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597. ANALYTIC INEQUALITIES OBTAINED BY QUADRATIC APPROXIMATION*

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In his paper *On Quadratic Approximation*, the author indicated several elementary quadratic approximations of selected functions without proof [1]. We shall establish these results as analytic inequalities

$$\operatorname{arctg} x > \frac{8x}{3+\sqrt{25+\frac{80}{3}x^2}} \qquad (x>0). \tag{1}$$

Using a method of L. S. GRINSTEIN, we determine an inequality that the derivative must satisfy:

$$\frac{1}{1+u^2} > \frac{8}{3+\sqrt{25+\frac{80}{3}u^2}} - \frac{640\,u^2}{3\sqrt{25+\frac{80}{3}u^2}\left[3+\sqrt{25+\frac{80}{3}u^2}\right]^2}. \tag{2}$$

Integrating the inequality (2) from 0 to x, we obtain the desired result.

Let $u^2 = v + \frac{4}{15}v^2$, $v \geqq 0$, defining the parameter v. Then (2) becomes

$$\frac{1}{1+v+\frac{4}{15}v^2} > \frac{1+\frac{v}{5}}{\left(1+\frac{8}{15}v\right)\left(1+\frac{1}{3}v\right)^2} \tag{3}$$

which is clearly evident. The error is $O(x^7)$ for small values of x. The largest error is at $x=\infty$ with an error of 1.5%.

A related inequality is:

$$\frac{1}{2}\log\frac{1+x}{1-x} = \operatorname{arth} x < \frac{8x}{3+\sqrt{25-\frac{80}{3}x^2}} \qquad \left(0<x\leqq\frac{\sqrt{15}}{4}\right). \tag{4}$$

In (2), replace u^2 by $-w^2$ and reverse the inequality. Let $w^2 = v - \frac{4}{15}v^2$, $v \geqq 0$ defining v as before. We get the inequality

$$\frac{1}{1-v+\frac{4}{15}v^2} < \frac{1-\frac{v}{5}}{\left(1-\frac{8}{15}v\right)\left(1-\frac{v}{3}\right)^2} \tag{5}$$

which may be easily verified. Again, by integration, we get (4).

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Making a change of variable $x=\frac{z}{2+z}$ we get from (4)

(6) $$\log(1+z)<\frac{16z}{6+3z+\sqrt{100+100z-\frac{5}{3}z^2}}\qquad (0<z\leqq 30+8\sqrt{15}).$$

The author is indebted to Y. L. Luke for this suggestion.

We directly obtain from (1), an inequality for arcsin z. Let $x=\operatorname{tg} z$. Then

(7) $$z>\frac{8\operatorname{tg} z}{3+\sqrt{25+\frac{80}{3}\operatorname{tg}^2 z}}=\frac{8\sin z}{3\cos z+\sqrt{25+\frac{5}{3}\sin^2 z}}$$

and so

(8) $$\arcsin z>\frac{8z}{3\sqrt{1-z^2}+\sqrt{25+\frac{5}{3}z^2}}$$

$$=\frac{8z}{3\sqrt{1-z^2}+5+\frac{z^2}{6}+\left[\sqrt{25+\frac{5}{3}z^2}-5-\frac{1}{6}z^2\right]}$$

$$>\frac{8z}{5+\frac{1}{6}z^2+3\sqrt{1-z^2}}\qquad (0<z\leqq 1).$$

The error in the last step of the inequality is negligible compared to the total error of the inequality (7).

REFERENCE

1. R. E. Shafer: *On Quadratic Approximation.* SIAM Journal of Numerical Analysis, April 1974 pp. 447—460.

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