

**INEQUALITIES OF HERMITE-HADAMARD'S TYPE FOR  
FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE  
 $m$ -CONVEX**

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ABSTRACT. In this paper, we establish several inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are  $m$ -convex. Some applications for special means of real numbers are also provided.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality(see, [8]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ .

In [4], G. Toader considered the class of  $m$ -convex functions: another intermediate between the usual convexity and starshaped convexity.

**Definition 1.** *The function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $m$ -convex, where  $m \in [0, 1]$ , if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ . We say that  $f$  is  $m$ -concave if  $(-f)$  is  $m$ -convex.

Obviously, for  $m = 1$  Definition 1 recaptures the concept of standard convex functions on  $[a, b]$ , and for  $m = 0$  the concept starshaped functions.

For recent results and generalizations concerning convex functions, see the references of [1],[2],[3] and [5].

In [6], U.S. Kırmacı gave the following lemma. Also, in [7], U.S. Kırmacı and M.E. Özdemir obtained the following inequality for differentiable mappings which are connected with Hermite-Hadamard's inequality and they used this lemma to prove it.

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**Lemma 1.** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  ( $I^\circ$  is the interior of  $I$ ) with  $a < b$ . If  $f' \in L([a, b])$ , then we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a) \left[ \int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned}$$

**Theorem 1.** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$  and  $p > 1$ . If the mapping  $|f'|^p$  is convex on  $[a, b]$ , then

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(3^{1-\frac{1}{q}})}{8} (b-a) (|f'(a)| + |f'(b)|).$$

In this article, using functions whose derivatives absolute values are  $m$ -convex, we obtained new inequalities related to the left side of Hermite-Hadamard inequality. Finally, we gave some applications for special means of real numbers.

## 2. MAIN RESULTS

We start with the following theorem:

**Theorem 2.** Let  $f : I^\circ \subset [0, b^*] \rightarrow \mathbb{R}$ ,  $b^* > 0$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is  $m$ -convex on  $[a, b]$  and  $m \in (0, 1]$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4} \left[ L_q \left( |f'(a)|, \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2} \right) + L_q \left( \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2}, m \left| f'\left(\frac{b}{m}\right) \right| \right) \right] \end{aligned}$$

where  $|f'(a)| \neq m|f'(\frac{b}{m})|$ ,  $\frac{b}{m} < b^*$  and  $L_q$  is a  $q$ -logarithmic mean of positive real numbers.

*Proof.* From Lemma 1, using Hölder integral inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq (b-a) \left[ \int_0^{\frac{1}{2}} t |f'(ta + (1-t)b)| dt + \int_{\frac{1}{2}}^1 |1-t| |f'(ta + (1-t)b)| dt \right] \\ & \leq (b-a) \left[ \left( \int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right] \\ & \quad + (b-a) \left[ \left( \int_{\frac{1}{2}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since  $|f'|$  is  $m$ -convex on  $[a, b]$ , we know that for  $t \in [0, 1]$

$$|f'(ta + (1-t)b)| = \left| f'\left( ta + m(1-t)\frac{b}{m} \right) \right| \leq t|f'(a)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right|,$$

hence

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
& \leq (b-a) \left( \int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} \left( t |f'(a)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right| \right)^q dt \right)^{\frac{1}{q}} \\
& \quad + (b-a) \left( \int_{\frac{1}{2}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 \left( t |f'(a)| + m(1-t) \left| f'\left(\frac{b}{m}\right) \right| \right)^q dt \right)^{\frac{1}{q}} \\
& = (b-a) \left( \frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \\
& \quad \times \left( \frac{\frac{1}{2} \left( \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2} \right)^{q+1} - \left( m \left| f'\left(\frac{b}{m}\right) \right| \right)^{q+1}}{(q+1) \left( \frac{|f'(a)| - m|f'(\frac{b}{m})|}{2} \right)} \right)^{\frac{1}{q}} \\
& \quad + (b-a) \left( \frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \\
& \quad \times \left( \frac{\frac{1}{2} \frac{(|f'(a)|)^{q+1} - \left( \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2} \right)^{q+1}}{(q+1) \left( \frac{|f'(a)| - m|f'(\frac{b}{m})|}{2} \right)}} \right)^{\frac{1}{q}} \\
& = (b-a) \frac{1}{(p+1)^{\frac{1}{p}}} \frac{1}{2^{\frac{p+1}{p}}} \frac{1}{2^{\frac{1}{q}}} \\
& \quad \times \left[ L_q \left( \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2}, m \left| f'\left(\frac{b}{m}\right) \right| \right) \right. \\
& \quad \left. + L_q \left( |f'(a)|, \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2} \right) \right].
\end{aligned}$$

Since  $\frac{1}{(p+1)^{\frac{1}{p}}} < 1$  if  $p > 1$ , we have

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{(b-a)}{4} \left[ L_q \left( \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2}, m \left| f'\left(\frac{b}{m}\right) \right| \right) \right. \\
& \quad \left. + L_q \left( |f'(a)|, \frac{|f'(a)| + m|f'(\frac{b}{m})|}{2} \right) \right]
\end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ , which completes the proof.  $\square$

**Theorem 3.** Let  $f : I^\circ \subset [0, b^*] \rightarrow \mathbb{R}$ ,  $b^* > 0$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is  $m$ -convex on  $[a, b]$ ,  $q > 1$  and  $m \in (0, 1]$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)}{4} \left\{ \left( |f'(a)|^q + 3m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left( 3|f'(a)|^q + m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where  $\frac{b}{m} < b^*$ .

*Proof.* From Lemma 1, using Hölder integral inequality, we get

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq (b-a) \left[ \int_0^{\frac{1}{2}} t |f'(ta + (1-t)b)| dt + \int_0^{\frac{1}{2}} |1-t| |f'(ta + (1-t)b)| dt \right] \\ & \leq (b-a) \left( \int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \quad + (b-a) \left( \int_{\frac{1}{2}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since  $|f'|^q$  is  $m$ -convex on  $[a, b]$ , we know that for  $t \in [0, 1]$

$$|f'(ta + (1-t)b)|^q = \left| f'\left( ta + m(1-t)\frac{b}{m} \right) \right|^q \leq t |f'(a)|^q + m(1-t) \left| f'\left(\frac{b}{m}\right) \right|^q,$$

hence

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq (b-a) \left( \int_0^{\frac{1}{2}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} \left( t |f'(a)|^q + m(1-t) \left| f'\left(\frac{b}{m}\right) \right|^q \right) dt \right)^{\frac{1}{q}} \\ & \quad + (b-a) \left( \int_{\frac{1}{2}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 \left( t |f'(a)|^q + m(1-t) \left| f'\left(\frac{b}{m}\right) \right|^q \right) dt \right)^{\frac{1}{q}} \\ & = (b-a) \left( \frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left[ \left( \frac{|f'(a)|^q}{8} + \frac{3m |f'\left(\frac{b}{m}\right)|^q}{8} \right)^{\frac{1}{q}} + \left( \frac{3|f'(a)|^q}{8} + \frac{m |f'\left(\frac{b}{m}\right)|^q}{8} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since  $\left(\frac{1}{p+1}\right)^{\frac{1}{p}} < 1$  and  $\frac{1}{4^{1/q}} < 1$  if  $p > 1$ , we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)}{4} \left\{ \left( |f'(a)|^q + 3m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left( 3|f'(a)|^q + m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ , which completes the proof.  $\square$

**Corollary 1.** *In theorem 3, if  $m = 1$ , we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4} \left\{ (|f'(a)|^q + 3|f'(b)|^q)^{\frac{1}{q}} + (3|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} \right\}. \end{aligned}$$

**Theorem 4.** *Let  $f : I^\circ \subset [0, b^*] \rightarrow \mathbb{R}$ ,  $b^* > 0$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is  $m$ -convex on  $[a, b]$  and  $m \in (0, 1]$ , then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{8} \left\{ \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( m \left| f'\left(\frac{b}{m}\right) \right|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( |f'(a)|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right\} \end{aligned}$$

where  $|f'(a)| \neq m |f'(\frac{b}{m})|$ ,  $\frac{b}{m} < b^*$  and  $L_q$  is a  $q$ -logarithmic mean of positive real numbers.

*Proof.* From Lemma 1, using well-known power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq (b-a) \left[ \int_0^{\frac{1}{2}} t |f'(ta + (1-t)b)| dt + \int_0^{\frac{1}{2}} |1-t| |f'(ta + (1-t)b)| dt \right] \\ & \leq (b-a) \left[ \left( \int_0^{\frac{1}{2}} t dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} t (|f'(ta + (1-t)b)|)^q dt \right)^{\frac{1}{q}} \right] \\ & \quad + (b-a) \left[ \left( \int_{\frac{1}{2}}^1 (1-t) dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 (1-t) (|f'(ta + (1-t)b)|)^q dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since  $|f'|$  is  $m$ -convex, using the elementary inequality  $cd \leq \frac{1}{2}(c^2 + d^2)$ , ( $c, d \in \mathbb{R}$ ), we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} t (|f'(ta + (1-t)b)|)^q dt \\
& \leq \int_0^{\frac{1}{2}} t \left( t |f'(a)| + m(1-t) \left| f' \left( \frac{b}{m} \right) \right| \right)^q dt \\
& \leq \frac{1}{2} \int_0^{\frac{1}{2}} t^2 dt + \frac{1}{2} \int_0^{\frac{1}{2}} \left( t |f'(a)| + m(1-t) \left| f' \left( \frac{b}{m} \right) \right| \right)^{2q} dt \\
& = \frac{1}{48} + \frac{1}{2} \left[ \frac{1}{2} \frac{(m |f'(\frac{b}{m})|)^{2q+1} - \left( \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right)^{2q+1}}{(2q+1) \left( \frac{m |f'(\frac{b}{m})| - |f'(a)|}{2} \right)} \right]^{2q} \\
& = \frac{1}{48} + \frac{1}{2^{2q+1}} L_{2q}^{4q^2} \left( m \left| f' \left( \frac{b}{m} \right) \right|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right)
\end{aligned}$$

and analogously

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 (1-t) (|f'(ta + (1-t)b)|)^q dt \\
& \leq \frac{1}{48} + \frac{1}{2^{2q+1}} L_{2q}^{4q^2} \left( |f'(a)|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
& \leq (b-a) \frac{1}{8^{1-\frac{1}{q}}} \left\{ \left[ \frac{1}{48} + \frac{1}{2^{2q+1}} L_{2q}^{4q^2} \left( m \left| f' \left( \frac{b}{m} \right) \right|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right. \\
& \quad \left. + \left[ \frac{1}{48} + \frac{1}{2^{2q+1}} L_{2q}^{4q^2} \left( |f'(a)|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right\} \\
& = \frac{b-a}{8} \left\{ \left[ \frac{1}{6} + \frac{1}{2^{2q-2}} L_{2q}^{4q^2} \left( m \left| f' \left( \frac{b}{m} \right) \right|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right. \\
& \quad \left. + \left[ \frac{1}{6} + \frac{1}{2^{2q-2}} L_{2q}^{4q^2} \left( |f'(a)|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since  $\frac{1}{2^{2q-2}} \leq 1$  if  $q \geq 1$ , we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{8} \left\{ \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( m \left| f'\left(\frac{b}{m}\right) \right|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( |f'(a)|, \frac{|f'(a)| + m |f'(\frac{b}{m})|}{2} \right) \right]^{\frac{1}{q}} \right\} \end{aligned}$$

which completes the proof.  $\square$

**Corollary 2.** *In theorem 4, if  $m = 1$ , we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{8} \left\{ \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( |f'(b)|, \frac{|f'(a)| + |f'(b)|}{2} \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[ \frac{1}{6} + L_{2q}^{4q^2} \left( |f'(b)|, \frac{|f'(a)| + |f'(b)|}{2} \right) \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

**Theorem 5.** *Let  $f : I^\circ \subset [0, b^*] \rightarrow \mathbb{R}$ ,  $b^* > 0$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is  $m$ -convex on  $[a, b]$ ,  $q > 1$  and  $m \in (0, 1]$ , then the following inequality holds:*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left( 3^{1-\frac{1}{q}} \right) \left( |f'(a)| + m^{\frac{1}{q}} \left| f'\left(\frac{b}{m}\right) \right| \right)$$

where  $\frac{b}{m} < b^*$ .

*Proof.* From Lemma 1, using well-known power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq (b-a) \left[ \int_0^{\frac{1}{2}} t |f'(ta + (1-t)b)| dt + \int_0^{\frac{1}{2}} |1-t| |f'(ta + (1-t)b)| dt \right] \\ & \leq (b-a) \left[ \left( \int_0^{\frac{1}{2}} t dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} t (|f'(ta + (1-t)b)|)^q dt \right)^{\frac{1}{q}} \right] \\ & \quad + (b-a) \left[ \left( \int_{\frac{1}{2}}^1 (1-t) dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 (1-t) (|f'(ta + (1-t)b)|)^q dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since  $|f'|^q$  is  $m$ -convex, we have

$$\begin{aligned}
 (2.1) \quad & \int_0^{\frac{1}{2}} t (|f'(ta + (1-t)b)|)^q dt \\
 & \leq \int_0^{\frac{1}{2}} t \left( t |f'(a)|^q + m(1-t) \left| f' \left( \frac{b}{m} \right) \right|^q \right) dt \\
 & = \int_0^{\frac{1}{2}} \left( t^2 |f'(a)|^q + m(t-t^2) \left| f' \left( \frac{b}{m} \right) \right|^q \right) dt \\
 & = \frac{|f'(a)|^q}{24} + \frac{m |f'(\frac{b}{m})|^q}{12}.
 \end{aligned}$$

Similarly, we have

$$(2.2) \quad \int_{\frac{1}{2}}^1 (1-t) (|f'(ta + (1-t)b)|)^q dt \leq \frac{|f'(a)|^q}{12} + \frac{m |f'(\frac{b}{m})|^q}{24}.$$

From (2.1) and (2.2), we obtain

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
 & \leq (b-a) \frac{1}{8^{1-\frac{1}{q}}} \left\{ \left[ \frac{|f'(a)|^q + 2m |f'(\frac{b}{m})|^q}{24} \right]^{\frac{1}{q}} + \left[ \frac{2|f'(a)|^q + m |f'(\frac{b}{m})|^q}{24} \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Let  $a_1 = |f'(a)|^q$ ,  $b_1 = 2m |f'(\frac{b}{m})|^q$ ,  $a_2 = 2|f'(a)|^q$  and  $b_2 = m |f'(\frac{b}{m})|^q$ . Here  $0 < \frac{1}{q} < 1$ , for  $q > 1$ . Using the fact that

$$\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n (a_k)^s + \sum_{k=1}^n (b_k)^s$$

for  $(0 \leq s < 1)$ ,  $a_1, a_2, \dots, a_n \geq 0$ ,  $b_1, b_2, \dots, b_n \geq 0$ , we obtain

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a+b}{2} \right) \right| \\
 & \leq (b-a) \frac{1}{8^{1-\frac{1}{q}}} \frac{3}{24^{\frac{1}{q}}} \left( |f'(a)| + m^{\frac{1}{q}} \left| f' \left( \frac{b}{m} \right) \right| \right) \\
 & \leq \frac{b-a}{8} \left( 3^{1-\frac{1}{q}} \right) \left( |f'(a)| + m^{\frac{1}{q}} \left| f' \left( \frac{b}{m} \right) \right| \right)
 \end{aligned}$$

which completes the proof.  $\square$

**Remark 1.** In Theorem 5, if  $m = 1$ , we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(3^{1-\frac{1}{q}})}{8} (b-a) (|f'(a)| + |f'(b)|)$$

which the inequality of Theorem 1.



## 3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of Corollary 1 and Theorem 5 to the following special means:

- (a) The arithmetic mean:  $A = A(a, b) := (a + b)/2$ ,  $a, b \geq 0$ ,  
 (b) The Identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a \neq b \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a = b \end{cases}, \quad a, b > 0,$$

- (c) The  $p$ -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; a, b > 0.$$

The following propositions hold:

**Proposition 1.** *Let  $a, b \in \mathbb{R}_+$ ,  $a < b$  and  $\frac{1}{q} \in \mathbb{N}$ . Then, for all  $q > 1$*

$$\left| L_{\frac{q}{q+1/q}}^{q/q+1}(a, b) - A^{q+1/q}(a, b) \right| \leq (b-a) A \left( (a+3b)^{\frac{1}{q}}, (3a+b)^{\frac{1}{q}} \right).$$

*Proof.* The assertion follows from Corollary 1 applied for  $f(x) = x^{\frac{1}{q}+1}$ ,  $x \in \mathbb{R}_+$ ,  $\frac{1}{q} \in \mathbb{N}$  and  $q > 1$ . Also, for  $q > 1$ ,  $\frac{1}{q} + 1 < 2$  used in the proof.  $\square$

**Proposition 2.** *Let  $a, b \in \mathbb{R}_+$ ,  $a < b$ ,  $n \in \mathbb{Z}$  and  $n \geq 2$  with  $m \in (0, 1]$ . Then, for all  $q > 1$*

$$\left| L_n^n(a, b) - A^n(a, b) \right| \leq 2n \frac{(3^{1-\frac{1}{q}})}{8} (b-a) A \left( a^{n-1}, m^{\frac{1}{q}} \left( \frac{b}{m} \right)^{n-1} \right).$$

*Proof.* The assertion follows by Theorem 5 on choosing  $f : [0, \infty) \rightarrow [0, \infty)$ ,  $f(x) = x^n$ ,  $n \in \mathbb{Z}$  and  $n \geq 2$  which is  $m$ -convex on  $[0, \infty)$ .  $\square$

**Proposition 3.** *Let  $a, b \in [0, \infty)$  and  $a < b$ . Then, for all  $q > 1$  we have*

$$\begin{aligned} & \ln \left[ \frac{A(a, b) + 1}{I(a+1, b+1)} \right] \\ & \leq \frac{b-a}{4} \left\{ \frac{[(b+1)^q + 3(a+1)^q]^{1/q} + [3(b+1)^q + (a+1)^q]^{1/q}}{(b+1)(a+1)} \right\}. \end{aligned}$$

*Proof.* The proof follows by corollary 1 on choosing  $f : [0, \infty) \rightarrow (-\infty, 0]$ ,  $f(x) = -\ln(x+1)$  which is  $m$ -convex on  $[0, \infty)$ , and we omit the details.  $\square$

**Proposition 4.** *With the above assumption for all  $q > 1$  and  $m \in (0, 1]$  we have*

$$\ln \left[ \frac{A(a, b) + 1}{I(a+1, b+1)} \right] \leq \frac{b-a}{8} \left( 3^{1-\frac{1}{q}} \right) \left( \frac{1}{a+1} + \frac{m^{1+\frac{1}{q}}}{b+m} \right).$$

*Proof.* The proof follows by Theorem 5 for the same function  $f : [0, \infty) \rightarrow (-\infty, 0]$ ,  $f(x) = -\ln(x+1)$  and the details are omitted.  $\square$

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