INEQUALITIES ON GENERALIZED TRIGONOMETRIC FUNCTIONS

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Abstract. The Sharp Cusa-Huygens inequality involving the generalized trigonometric functions are established.

Introduction

It is well known from basic calculus that

$$\arcsin(x) = \int_0^x \frac{1}{(1-t^2)^{1/2}}\mathrm{d}t, \quad 0 \le x \le 1,$$

And $\dfrac{\pi}{2} = \arcsin(1) = \int_0^1 \dfrac{1}{(1-t^2)^{1/2}}\mathrm{d}t.$

For $1 < p < \infty$, We can generalize the above function as follows:

$$\arcsin_p(x) \equiv \int_0^x \frac{1}{(1-t^p)^{1/p}}\mathrm{d}t, \quad 0 \le x \le 1,$$

and $\dfrac{\pi_p}{2} = \arcsin_p(1) = \int_0^1 \dfrac{1}{(1-t^p)^{1/p}}\mathrm{d}t.$

where $\pi_p = \dfrac{2\pi}{p\sin(\pi/p)}$ is decreasing on $(1,\infty)$.

The inverse of $\arcsin_p$ on $[0, \pi_p/2]$ is called the generalized sine function and denoted by $\sin_p$. The generalized cosine function $\cos_p$ is defined as

$$\cos_p(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\sin_p(x).$$

It is clear from the definition that

$$\cos_p(x) = \left(1-\sin_p(x)^p\right)^{1/p}.$$

The generalized tangent function $\tan_p$ is defined as

$$\tan_p(x) \equiv \frac{\sin_p(x)}{\cos_p(x)}.$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}, \ \frac{\mathrm{d}}{\mathrm{d}x}\tan_p(x) = 1+\tan_p(x)^p,$$

when $p = 2$, the $p-$ functions $\sin_p, \cos_p, \tan_p$ become our familiar trigonometric functions.

Recently, the generalized trigonometric functions have been studied by many mathematicians from different viewpoints(see [2,4,5,6,7]). In [5,9], the authors gave basic properties of the generalized trigonometric functions. In [6], Klén, Vuorinen and Zhang generalized some classical inequalities for trigonometric functions, such as Mitrinović-Adamović's inequality, Lazarević's inequality, Huygens-type inequalities, and Wilker-type inequalities, to the case of generalized functions.

The main results of this paper are the following theorems.

Theorem 1 For $1 < p \le 2$, the function

$$f(x) = \frac{x(p + \cos_p(x))}{\sin_p(x)}$$

is strictly increasing from $(0, \pi_p/2)$ onto $(p+1, p\pi_p/2)$.

Theorem 2 For $1 < p \le 2$, the function

$$F(x) = \frac{\sin_p(x) - x\cos_p(x)}{x^2 \sin_p(x)^{p-1} \cos_p(x)^{2-p}}$$

is strictly increasing from $(0, \pi_p/2)$ onto $(\frac{1}{p+1}, b_p)$.

Where $$b_p = \begin{cases} \infty, 1 < p < 2, \\ \frac{4}{\pi^2}, \quad p = 2. \end{cases}$$

Theorem 3 For $1 < p \le 2$, the function

$$G(x) = \frac{\ln(\sin_p(x)/x)}{\ln[(p + \cos_p(x))/(p+1)]}$$

is strictly increasing from $(0, \pi_p/2)$ onto $(1, (\log(\pi_p/2))/\log[(p+1)/p])$.

In particular, for all $p \in (1,2]$, $x \in (0, \pi_p/2)$,

$$\left(\frac{p + \cos_p(x)}{p+1}\right)^{\alpha} < \frac{\sin_p x}{x} < \left(\frac{p + \cos_p(x)}{p+1}\right)^{\beta},$$

Where $a = (\log(\pi_p/2))/\log[(p+1)/p]$ and $\beta = 1$ are the best constants.

Remark 4 For $p = 2$, the above inequalities are due to C.-P. Chen and W.-S. Cheung [8].

Proof of theorems

In order to establish our main results we need following lemma:

Lemma 5 (L'Hopital Monotone Rule see [1]) *Let* $-\infty < a < b < \infty$, *and let* $f, g : [a,b] \to ¡$ *be continuous functions that are differentiable on* (a,b), *with* $f(a) = g(a) = 0$ *or* $f(b) = g(b) = 0$. *Assume that* $g'(x) \ne 0$ *for each* $x \in (a,b)$.
If f'/g' *is increasing (decreasing) on* (a,b), *then so is* f/g.

*Proof of Theorem*1 By differention, we have

$$f'(x) = \frac{1}{\sin_p(x)^2} g(x),$$

With $g(x) = p\sin_p(x) + \sin_p(x)\cos_p(x) - px\cos_p(x) - x\cos_p(x)^{2-p}$.
a simple computation leads to

$$g'(x) = \cos_p(x)^{2-p}[-2\sin_p(x)^p + px\sin_p(x)^{p-1} + (2-p)x\cos_p(x)^{1-p}\sin_p(x)^{p-1}]$$
$$= \cos_p(x)^{2-p}\sin_p(x)^{p-1}[px - 2\sin_p(x) + (2-p)x\cos_p(x)^{1-p}]$$

$$= \cos_p(x)^{2-p} \sin_p(x)^{p-1} h(x),$$

where

$$h(x) = px - 2\sin_p(x) + (2-p)x\cos_p(x)^{1-p},$$

and

$$h'(x) = p - 2\cos_p(x) + (2-p)\cos_p(x)^{1-p} + (2-p)(p-1)x\cos_p(x)^{2-2p}\sin_p(x)^{p-1}$$
$$> (2-p)(\cos_p(x)^{1-p} - \cos_p(x)) > 0.$$

Hence $h(x) > h(0) = 0$, therefor $g'(x) > 0$, $f(x)$ is strictly increasing on $(0, \pi_p/2)$,

$$p + 1 = f(0^+) < f(x) < f(\pi_p/2) = p\pi_p/2.$$

Proof of Theorem 2. Write

$$F_1(x) \equiv \sin_p(x) - x\cos_p(x), \text{and } F_2(x) \equiv x^2 \sin_p(x)^{p-1}\cos_p(x)^{2-p},$$

then $F_1(0) = 0$, $F_2(0) = 0$, by simple computations,

$$\frac{F_2'(x)}{F_1'(x)}$$
$$= \frac{2x\cos_p(x)^{2-p}\sin_p(x)^{p-1} + (p-1)x^2\cos_p(x)^{3-p}\sin_p(x)^{p-2} + (p-2)x^2\cos_p(x)^{3-2p}\sin_p(x)^{2p-2}}{x\cos_p(x)^{2-p}\sin_p(x)^{p-1}}$$
$$= 2 + (p-1)x/\tan_p(x) + (p-2)x\tan_p(x)^{p-1}.$$

Which is strictly decreasing, by lamma 5, $\frac{F_2(x)}{F_1(x)}$ is strictly decreasing on $(0, \pi_p/2)$,

$F(x)$ is strictly increasing on $(0, \pi_p/2)$, leads to

$F(0^+) < F(x) < F(\pi_p/2)$. But

$$F(0^+) = \lim_{x\to 0^+} \frac{F_1(x)}{F_2(x)} = \lim_{x\to 0^+} \frac{F_1'(x)}{F_2'(x)}$$
$$= \lim_{x\to 0^+} \frac{1}{2 + (p-1)x/\tan_p(x) + (p-2)x\tan_p(x)^{p-1}} = \frac{1}{p+1}.$$

Theorem 2 is proved.

Proof of Theorem 3. Write

$$G_1(x) \equiv \ln\frac{\sin_p x}{x}, \text{and } G_2(x) \equiv \ln\left(\frac{p + \cos_p(x)}{p+1}\right),$$

then $G_1(0) = 0$, $G_2(0) = 0$, by simple computations,

$$\frac{G_1'(x)}{G_2'(x)} = \frac{\sin_p(x) - x\cos_p(x)}{x\sin_p(x)} \cdot \frac{p + \cos_p(x)}{\sin_p(x)^{p-1}\cos_p(x)^{2-p}}$$
$$= \frac{x(p + \cos_p(x))}{\sin_p(x)} \cdot \frac{\sin_p(x) - x\cos_p(x)}{x^2\sin_p(x)^{p-1}\cos_p(x)^{2-p}} = f(x)F(x).$$

By theorem 1 and theorem 2, the functions $f(x), F(x)$ are strictly increasing on $(0, \pi_p/2)$, $f(x) \geq 0, F(x) \geq 0$. Thus

$\frac{G_1'(x)}{G_2'(x)}$ is strictly increasing on $(0, \pi_p/2)$, by lamma 5, the function

$\frac{G_1(x)}{G_2(x)}$ is strictly increasing on $(0, \pi_p/2)$, and we have

$$1 = G(0^+) < G(x) = \frac{\ln(\sin_p(x)/x)}{\ln[(p+\cos_p(x))/(p+1)]} < G(\pi_p/2) = \frac{\ln(\pi_p/2)}{\ln((p+1)/p)}.$$

Theorem 3 is proved.

References

[1] G. D. Anderson, M. K. Vamanamurthy, M.Vuorinen, Monotonicity rules in calculus, Amer. Math. Monthly133(2006) 805-816.

[2] G. D. Anderson, M.Vuorinen, X.-H. Zhang , Topics on special functions III. arXiv:1209. 1696v1[math.CA].

[3] E. Neuman, J. Sandor, On Some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inequal. Appl. 13(2010), 715-423.

[4] B. A. Bhayo, M.Vuorinen, On generalized trigonometric functions with two parameters, J. Approx.Theory 164(2012), 1415-1426.

[5] B. A. Bhayo, M. Vuorinen, Inequalities for eigenfunctions of the p-Laplacian. arXiv:1101.3911v3 [math.CA].

[6] R. Klén, M. Vuorinen, X. –H. Zhang, Inequalities for the generalized trigonometric and hyperbolic functions, J. Math.Anal. and Appl.409 (2014), 521-529.

[7] P. Lindqvist, Note on a nonlinear eigenvalue problem , Rocky Mountain J.Math.23(1993), 281-288.

[8] C. –P. Chen, W. –S. Cheug, Sharp Cusa and Becker-Stark inequalities, J. Inequal. Appl.(2011), Article136(2011).

[9] D. E. Edmunds, P. Gurka, J. Lang. Properties of generalized trigonometric functions, J. Approx. Theory 164(2012), 47-56.