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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GA-CONVEX FUNCTIONS

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ABSTRACT. Some inequalities of Hermite-Hadamard type for GA-convex functions defined on positive intervals are given.

1. Introduction

Let $I \subset (0, \infty)$ be an interval; a real-valued function $f : I \to \mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq (\geq)\,(1-\lambda)\,f(x) + \lambda f(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Since the condition (1.1) can be written as

$$f \circ \exp\left((1-\lambda)\ln x + \lambda \ln y\right) \leq (\geq)\,(1-\lambda)\,f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y)\,, \tag{1.2}$$

then we observe that $f : I \to \mathbb{R}$ is *GA-convex* (concave) on I if and only if $f \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a, b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is GA-convex on $(0, \infty)$ [4].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [54] that the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0, \infty)$ while the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0, \infty)$.

If $[a, b] \subset (0, \infty)$ and the function $g : [\ln a, \ln b] \to \mathbb{R}$ is convex (concave) on $[\ln a, \ln b]$, then the function $f : [a, b] \to \mathbb{R}$, $f(t) = g(\ln t)$ is GA-convex (concave) on $[a, b]$.

Indeed, if $x, y \in [a, b]$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} f\left(x^{1-\lambda}y^{\lambda}\right) &= g\left(\ln\left(x^{1-\lambda}y^{\lambda}\right)\right) = g\left[(1-\lambda)\ln x + \lambda \ln y\right] \\ &\leq (\geq)\,(1-\lambda)\,g(\ln x) + \lambda g(\ln y) = (1-\lambda)\,f(x) + \lambda f(y) \end{aligned}$$

showing that f is GA-convex (concave) on $[a, b]$.

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We recall that the classical Hermite-Hadamard inequality that states that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{f(a)+f(b)}{2} \tag{1.3}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1]-[20], [23]-[25], [26]-[35] and [36]-[46].

In [54] the authors obtained the following Hermite-Hadamard type inequality.

Theorem 1. *If $b>a>0$ and $f:[a,b]\to\mathbb{R}$ is a differentiable GA-convex (concave) function on $[a,b]$, then*

$$f(I(a,b))\leq(\geq)\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq(\geq)\frac{b-L(a,b)}{b-a}f(b)+\frac{L(a,b)-a}{b-a}f(a). \tag{1.4}$$

The *identric mean* $I(a,b)$ is defined by

$$I(a,b):=\frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b):=\frac{b-a}{\ln b-\ln a}.$$

The differentiability of the function is not necessary in Theorem 1 for the first inequality (1.4) to hold. A proof of this fact is proved below after some short preliminaries. The second inequality in (1.4) has been proved in [54] without differentiability assumption.

2. Preliminaries

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f:I\to\mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $f'_{-}(x)\leq f'_{+}(x)\leq f'_{-}(y)\leq f'_{+}(y)$ which shows that both f'_{-} and f'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f:I\to\mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi:I\to[-\infty,\infty]$ such that $\varphi\left(\mathring{\mathrm{I}}\right)\subset\mathbb{R}$ and

$$f(x)\geq f(a)+(x-a)\varphi(a)\ \text{ for any } x,a\in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_{-}, $f'_{+}\in\partial f$ and if $\varphi\in\partial f$, then

$$f'_{-}(x)\leq\varphi(x)\leq f'_{+}(x)\ \text{ for any } x\in\mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f=\{f'\}$.

Now, since $f\circ\exp$ is convex on $[\ln a,\ln b]$ it follows that f has finite lateral derivatives on $(\ln a,\ln b)$ and by gradient inequality for convex functions we have

$$f\circ\exp(x)-f\circ\exp(y)\geq(x-y)\,\varphi(\exp y)\exp y \tag{2.1}$$

where $\varphi(\exp y)\in\left[f'_{-}(\exp y),f'_{+}(\exp y)\right]$ for any $x,y\in(\ln a,\ln b)$.

If $s,t\in(a,b)$ and we take in (2.1) $x=\ln t, y=\ln s$, then we get

$$f(t)-f(s)\geq(\ln t-\ln s)\varphi(s)s \tag{2.2}$$

where $\varphi(s)\in\left[f'_{-}(s),f'_{+}(s)\right].$

Now, if we take the integral mean on $[a,b]$ in the inequality (2.2) we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f(s)\geq\left(\frac{1}{b-a}\int_a^b\ln t dt-\ln s\right)\varphi(s)s$$

and since

$$\frac{1}{b-a}\int_a^b\ln t dt=\ln I(a,b)$$

then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(s)+(\ln I(a,b)-\ln s)\varphi(s)s \tag{2.3}$$

for any $s\in(a,b)$ and $\varphi(s)\in\left[f'_{-}(s),f'_{+}(s)\right].$ This is an inequality of interest in itself.

Now, if we take in (2.3) $s=I(a,b)\in(a,b)$ then we get the first inequality in (1.4) for GA-convex functions.

If f is differentiable and GA-convex on (a,b), then we have from (2.3) the inequality

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(s)+(\ln I(a,b)-\ln s)f'(s)s \tag{2.4}$$

for any $s\in(a,b)$.

If we take in (2.4) $s=\frac{a+b}{2}=A(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(A(a,b))-f'(A(a,b))A(a,b)\ln\left(\frac{A(a,b)}{I(a,b)}\right). \tag{2.5}$$

If we assume that $f'(A(a,b))\leq 0$, then, since $I(a,b)\leq A(a,b)$, we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(A(a,b)) \tag{2.6}$$

provided that f is differentiable and GA-convex on (a,b).

Also, if we take in (2.4) $s=L(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(L(a,b))+f'(L(a,b))L(a,b)\ln\left(\frac{I(a,b)}{L(a,b)}\right). \tag{2.7}$$

If we assume that $f'(L(a,b))\geq 0$, then we get from (2.7) that

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(L(a,b)) \tag{2.8}$$

provided that f is differentiable and GA-convex on (a,b).

Now, if we take in (2.4) $s=\sqrt{ab}=G(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq f(G(a,b))+f'(G(a,b))G(a,b)\ln\left(\frac{I(a,b)}{G(a,b)}\right). \tag{2.9}$$

Since

$$\ln\left(\frac{I(a,b)}{G(a,b)}\right) = \ln I(a,b) - \ln G(a,b)$$
$$= \frac{b\ln b - a\ln a}{b-a} - 1 - \frac{\ln a + \ln b}{2}$$
$$= \frac{a+b}{2}\frac{\ln b - \ln a}{b-a} - 1 = \frac{A(a,b) - L(a,b)}{L(a,b)},$$

then (2.9) is equivalent to

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) + f'(G(a,b))\,G(a,b)\,\frac{A(a,b)-L(a,b)}{L(a,b)}. \tag{2.10}$$

If $f'(G(a,b)) \geq 0$, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) \tag{2.11}$$

provided that f is differentiable and GA-convex on (a,b).

Motivated by the above results we establish in this paper other inequalities of Hermite-Hadamard type for GA-convex functions. Applications for special means are also provided.

3. New Results

We start with the following result that provide in the right side of (1.4) a bound in terms of the identric mean.

Theorem 2. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq (\geq) \frac{(\ln b - \ln I(a,b))\,f(a) + (\ln I(a,b) - \ln a)\,f(b)}{\ln b - \ln a} \tag{3.1}$$
$$= \frac{b - L(a,b)}{b-a} f(b) + \frac{L(a,b) - a}{b-a} f(a).$$

Proof. Since is a GA-convex (concave) function on $[a,b]$ then $f \circ \exp$ is convex (concave) and we have

$$f(t) = f\circ\exp(\ln t) = f\circ\exp\left(\frac{(\ln b - \ln t)\ln a + (\ln t - \ln a)\ln b}{\ln b - \ln a}\right) \tag{3.2}$$
$$\leq (\geq) \frac{(\ln b - \ln t)\, f\circ\exp(\ln a) + (\ln t - \ln a)\, f\circ\exp(\ln b)}{\ln b - \ln a}$$
$$= \frac{(\ln b - \ln t)\, f(a) + (\ln t - \ln a)\, f(b)}{\ln b - \ln a}$$

for any $t \in [a,b]$.

This inequality is of interest in itself as well.

If we take the integral mean in (3.2) we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\leq (\geq) \frac{\left(\ln b - \frac{1}{b-a}\int_a^b \ln t dt\right) f(a) + \left(\frac{1}{b-a}\int_a^b \ln t dt - \ln a\right) f(b)}{\ln b - \ln a}$$

and since

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b),$$

then we obtain the desired result (3.1).

Now, we observe that

$$\begin{aligned}\frac{\ln b - \ln I(a,b)}{\ln b - \ln a} &= \frac{\ln b - \frac{b\ln b - a\ln a}{b-a} + 1}{\ln b - \ln a} \\ &= \frac{(b-a)\ln b - b\ln b + a\ln a + b - a}{(b-a)(\ln b - \ln a)} \\ &= \frac{b-a-a(\ln b - \ln a)}{(b-a)(\ln b - \ln a)} \\ &= \frac{L(a,b)-a}{b-a}\end{aligned}$$

and, similarly

$$\frac{\ln I(a,b) - \ln a}{\ln b - \ln a} = \frac{b - L(a,b)}{b-a},$$

which proves the last part of (3.1). □

If $f : I \subset (0,\infty) \to \mathbb{R}$ is a GA-convex (concave) on I then we have the inequality

$$f(\sqrt{xy}) \le (\ge) \frac{f(x)+f(y)}{2} \tag{3.3}$$

for any $x, y \in I$.

The following refinement of (3.3), which is an inequality of Hermite-Hadamard type, holds (see [44] for an extension for GA h-convex functions). For the sake of completeness we give here a short proof.

Lemma 1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$f\left(\sqrt{ab}\right) \le (\ge) \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \le (\ge) \frac{f(a)+f(b)}{2}. \tag{3.4}$$

Proof. By the definition of GA-convex (concave) functions on $[a,b]$ we have

$$f\left(a^{1-\lambda}b^{\lambda}\right) \le (\ge) (1-\lambda) f(a) + \lambda f(b) \tag{3.5}$$

for any $\lambda \in [0,1]$.

Integrating the inequality (3.5) on $[0,1]$ we get

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda \le (\ge) f(a)\int_0^1 (1-\lambda)\, d\lambda + f(b)\int_0^1 \lambda d\lambda. \tag{3.6}$$

Since

$$\int_0^1 (1-\lambda)\, d\lambda = \int_0^1 \lambda d\lambda = \frac{1}{2}$$

and, by changing the variable $t = a^{1-\lambda}b^{\lambda}$, $\lambda \in [0,1]$, we have

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt$$

then by (3.6) we get the second inequality in (3.4).

By the inequality (3.3) we have

$$f\left(\sqrt{ab}\right)=f\left(\sqrt{a^{1-\lambda}b^{\lambda}a^{\lambda}b^{1-\lambda}}\right)\leq(\geq)\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+f\left(a^{\lambda}b^{1-\lambda}\right)\right] \tag{3.7}$$

for any $\lambda\in[0,1]$.

Integrating the inequality (3.7) $[0,1]$ we get

$$f\left(\sqrt{ab}\right)\leq(\geq)\frac{1}{2}\left[\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda+\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda\right]. \tag{3.8}$$

Since

$$\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right)d\lambda=\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda=\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt,$$

then by (3.8) we get the first inequality in (3.4). □

Remark 1. *The inequality (3.3) can be also written for any $d>c>0$ with $c,d\in I$ as*

$$f\left(\sqrt{cd}\right)\leq(\geq)\int_0^1 f\left(c^{1-\lambda}d^{\lambda}\right)d\lambda\leq(\geq)\frac{f(c)+f(d)}{2}, \tag{3.9}$$

provided GA-convex (concave) function on I.

We have the following representation result:

Lemma 2. *Let $g:[x,y]\to\mathbb{C}$ be a Lebesgue integrable function on $[x,y]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$\int_0^1 g[(1-t)x+ty]\,dt=(1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x+\lambda y)+ty]\,dt \tag{3.10}$$
$$+\lambda\int_0^1 g[(1-t)x+t((1-\lambda)x+\lambda y)]\,dt.$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (3.3) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 g[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt$$
$$=\int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt$$

and

$$\int_0^1 g[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt=\int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt=\frac{1}{1-\lambda}\int_\lambda^1 g[uy+(1-u)x]\,du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt=\frac{1}{\lambda}\int_0^\lambda g[uy+(1-u)x]\,du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 g\left[(1-t)(\lambda y+(1-\lambda)x)+ty\right]dt \\
&+\lambda\int_0^1 g\left[t(\lambda y+(1-\lambda)x)+(1-t)x\right]dt \\
&=\int_\lambda^1 g\left[uy+(1-u)x\right]du+\int_0^\lambda g\left[uy+(1-u)x\right]du \\
&=\int_0^1 g\left[uy+(1-u)x\right]du
\end{aligned}$$

and the identity (3.3) is proved. □

Corollary 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{C}$ be a Lebesgue integrable function on $[a,b]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$\begin{aligned}
&\int_0^1 f\left(a^{1-s}b^s\right)ds \\
&=(1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^\lambda\right]^{1-s}b^s\right)ds+\lambda\int_0^1 f\left(a^{1-s}\left[a^{(1-\lambda)}b^\lambda\right]^s\right)ds.
\end{aligned}\tag{3.11}$$

Proof. Using (3.10) we have

$$\begin{aligned}
&\int_0^1 f\left(a^{1-s}b^s\right)ds \\
&=\int_0^1 f\circ\exp\left((1-s)\ln a+s\ln b\right)ds \\
&=(1-\lambda)\int_0^1 f\circ\exp\left[(1-t)\left((1-\lambda)\ln a+\lambda\ln b\right)+t\ln b\right]dt \\
&+\lambda\int_0^1 f\circ\exp\left[(1-t)\ln a+t\left((1-\lambda)\ln a+\lambda\ln b\right)\right]dt \\
&=(1-\lambda)\int_0^1 f\circ\exp\left[(1-t)\ln\left[a^{(1-\lambda)}b^\lambda\right]+t\ln b\right]dt \\
&+\lambda\int_0^1 f\circ\exp\left[(1-t)\ln a+t\ln\left[a^{(1-\lambda)}b^\lambda\right]\right]dt \\
&=(1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^\lambda\right]^{1-t}b^t\right)dt+\lambda\int_0^1 f\left(a^{1-t}\left[a^{(1-\lambda)}b^\lambda\right]^t\right)dt \\
&=(1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^\lambda\right]^{1-s}b^s\right)ds+\lambda\int_0^1 f\left(a^{1-s}\left[a^{(1-\lambda)}b^\lambda\right]^s\right)ds
\end{aligned}$$

and the identity (3.11) is proved. □

We are able now to provide a refinement of (3.4) as follows:

Theorem 3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $\lambda\in[0,1]$ we have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq (\geq)\,(1-\lambda)\,f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\\
&\leq (\geq)\,\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\\
&\leq (\geq)\,\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)\,f(b)+\lambda f(a)\right]\\
&\leq (\geq)\,\frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{3.12}
$$

Proof. We prove the inequalities only for the GA-convex case.

Using the inequality (3.9) we have

$$
f\left(\sqrt{a^{1-\lambda}b^{\lambda}b}\right)\leq\int_0^1 f\left(\left[a^{1-\lambda}b^{\lambda}\right]^{1-s}b^s\right)ds\leq\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2},
$$

that is equivalent to

$$
f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\leq\int_0^1 f\left(\left[a^{1-\lambda}b^{\lambda}\right]^{1-s}b^s\right)ds\leq\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2},
\tag{3.13}
$$

for any $\lambda\in[0,1]$.

We also have

$$
f\left(\sqrt{aa^{1-\lambda}b^{\lambda}}\right)\leq\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^{\lambda}\right]^s\right)ds\leq\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2}
$$

that is equivalent to

$$
f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\leq\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^{\lambda}\right]^s\right)ds\leq\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2}
\tag{3.14}
$$

for any $\lambda\in[0,1]$.

If we multiply (3.13) by $1-\lambda$ and (3.14) by λ and add the obtained inequalities we get, by the identity (3.11), that

$$
\begin{aligned}
&(1-\lambda)\,f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\\
&\leq\int_0^1 f\left(a^{1-s}b^s\right)ds\\
&\leq(1-\lambda)\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2}+\lambda\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2}\\
&=\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)\,f(b)+\lambda f(a)\right]
\end{aligned}
$$

for any $\lambda\in[0,1]$.

This proves the second and third inequalities in (3.12).

By the GA-convexity we have

$$
\begin{aligned}
(1-\lambda)\,f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)&\geq f\left[\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)^{1-\lambda}\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)^{\lambda}\right]\\
&=f\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right),
\end{aligned}
$$

which proves the first inequality in (3.12).

By the GA-convexity we also have

$$\begin{aligned}&\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)f(b)+\lambda f(a)\right]\\&\leq\frac{1}{2}\left[(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)f(b)+\lambda f(a)\right]\\&=\frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the last inequality in (3.12). □

Corollary 2. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq(\geq)\frac{1}{2}\left[f\left(a^{\frac{1}{4}}b^{\frac{3}{4}}\right)+f\left(a^{\frac{3}{4}}b^{\frac{1}{4}}\right)\right]\\&\leq(\geq)\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\\&\leq(\geq)\frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(b)+f(a)}{2}\right]\leq(\geq)\frac{f(a)+f(b)}{2}.\end{aligned}\tag{3.15}$$

4. Related Results

The following result also holds:

Theorem 4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $t\in[a,b]$ we have*

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds\\&\leq(\geq)\frac{1}{2}\left[f(t)+\frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a}\right]\\&\leq(\geq)\frac{f(a)+f(b)}{2}.\end{aligned}\tag{4.1}$$

Proof. We give a proof only for the GA-convex case.

From the inequality (2.2) we have that

$$f(t)-f(s)\geq(\ln t-\ln s)f'_+(s)s\tag{4.2}$$

for any $s\in(a,b)$ and $t\in[a,b]$.

We divide by $s>0$ and integrate on $[a,b]$ over s to get

$$f(t)\int_a^b\frac{1}{s}ds-\int_a^b\frac{f(s)}{s}ds\geq\left(\int_a^b f'_+(s)ds\right)\ln t-\int_a^b f'_+(s)\ln s ds\tag{4.3}$$

for any $t\in[a,b]$.

However

$$\int_a^b\frac{1}{s}ds=\ln b-\ln a,\quad\int_a^b f'_+(s)ds=f(b)-f(a)$$

and

$$\int_a^b f'_+(s)\ln s ds=\left.f(s)\ln s\right|_a^b-\int_a^b\frac{f(s)}{s}ds=f(b)\ln b-f(a)\ln a-\int_a^b\frac{f(s)}{s}ds.$$

Therefore, by (4.3) we get

$$f(t)(\ln b-\ln a)-\int_a^b \frac{f(s)}{s}ds$$
$$\geq (f(b)-f(a))\ln t - f(b)\ln b + f(a)\ln a + \int_a^b \frac{f(s)}{s}ds$$

namely

$$f(t)(\ln b-\ln a)-(f(b)-f(a))\ln t + f(b)\ln b - f(a)\ln a \geq 2\int_a^b \frac{f(s)}{s}ds,$$

which can be written as

$$f(t)(\ln b-\ln a)+f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)\geq 2\int_a^b \frac{f(s)}{s}ds$$

and the first inequality in (4.1) is proved.

Using (3.2) we have

$$\begin{aligned} f(t)+\frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a} &\\ \leq \frac{(\ln b-\ln t)f(a)+(\ln t-\ln a)f(b)}{\ln b-\ln a} &\\ +\frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a} &\\ = f(a)+f(b) & \end{aligned}$$

for any $t\in[a,b]$ that proves the last part of (4.1). □

By taking the integral mean in the inequality (4.1) we have:

Corollary 3. *With the assumptions in Theorem 4 we have*

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \tag{4.4}$$
$$\leq (\geq)\,\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right]$$
$$\leq (\geq)\,\frac{f(a)+f(b)}{2}.$$

Since a simple calculation reveals (see the proof of Theorem 2) that

$$\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}$$
$$=\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a),$$

then the inequality (4.4) is equivalent to

$$\begin{aligned}(4.5)\quad &\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq (\geq)\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right] \\ &\leq (\geq)\frac{f(a)+f(b)}{2}.\end{aligned}$$

Remark 2. *Taking specific values for $t\in[a,b]$ in (4.1) we get the following results*

$$\begin{aligned}(4.6)\quad &\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq (\geq)\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)\left(\ln b-\ln\frac{a+b}{2}\right)+f(a)\left(\ln\frac{a+b}{2}-\ln a\right)}{\ln b-\ln a}\right] \\ &\leq (\geq)\frac{f(a)+f(b)}{2},\end{aligned}$$

$$\begin{aligned}(4.7)\quad \frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\leq (\geq)\frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\leq (\geq)\frac{f(a)+f(b)}{2}\end{aligned}$$

$$\begin{aligned}(4.8)\quad &\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq (\geq)\frac{1}{2}\left[f(I(a,b))+\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right] \\ &=\frac{1}{2}\left[f(I(a,b))+\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right] \\ &\leq (\geq)\frac{f(a)+f(b)}{2},\end{aligned}$$

and

$$\begin{aligned}(4.9)\quad &\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq (\geq)\frac{1}{2}\left[f(L(a,b))+\frac{f(b)(\ln b-\ln L(a,b))+f(a)(\ln L(a,b)-\ln a)}{\ln b-\ln a}\right] \\ &\leq (\geq)\frac{f(a)+f(b)}{2}.\end{aligned}$$

Now, observe that

$$f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)=0$$

iff

$$\ln t=\frac{f(b)\ln b-f(a)\ln a}{f(b)-f(a)}=\ln\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}},$$

which is equivalent to

$$t=\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}.$$

Therefore, if

$$t=\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}\in[a,b]$$

then by (4.1) we get

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds\le(\ge)\frac{1}{2}f\left(\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}\right)\le(\ge)\frac{f(a)+f(b)}{2}. \tag{4.10}$$

The following result also holds

Theorem 5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $t\in[a,b]$ we have*

$$\begin{aligned}&\frac{1}{2}\left[f(t)+\frac{f(b)\,b(\ln b-\ln t)+af(a)(\ln t-\ln a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(s)\,ds\\&\ge(\le)\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln t\right].\end{aligned} \tag{4.11}$$

Proof. We give a proof only for the GA-convex case.

Integrate over s in the inequality (4.2) to get

$$f(t)(b-a)-\int_a^b f(s)\,ds\ge\ln t\int_a^b f'_+(s)\,sds-\int_a^b f'_+(s)\,s\ln sds \tag{4.12}$$

for any $t\in[a,b]$.

Observe that, integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_+(s)\,sds=bf(b)-af(a)-\int_a^b f(s)\,ds$$

and

$$\begin{aligned}\int_a^b f'_+(s)\,s\ln sds&=f(b)\,b\ln b-f(a)\,a\ln a-\int_a^b(s\ln s)'f(s)\,ds\\&=f(b)\,b\ln b-f(a)\,a\ln a-\int_a^b(\ln s+1)\,f(s)\,ds\\&=f(b)\,b\ln b-f(a)\,a\ln a-\int_a^b f(s)\ln sds-\int_a^b f(s)\,ds.\end{aligned}$$

Using the inequality (4.12) we get

$$\begin{aligned}
&f(t)(b-a)-\int_a^b f(s)\,ds\\
&\geq \ln t\left(bf(b)-af(a)-\int_a^b f(s)\,ds\right)\\
&-f(b)b\ln b+f(a)a\ln a+\int_a^b f(s)\ln s ds+\int_a^b f(s)\,ds\\
&=bf(b)\ln t-af(a)\ln t-\ln t\int_a^b f(s)\,ds\\
&-f(b)b\ln b+f(a)a\ln a+\int_a^b f(s)\ln s ds+\int_a^b f(s)\,ds
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&f(t)(b-a)-bf(b)\ln t+af(a)\ln t+f(b)b\ln b-f(a)a\ln a-2\int_a^b f(s)\,ds\\
&\geq \int_a^b f(s)\ln s ds-\ln t\int_a^b f(s)\,ds,
\end{aligned}$$

namely

$$\begin{aligned}
&f(t)(b-a)+f(b)b(\ln b-\ln t)+af(a)(\ln t-\ln a)-2\int_a^b f(s)\,ds\\
&\geq \int_a^b f(s)\ln s ds-\ln t\int_a^b f(s)\,ds,
\end{aligned}$$

for any $t\in[a,b]$ and the inequality (4.11) is proved. □

Corollary 4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\begin{aligned}
&\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds\\
&\geq \frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b).
\end{aligned}\tag{4.13}$$

Moreover, if f is monotonic nondecreasing, then

$$\begin{aligned}
&\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds\\
&\geq \frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b)\geq 0.
\end{aligned}\tag{4.14}$$

Proof. Integrating over t on $[a,b]$ and dividing by $b-a$ in (4.11) we get

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\,ds+\frac{f(b)\,b\left(\ln b-\frac{1}{b-a}\int_a^b \ln t dt\right)+af(a)\left(\frac{1}{b-a}\int_a^b \ln t dt-\ln a\right)}{b-a}\right]$$
$$-\frac{1}{b-a}\int_a^b f(s)\,ds$$
$$\geq(\leq)\,\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\frac{1}{b-a}\int_a^b \ln t dt\right],$$

that is equivalent to (4.13).

Now, since f is monotonic nondecreasing on $[a,b]$, then by Čebyšev inequality for synchronous functions, we have

$$\frac{1}{b-a}\int_a^b f(s)\ln s ds\geq\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\frac{1}{b-a}\int_a^b \ln t dt$$

that proves (4.14). □

Corollary 5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\frac{1}{2}\left[f\left(\exp\left(\mu_f\right)\right)+\frac{f(b)\,b\left(\ln b-\mu_f\right)+af(a)\left(\mu_f-\ln a\right)}{b-a}\right]\geq\frac{1}{b-a}\int_a^b f(s)\,ds, \tag{4.15}$$

where

$$\mu_f:=\frac{\int_a^b f(s)\ln s ds}{\int_a^b f(s)\,ds}\in[\ln a,\ln b].$$

Proof. Follows by (4.11) on taking

$$\ln t=\frac{\int_a^b f(s)\ln s ds}{\int_a^b f(s)\,ds}\in[\ln a,\ln b].$$

□

Remark 3. *If we take $t=\sqrt{ab}$ in (4.11), then we get*

$$\frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(b)\,b+af(a)}{2L(a,b)}\right]-\frac{1}{b-a}\int_a^b f(s)\,ds\geq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln\sqrt{ab}\right]. \tag{4.16}$$

If we take $t = I(a,b)$ *in (4.11), then we get*

$$\begin{aligned}(4.17)\quad & \frac{1}{2}\left[f(I(a,b)) + \frac{f(b)\,b(\ln b - \ln I(a,b)) + af(a)(\ln I(a,b) - \ln a)}{b-a}\right] \\ & - \frac{1}{b-a}\int_a^b f(s)\,ds \\ & \geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b)\right].\end{aligned}$$

We use the following results obtained by the author in [21] and [22]

Lemma 3. *Let* $h : [\alpha, \beta] \to \mathbb{R}$ *be a convex function on* $[\alpha, \beta]$. *Then we have the inequalities*

$$\begin{aligned}(4.18)\quad & \frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ & \leq \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \\ & \leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha)\end{aligned}$$

and

$$\begin{aligned}(4.19)\quad & \frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ & \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \\ & \leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha).\end{aligned}$$

The constant $\frac{1}{8}$ *is best possible in (4.18) and (4.19).*

Finally, we have

Theorem 6. *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$. *Then we have*

$$\begin{aligned}(4.20)\quad & \frac{1}{8}\left[f'_+\left(\sqrt{ab}\right) - f'_-\left(\sqrt{ab}\right)\right](\ln b - \ln a) \\ & \leq (\geq) \frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds \\ & \leq (\geq) \frac{1}{8}\left[f'_-(b)\,b - f'_+(a)\,a\right](\ln b - \ln a)\end{aligned}$$

and

$$\begin{aligned}(4.21)\quad & \frac{1}{8}\left[f'_+\left(\sqrt{ab}\right) - f'_-\left(\sqrt{ab}\right)\right](\ln b - \ln a) \\ & \leq (\geq) \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds - f\left(\sqrt{ab}\right) \\ & \leq (\geq) \frac{1}{8}\left[f'_-(b)\,b - f'_+(a)\,a\right](\ln b - \ln a).\end{aligned}$$

Proof. Consider the function $h:[\ln a,\ln b]\to\mathbb{R}$ defined by $h(t)=f\circ\exp(t)$. Since f is a GA-convex (concave) function on $[a,b]$, then we have the lateral derivatives

$$h'_{\pm}(t)=\left(f'_{\pm}\circ\exp(t)\right)\exp t,\ \ t\in[\ln a,\ln b].$$

If we apply the inequality (4.18) for the convex function $f\circ\exp$ on the interval $[\ln a,\ln b]$, then we have

$$\begin{aligned}&\frac{1}{8}\left[f'_{+}\circ\exp\left(\frac{\ln a+\ln b}{2}\right)-f'_{-}\circ\exp\left(\frac{\ln a+\ln b}{2}\right)\right](\ln b-\ln a)\\&\leq\frac{f\circ\exp(\ln a)+f\circ\exp(\ln b)}{2}-\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f\circ\exp(t)\,dt\\&\leq\frac{1}{8}\left[\left(f'_{-}\circ\exp(\ln b)\right)\exp(\ln b)-\left(f'_{+}\circ\exp(\ln a)\right)\exp(\ln a)\right](\ln b-\ln a)\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{8}\left[f'_{+}\left(\sqrt{ab}\right)-f'_{-}\left(\sqrt{ab}\right)\right](\ln b-\ln a)\\&\leq\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f\circ\exp(t)\,dt\\&\leq\frac{1}{8}\left[f'_{-}(b)\,b-f'_{+}(a)\,a\right](\ln b-\ln a).\end{aligned}\tag{4.22}$$

If we change the variable $s=\exp t$, then $t=\ln s$ and $dt=\frac{ds}{s}$. Therefore

$$\int_{\ln a}^{\ln b}f\circ\exp(t)\,dt=\int_a^b\frac{f(s)}{s}ds$$

and by (4.22) we get the desired inequality (4.20).

The inequality (4.21) follows by (4.19). □

Remark 4. *If the function $f:I\subset(0,\infty)\to\mathbb{R}$ is differentiable and a GA-convex function on $[a,b]\subset\mathring{I}$ then we have the following inequalities*

$$\begin{aligned}0&\leq\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds\\&\leq\frac{1}{8}\left[f'(b)\,b-f'(a)\,a\right](\ln b-\ln a)\end{aligned}\tag{4.23}$$

and

$$\begin{aligned}0&\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds-f\left(\sqrt{ab}\right)\\&\leq\frac{1}{8}\left[f'(b)\,b-f'(a)\,a\right](\ln b-\ln a).\end{aligned}\tag{4.24}$$

5. Some Applications

Let $p\neq0$ and consider the convex function $g(t)=\exp(pt),t\in\mathbb{R}$. Then the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=g(\ln t)=\exp(p\ln t)=t^p$ is a GA-convex function on $(0,\infty)$. We observe that for $0<a<b$ we have

$$\frac{1}{b-a}\int_a^b t^p dt = \begin{cases} \frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a}, \ p\neq -1 \\ \frac{\ln b-\ln a}{b-a}, \ p=-1 \end{cases} = \begin{cases} L_p^p(a,b), \ p\neq -1 \\ L^{-1}(a,b), \ p=-1 \end{cases}$$

where $L_p(a,b)$ $(p\neq -1)$ is the p-*Logarithmic mean* and L is the logarithmic mean defined in the introduction.

Using the inequality

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{b-L(a,b)}{b-a}f(b)+\frac{L(a,b)-a}{b-a}f(a)$$

for $f(t)=t^p$ $(p\neq 0)$, we get

$$L_p^p(a,b)\leq \frac{b-L(a,b)}{b-a}b^p+\frac{L(a,b)-a}{b-a}a^p \tag{5.1}$$

for $p\neq 0$.

Observe that

$$\frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt = \frac{1}{b-a}\int_a^b t^{p-1}dt = \frac{1}{p}\frac{b^p-a^p}{b-a}=L_{p-1}^{p-1}(a,b), \ p\neq 0.$$

If we use the inequality

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq (1-\lambda)f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}dt \\ &\leq \frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)f(b)+\lambda f(a)\right] \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned}$$

for $\lambda\in[0,1]$ and $f(t)=t^p$ $(p\neq 0)$, then we get

$$\begin{aligned} G^p(a,b) &\leq (1-\lambda)G^p\left(a^{1-\lambda},b^{\lambda+1}\right)+\lambda G^p\left(a^{2-\lambda},b^{\lambda}\right) \\ &\leq L(a,b)L_{p-1}^{p-1}(a,b) \\ &\leq \frac{1}{2}\left[G^p\left(a^{2(1-\lambda)},b^{2\lambda}\right)+(1-\lambda)b^p+\lambda a^p\right]\leq\frac{a^p+b^p}{2} \end{aligned} \tag{5.2}$$

for $\lambda\in[0,1]$.

If we use the inequalities (4.23) and (4.24) for $f(t)=t^p$ $(p\neq 0)$, then we get

$$0\leq \frac{a^p+b^p}{2}-L(a,b)L_{p-1}^{p-1}(a,b)\leq \frac{1}{8}p^2\frac{L_{p-1}^{p-1}(a,b)}{L(a,b)}(b-a)^2 \tag{5.3}$$

and

$$0\leq L(a,b)L_{p-1}^{p-1}(a,b)-G^p(a,b)\leq \frac{1}{8}p^2\frac{L_{p-1}^{p-1}(a,b)}{L(a,b)}(b-a)^2. \tag{5.4}$$

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