choose* $x = \frac{a+b}{2}$, *then we have*

$$\begin{aligned}
&\Big|\frac{2}{b-a}\int_a^b f(u)\mathrm{d}u - f(\frac{a+b}{2}) - \frac{f(a)+f(b)}{2} \\
&\qquad + \frac{(b-a)}{16}\big\{f'(b)-f'(a)\big\}\Big| \\
&\le \frac{(b-a)^2}{192}\Big[\Big|f''(\frac{9a+7b}{16})\Big| + \Big|f''(\frac{15a+b}{16})\Big| \\
&\qquad + \Big|f''(\frac{7a+9b}{16})\Big| + \Big|f''(\frac{a+15b}{16})\Big|\Big].
\end{aligned}$$

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