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Research Article

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Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings

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Abstract: In this paper, we introduce a shrinking projection method of an inertial type with self-adaptive step size for finding a common element of the set of solutions of a split generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces. The self-adaptive step size incorporated helps to overcome the difficulty of having to compute the operator norm, while the inertial term accelerates the rate of convergence of the proposed algorithm. Under standard and mild conditions, we prove a strong convergence theorem for the problems under consideration and obtain some consequent results. Finally, we apply our result to solve split mixed variational inequality and split minimization problems, and we present numerical examples to illustrate the efficiency of our algorithm in comparison with other existing algorithms. Our results complement and generalize several other results in this direction in the current literature.

Keywords: inertial, split generalized equilibrium problems, self-adaptive, step size, nonexpansive multivalued mappings, firmly nonexpansive mapping, fixed point problems

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$, $F : C \times C \to \mathbb{R}$ be two bifunctions. The generalized equilibrium problem (GEP) is to find a point $x^* \in C$ such that

$$F(x^*, y) + \phi(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

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The solution set of the GEP is denoted by $\mathrm{GEP}(F, \phi)$. In particular, if we set $\phi = 0$ in (1), then the GEP reduces to the classical equilibrium problem (EP), which is to find a point $x^* \in C$ such that $F(x^*, y) \geq 0$, $\forall\, y \in C$. The solution set of EP is denoted by $\mathrm{EP}(F)$.

The EP is a generalization of many mathematical models such as variational inequality problems (VIPs), fixed point problems (FPPs), certain optimization problems (OPs), Nash EPs, minimization problems (MPs), and others, see [1,2]. Many authors have studied and proposed several iterative algorithms for solving EPs and related OPs, see [3–18].

In 2013, Kazmi and Rizvi [19] introduced and studied the following split generalized equilibrium problem (SGEP): let $C \subseteq H_1$ and $Q \subseteq H_2$, where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The SGEP is defined as follows: find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{2}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{3}$$

We denote the solution set of SGEP (2)–(3) by

$$\mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2) \coloneqq \{x^* \in C : x^* \in \mathrm{GEP}(F_1, \phi_1) \text{ and } Ax^* \in \mathrm{GEP}(F_2, \phi_2)\}.$$

Furthermore, an iterative algorithm was also presented by the authors for approximating the solution of SGEP in a real Hilbert space. If $\phi_1 = 0$ and $\phi_2 = 0$, then the SGEP reduces to split equilibrium problem (SEP), which is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{4}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{5}$$

Observe that (4) is the classical EP. Therefore, the inequalities (4) and (5) comprise a pair of EPs, which involves finding the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of (4) in H_1, which is the solution of (5) in H_2. The solution set of SEP (4)–(5) is denoted by $\mathrm{SEP}(F_1, F_2) \coloneqq \{z \in \mathrm{EP}(F_1) : Az \in \mathrm{EP}(F_2)\}$.

Another important problem in fixed point theory is the fixed point problem (FPP), which is defined as follows:

$$\text{Find a point } x^* \in C \text{ such that } Sx^* = x^*, \tag{6}$$

where $S : C \to C$ is a nonlinear operator. If S is a multivalued mapping, i.e., $S : C \to 2^C$, then $x^* \in C$ is called a fixed point of S if

$$x^* \in Sx^*. \tag{7}$$

We denote the set of fixed points of S by $F(S)$. The fixed point theory for multivalued mappings can be utilized in various areas such as game theory, control theory, and mathematical economics.

In this article, we are interested in studying the problem of finding a common solution for both the SGEP (2)–(3) and the common FPP for multivalued mappings. The motivation for studying such problems is in its potential application to mathematical models whose constraints can be expressed as FPP and SGEP. This occurs, in particular, in practical problems such as signal processing, network resource allocation, and image recovery. A scenario is in network bandwidth allocation problem for two services in heterogeneous wireless access networks in which the bandwidth of the services is mathematically related (see, for instance, [20,21] and references therein).

In 2016, Suantai et al. [22] introduced the following iterative scheme for solving SEP and FPP of non-spreading multi-valued mapping in Hilbert spaces:

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}\big(I - \gamma A^*(I - T_{r_n}^{F_2})A\big)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \end{cases} \tag{8}$$

for all $n \geq 1$, where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$, S is a nonspreading multivalued mapping, and $\gamma \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of the bounded linear operator A. Under the following conditions on the control sequences:
(i) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$; and
(ii) $\liminf_{n\to\infty} r_n > 0$,
the authors proved that the sequence $\{x_n\}$ defined by (8) converges weakly to $p \in F(S) \cap \mathrm{SEP}(F_1, F_2) \neq \varnothing$.

Bauschke and Combettes [23] pointed out that in solving OPs, strong convergence of iterative schemes is more desirable than their weak convergence counterparts. Hence, there is a the need to construct iterative schemes that generate a strong convergence sequence.

Takahashi et al. [24] introduced an iterative scheme known as the shrinking projection method for finding a fixed point of a nonexpansive single-valued mapping in Hilbert spaces. The shrinking projection method is a famous method, which plays a significant role in mastering strong convergence for finding fixed points of nonlinear mappings. The method has received much attention due to its applications, and it has been developed to solve many problems, such as, EPs, VIPs, and FPPs in Hilbert spaces (see, for example, [25]).

Very recently, Phuengrattana and Lerkchaiyaphum [26] introduced the following shrinking projection method for solving SGEP and FPP of a countable family of nonexpansive multivalued mappings: for $x_1 \in C$ and $C_1 = C$, then

$$\begin{cases} u_n = T_{r_n}^{(F_1,\phi_1)}\left(I - \gamma A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)A\right)x_n, \\ z_n = \alpha_n^{(0)}x_n + \alpha_n^{(1)}y_n^{(1)} + \cdots + \alpha_n^{(n)}y_n^{(n)}, \quad y_n^{(i)} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}. \end{cases} \tag{9}$$

They proved that if
(i) $\liminf_{n\to\infty} r_n > 0$,
(ii) The limits $\lim_{n\to\infty}\alpha_n^{(i)} \in (0, 1)$ exist for all $i \geq 0$,
then the sequence $\{x_n\}$ generated by (9) converges strongly to $P_\Gamma x_1$, where $\Gamma = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2) \neq \varnothing$ and S_i is a countable family of nonexpansive multivalued mappings.

It is important to point out at this point that the step size γ of the aforementioned algorithm plays an essential role in the convergence properties of iterative methods. The result obtained by the authors in [22,26] and several other related results in the literature involve step size that requires prior knowledge of the operator norm $\|A\|$. One of the drawbacks of such algorithms is that they are usually not easy to implement because they require computation of the operator norm $\|A\|$, which is very difficult if not impossible to calculate or even estimate. Moreover, the step size defined by such algorithms is often very small and deteriorates the convergence rate of the algorithm. In practice, a larger stepsize can often be used to yield better numerical results.

Based on the heavy ball methods of a two-order time dynamical system, Polyak [27] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative algorithms by using inertial extrapolation (see, e.g., [1,28–32]).

Motivated by the above results and the ongoing research interest in this direction, in this paper, we present a new inertial shrinking projection algorithm, which does not require any prior knowledge of the operator norm for finding a common element of the set of solutions of SGEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings in Hilbert spaces. We prove strong convergence theorem for the proposed algorithm and obtain some consequent results. Moreover, we apply our results to solve split mixed variational inequality problem (SMVIP) and split minimization problem (SMP), and we provide numerical examples to illustrate the efficiency of the proposed algorithm in comparison with the existing results in the current literature.

The remaining sections of the paper are organized as follows. In Section 2, we recall some basic definitions and results that will be employed in the convergence analysis of our proposed algorithm. Our new inertial shrinking projection algorithm is presented and analyzed in Section 3, and we also obtain some consequent results. In Section 4, we apply our result to solve SMVIP and SMP. In Section 5, we present some numerical experiments to demonstrate the validity and efficiency of our proposed method in comparison with some recent results in the literature. Finally, in Section 6, we give the concluding remarks.

2 Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with an inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is *firmly nonexpansive*, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle, \tag{10}$$

for all $x, y \in H$. Moreover, $\langle x - P_C x, y - P_C x \rangle \le 0$ holds for all $x \in H$ and $y \in C$, see [33,34]. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point x in a Hilbert space H by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is well known [35] that a Hilbert space H satisfies *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{11}$$

holds for every $y \in H$ with $y \ne x$.

Definition 2.1. A single-valued mapping $S : C \to C$ is said to be

- *nonexpansive*, if and only if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C;$$

- *δ-inverse strongly monotone* [36], if there exists a positive real number δ such that

$$\langle x - y, Sx - Sy \rangle \ge \delta \|Sx - Sy\|^2, \quad \forall x, y \in C;$$

- *monotone*, if and only if

$$\langle y - x, Sy - Sx \rangle \ge 0, \quad \forall x, y \in C.$$

If S is δ-inverse strongly monotone, for each $\gamma \in (0, 2\delta]$, it is known [26] that $I - \gamma S$ is a nonexpansive single-valued mapping.

A subset K of H is called *proximal* if for each $x \in H$, there exists $y \in K$ such that

$$\|x - y\| = \mathrm{d}(x, K).$$

We denote by $CB(C)$, $CC(C)$, $K(C)$, and $P(C)$ the families of all nonempty closed bounded subsets of C, nonempty closed convex subset of C, nonempty compact subsets of C, and nonempty proximal bounded subsets of C, respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} \mathrm{d}(x, B), \sup_{y\in B} \mathrm{d}(y, A)\},$$

for all $A, B \in CB(C)$. Let $S : C \to 2^C$ be a multivalued mapping. We say that S satisfies the *endpoint condition* if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : C \to 2^C (i \in \mathbb{N})$ with $\bigcap_{i=1}^{\infty} F(S_i) \ne \emptyset$, we say S_i satisfies the *common endpoint condition* if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}$, $p \in \bigcap_{i=1}^{\infty} F(S_i)$. We recall some basic and useful definitions on multivalued mappings.

Definition 2.2. A multivalued mapping $S : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Sx, Sy) \leq \|x - y\|, \quad \forall x, y \in C.$$

The class of nonexpansive multivalued mappings contains the class of nonexpansive single-valued mappings. If S is a nonexpansive single-valued mapping on a closed convex subset of a Hilbert space, then $F(S)$ is closed and convex. The closedness of $F(S)$ can easily be extended to the multivalued case. However, the convexity of $F(S)$ cannot be extended (see, e.g., [37]). But, if S is a nonexpansive multivalued mapping which satisfies the endpoint condition, then $F(S)$ is always closed and convex as shown by the following result:

Lemma 2.3. [38] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to CB(C)$ be a nonexpansive multivalued mapping with $F(S) \neq \emptyset$ and $Sp = \{p\}$ for each $p \in F(S)$. Then, $F(S)$ is a closed and convex subset of C.*

The *best approximation operator* P_S for a multivalued mapping $S : C \to P(C)$ is defined by

$$P_S(x) := \{y \in Sx : \|x - y\| = d(x, Sx)\}.$$

It is known that $F(S) = F(P_S)$ and P_S satisfies the endpoint condition. Song and Cho [39] gave an example of a best approximation operator P_S which is nonexpansive, but where S is not necessarily nonexpansive.

The following results will be needed in the sequel:

Lemma 2.4. [40] *In a real Hilbert space H, the following inequalities hold for all $x, y \in H$:*
(*i*) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(*ii*) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(*iii*) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.

Lemma 2.5. [41] *Let H be a Hilbert space, $\{x_n\}$ be a sequence in H, and $\alpha_1, \alpha_2, \ldots, \alpha_N$* be real numbers such that $\sum_{i=1}^{N} \alpha_i = 1$. *Then,*

$$\left\| \sum_{i=1}^{N} \alpha_i x_i \right\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2. \tag{12}$$

Lemma 2.6. [42] *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Let $u, v \in H$ be such that* $\lim_{n\to\infty}\|x_n - u\|$ *and* $\lim_{n\to\infty}\|x_n - v\|$ *exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v respectively, then $u = v$.*

Lemma 2.7. [43] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and a real number α, the set $\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u\rangle + \alpha\}$ is closed and convex.*

Lemma 2.8. [44,45] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then,*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in C.$$

Assumption 2.9. Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following conditions:
(A1) $F_1(x, x) = 0$ for all $x \in C$,
(A2) F_1 is monotone, that is, $F_1(x, y) + F_1(y, x) \leq 0$ for all $x, y \in C$,
(A3) F_1 is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t\downarrow 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$,
(A4) for each $x \in C$, $y \mapsto F_1(x, y)$ is convex and lower semicontinuous,
(A5) $\phi_1(x, x) \geq 0$ for all $x \in C$,
(A6) for each $y \in C$, $x \mapsto \phi_1(x, y)$ is upper semicontinuous,
(A7) for each $x \in C \mapsto \phi_1(x, y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y, x) + \phi_1(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C\backslash K.$$

Lemma 2.10. [46] *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying* Assumption 2.9. *Assume that ϕ_1 is monotone. For $r > 0$ and $x \in H_1$, define a mapping $T_r^{(F_1,\phi_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,\phi_1)}(x) = \left\{z \in C : F_1(z, y) + \phi_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}, \tag{13}$$

for all $x \in H_1$, Then,

(i) for each $x \in H_1$, $T_r^{(F_1,\phi_1)} \neq \emptyset$,
(ii) $T_r^{(F_1,\phi_1)}$ is single-valued,
(iii) $T_r^{(F_1,\phi_1)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,

$$\|T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y\|^2 \leq \langle T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y, x - y\rangle,$$

(iv) $F(T_r^{(F_1,\phi_1)}) = \mathrm{GEP}(F_1, \phi_1)$,
(v) $\mathrm{GEP}(F_1, \phi_1)$ is compact and convex.

Furthermore, assume that $F_2 : Q \times Q \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$ satisfy Assumption 2.9, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For all $s > 0$ and $w \in H_2$, define the mapping $T_s^{(F_2,\phi_2)} : H_2 \to Q$ by

$$T_s^{(F_2,\phi_2)}(v) = \left\{w \in Q : F_2(w, d) + \phi_2(w, d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall d \in Q\right\}. \tag{14}$$

Then, we have

(vi) for each $v \in H_2$, $T_s^{(F_2,\phi_2)} \neq \emptyset$,
(vii) $T_s^{(F_2,\phi_2)}$ is single-valued,
(viii) $T_s^{(F_2,\phi_2)}$ is firmly nonexpansive,
(ix) $F(T_s^{(F_2,\phi_2)}) = \mathrm{GEP}(F_2, \phi_2)$,
(x) $\mathrm{GEP}(F_2, \phi_2)$ is closed and convex,

where $\mathrm{GEP}(F_2, \phi_2)$ is the solution set of the following GEP: find $y^* \in Q$ such that

$$F_2(y^*, y) + \phi_2(y^*, y) \geq 0 \quad \forall y \in Q.$$

Moreover, $\mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2)$ is a closed and convex set.

Lemma 2.11. [47] *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ and $\phi_1 : C \times C \to \mathbb{R}$ be two bifunctions satisfying* Assumption 2.9, *and let $T_r^{(F_1,\phi_1)}$ be defined as in* Lemma 2.10 *for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then,*

$$\left\|T_{r_2}^{(F_1,\phi_1)}y - T_{r_1}^{(F_1,\phi_1)}x\right\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \left\|T_{r_2}^{(F_1,\phi_1)}y - y\right\|.$$

3 Main results

In this section, we state and prove our strong convergence theorem for finding a common element of the set of solutions of SGEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings in real Hilbert spaces.

Theorem 3.1. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$, $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying* Assumption 2.9. *Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2) \neq \varnothing$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}\left(I - \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}y_{n,i}, \quad y_{n,i} \in S_i u_n, \\ C_{n+1} = \{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{15}$$

$$\gamma_n = \begin{cases} \dfrac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})\,Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})\,Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}), \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1$, $\{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0,1)$, such that $\sum_{i=0}^{n} \alpha_{n,i} = 1$, and $\{r_n\} \subset (0, \infty)$. Suppose that the following conditions hold:
(C1) $\liminf_{n\to\infty} r_n > 0$,
(C2) the limits $\lim_{n\to\infty}\alpha_{n,i} \in (0,1)$ *exist for all $i \geq 0$.*
Then, the sequence $\{x_n\}$ generated by (15), *converges strongly to $P_\Omega x_1$.*

Proof. We divide the proof into several steps as follows:
Step 1: First, we show that $\{x_n\}$ is well-defined for every $n \in \mathbb{N}$.
By Lemmas 2.3 and 2.10, we have that $\mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2)$ and $\bigcap_{i=1}^{\infty} F(S_i)$ are closed and convex subsets of C. Therefore, the solution set Ω is a closed and convex subset of C. By Lemma 2.7, it then follows that C_{n+1} is closed and convex for each $n \in \mathbb{N}$. Let $p \in \Omega$, then we have $p = T_{r_n}^{F_1,\phi_1}p$ and $Ap = T_{r_n}^{(F_2,\phi_2)}(Ap)$. Since $T_{r_n}^{(F_1,\phi_1)}$ is nonexpansive, by Lemma 2.4, we have

$$\begin{aligned} \|u_n - p\|^2 &= \left\|T_{r_n}^{(F_1,\phi_1)}\left(w_n - \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right) - p\right\|^2 \\ &\leq \left\|w_n - \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n - p\right\|^2 \\ &= \left\|w_n - p\|^2 + \gamma_n^2\left\|A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2 - 2\gamma_n\left\langle w_n - p, A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle. \end{aligned} \tag{16}$$

By the firmly nonexpansivity of $I - T_{r_n}^{(F_2,\phi_2)}$, we get

$$\begin{aligned} \left\langle w_n - p, A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle &= \left\langle Aw_n - Ap, \left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle \\ &= \left\langle Aw_n - Ap, \left(I - T_{r_n}^{(f_2,\phi_2)}\right)Aw_n - \left(I - T_{r_n}^{(F_2,\phi_2)}\right)Ap\right\rangle \\ &\geq \left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2. \end{aligned} \tag{17}$$

By substituting (17) into (16), applying the definition of γ_n and the condition on τ_n, we obtain

$$\begin{aligned} \|u_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\left\|A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2 - 2\gamma_n\left\|\left(I - T_{r_n}^{(F_1,\phi_1)}\right)Aw_n\right\|^2 \\ &= \|w_n - p\|^2 - \gamma_n\left[2\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2 - \gamma_n\left\|A^*\left(I - T_{r_n}^{(F_1,\phi_1)}\right)Aw_n\right\|^2\right] \\ &= \|w_n - p\|^2 - \gamma_n(2 - \tau_n)\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2 && (18) \\ &\leq \|w_n - p\|^2. && (19) \end{aligned}$$

Applying Lemma 2.5 and using (19), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}y_{n,i} - p \right\|^2 \\
&\le \alpha_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{n} \alpha_{n,i}\|y_{n,i} - p\|^2 - \alpha_{n,0}\sum_{i=1}^{n} \alpha_{n,i}\|u_n - y_{n,1}\|^2 \\
&= \alpha_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{n} \alpha_{n,i}d(y_{n,i}, S_ip)^2 - \alpha_{n,0}\sum_{i=1}^{n} \alpha_{n,i}\|u_n - y_{n,i}\|^2 \\
&\le \alpha_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{n} \alpha_{n,i}H(S_iu_n, S_ip)^2 - \alpha_{n,0}\sum_{i=1}^{n} \alpha_{n,i}\|u_n - y_{n,i}\|^2 \\
&\le \alpha_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{n} \alpha_{n,i}\|u_n - p\|^2 - \alpha_{n,0}\sum_{i=1}^{n} \alpha_{n,i}\|u_n - y_{n,i}\|^2 \\
&\le \|u_n - p\|^2 - \alpha_{n,0}\sum_{i=1}^{n} \alpha_{n,i}\|u_n - y_{n,i}\|^2 \qquad (20) \\
&\le \|u_n - p\|^2. \qquad (21)
\end{aligned}$$

Also, by applying Lemma 2.4(iii), we get

$$\|w_n - p\|^2 = \|(x_n - p - \theta_n(x_{n-1} - x_1))\|^2 = \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2. \tag{22}$$

By using (19) and (22) in (21), we have

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2. \tag{23}$$

This shows that $p \in C_{n+1}$, and it follows that $\Omega \subset C_{n+1} \subset C_n$. Therefore, $P_{C_{n+1}}x_1$ is well-defined for every $x_1 \in C$ and the sequence $\{x_n\}$ is well defined.

Step 2: Next, we show that $\lim_{n\to\infty} x_n = q$ for some $q \in C$.
We know that Ω is a nonempty closed convex subset of H_1, then there exists a unique $w \in \Omega$ such that $w = P_\Omega x_1$. Since $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$, we have

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|, \quad \forall n \in \mathbb{N}. \tag{24}$$

Similarly, since $\Omega \subset C_n$, we have

$$\|x_n - x_1\| \le \|w - x_1\|, \quad \forall n \in \mathbb{N}. \tag{25}$$

Therefore, $\{\|x_n - x_1\|\}$ is bounded, and it follows that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{y_{n,i}\}$ are bounded. Hence, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. From the construction of C_n, it is clear that $x_m = P_{C_m}x_1 \in C_m \subset C_n$ for $m > n \ge 1$. By Lemma 2.8, we have that

$$\|x_m - x_n\|^2 \le \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \to 0 \quad \text{as } m, n \to \infty. \tag{26}$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, then it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of H_1 and the closedness of C, we have that there exists an element $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 3: We next show that $\lim_{n\to\infty}\|y_{n,i} - u_n\| = 0$ for all $i \in \mathbb{N}$.
From (26), we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{27}$$

Since $x_{n+1} \in C_{n+1}$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - 2\theta_n\langle x_n - x_{n+1}, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2.$$

By (27), we obtain

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0. \tag{28}$$

By applying (27) and (28), we get

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad n \to \infty. \tag{29}$$

Hence, $\lim_{n\to\infty} z_n = q$.

By the triangle inequality, we have that

$$\|w_n - x_n\| = \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| \le \|x_n - x_n\| + \theta_n\|x_n - x_{n-1}\|.$$

By (27), we obtain

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{30}$$

Applying (29) and (30), we get

$$\|z_n - w_n\| \le \|z_n - x_n\| + \|x_n - w_n\| \to 0, \quad n \to \infty. \tag{31}$$

From (19) and (20), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n - y_{n,i}\|^2,$$

which implies that

$$\alpha_{n,0}\alpha_{n,i}\|u_n - y_{n,i}\|^2 \le \alpha_{n,0}\sum_{i=1}^{n}\alpha_{n,i}\|u_n - y_{n,i}\|^2 \le \|w_n - p\|^2 - \|z_n - p\|^2 \le \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).$$

By the conditions on $\{\alpha_{n,i}\}$ and using (31), we get

$$\lim_{n\to\infty}\|u_n - y_{n,i}\| = 0, \quad \forall i \in \mathbb{N}. \tag{32}$$

Step 4: We show that $\|u_n - x_n\| = 0$.
Substituting (18) into (21), we have

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \gamma_n(2 - \tau_n)\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2. \tag{33}$$

From this, we obtain

$$\gamma_n(2 - \tau_n)\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2 \le \|w_n - p\|^2 - \|z_n - p\|^2 \le \|w_n - z_n\|(\|w_n - p\| + \|z_n - p\|).$$

By the definition of γ_n, condition on τ_n and (31), we get

$$\frac{\tau_n(2 - \tau_n)\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^4}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} \to 0, \quad n \to \infty,$$

which implies that

$$\frac{\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|} \to 0, \quad n \to \infty.$$

Since $\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|$ is bounded, it follows that

$$\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\| \to 0, \quad n \to \infty. \tag{34}$$

From this, we obtain

$$\left\|A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\| \le \left\|A^*\right\|\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\| = \left\|A\right\|\left\|\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\| \to 0, \quad n \to \infty. \tag{35}$$

Since $T_{r_n}^{(F_1,\phi_1)}$ is firmly nonexpansive and $I-\gamma_n A^*(I-T_{r_n}^{(F_2,\phi_2)})A$ is non expansive by invoking Lemma 2.4(ii), we obtain

$$\begin{aligned}
\|u_n-p\|^2 &= \left\|T_{r_n}^{(F_1,\phi_1)}\left(I-\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n - T_{r_n}^{(F_1,\phi_1)}p\right\|^2\\
&\le \left\langle T_{r_n}^{(F_1,\phi_1)}\left(I-\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n - T_{r_n}^{(F_1,\phi_1)}p,\ \left(I-\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n - p\right\rangle\\
&= \left\langle u_n-p,\ \left(I-\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n - p\right\rangle\\
&= \frac{1}{2}\left[\|u_n-p\|^2 + \left\|\left(I-\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n-p\right\|^2 - \left\|u_n-w_n+\gamma_n A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2\right]\\
&\le \frac{1}{2}\Bigg[\|u_n-p\|^2 + \|w_n-p\|^2 - \Bigg(\|u_n-w_n\|^2 + \gamma_n^2\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|^2\\
&\quad + 2\gamma_n\left\langle u_n-w_n, A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle\Bigg)\Bigg],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_n-p\|^2 &\le \|w_n-p\|^2 - \|u_n-w_n\|^2 - \gamma_n^2\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\| + 2\gamma_n\left\langle w_n-u_n, A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle\\
&\le \|w_n-p\|^2 - \|u_n-w_n\|^2 + 2\gamma_n\|w_n-u_n\|\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|.
\end{aligned}\tag{36}$$

Substituting (36) into (20), we have

$$\|z_n-p\|^2 \le \|w_n-p\|^2 - \|u_n-w_n\|^2 + 2\gamma_n\|w_n-u_n\|\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|.$$

From this, we get

$$\begin{aligned}
\|u_n-w_n\|^2 &\le \|w_n-p\|^2 - \|z_n-p\|^2 + 2\gamma_n\|w_n-u_n\|\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|\\
&\le \|w_n-p\|^2 - \|z_n-p\|^2 + 2\gamma_n M\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|\\
&\le \|w_n-z_n\|(\|w_n-p\|+\|z_n-p\|) + 2\gamma_n M\left\|A^*\left(I-T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\|,
\end{aligned}\tag{37}$$

where $M=\sup\{\|w_n-u_n\| : n\in\mathbb{N}\}$.

By applying (31) and (35) in (37), we get

$$\lim_{n\to\infty}\|u_n-w_n\|=0. \tag{38}$$

Combining this together with (30) and (31), we have

$$\|u_n-z_n\| \le \|u_n-w_n\| + \|w_n-z_n\| \to 0, \quad n\to\infty \tag{39}$$

and

$$\|u_n-x_n\| \le \|u_n-w_n\| + \|w_n-x_n\| \to 0, \quad n\to\infty. \tag{40}$$

Step 5: Next, we show that $q\in\bigcap_{i=1}^{\infty}F(S_i)$.
By (32), for all $i\in\mathbb{N}$, we get that

$$\lim_{n\to\infty} d(u_n, S_i u_n) \le \lim_{n\to\infty}\|u_n-y_{n,i}\| = 0. \tag{41}$$

For each $i\in\mathbb{N}$, we have

$$\begin{aligned}
d(q,S_iq) &\le \|q-u_n\| + \|u_n-y_{n,i}\| + d(y_{n,i}, S_iq)\\
&\le \|q-u_n\| + d(u_n,S_iu_n) + H(S_iu_n,S_iq)\\
&\le 2\|q-u_n\| + d(u_n,S_iu_n).
\end{aligned}$$

By (40), we have that $\lim_{n\to\infty}u_n=q$. Then, it follows from (41) that

$$d(q,S_iq)=0 \quad \forall i\in\mathbb{N}.$$

This shows that $q \in S_i q$ for all $i \in \mathbb{N}$, which implies that $q \in \bigcap_{i=1}^{\infty} F(S_i)$.

Step 6: Next, we show that $q \in \mathrm{GEP}(F_1, \phi_1, F_2, \phi_2)$.
First, we will show that $q \in \mathrm{GEP}(F_1, \phi_1)$. Since $u_n = T_{r_n}^{(F_1,\phi_1)}\,(I - \gamma_n A^*(I - T_{r_n}^{(F_2,\phi_2)})A)\,w_n$, then by Lemma 2.10, we obtain

$$F_1(u_n, y) + \phi_1(u_n, y) + \frac{1}{r_n}\left\langle y - u_n, u_n - w_n - \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle \geq 0, \quad \forall y \in C,$$

which implies that

$$F_1(u_n, y) + \phi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle - \frac{1}{r_n}\left\langle y - u_n, \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle \geq 0, \quad \forall y \in C.$$

Since F_1 and ϕ_1 are monotone, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle - \frac{1}{r_n}\left\langle y - u_n, \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)Aw_n\right\rangle \geq F_1(y, u_n) + \phi_1(y, u_n), \quad \forall y \in C.$$

By (30) and (38), and $\lim_{n\to\infty} x_n = q$, we obtain $\lim_{n\to\infty} u_n = q$. Then, by Condition (C1), (34), (38), Assumption 2.9, (A4) and (A7), it follows that

$$0 \geq F_1(y, q) + \phi_1(y, q) \quad \forall y \in C.$$

Let $y_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Then, $y_t \in C$, and thus, $F_1(y_t, q) + \phi_1(y_t, q) \leq 0$. Therefore, by Assumption 2.9, (A1)–(A7), we obtain

$$\begin{aligned}
0 &\leq F_1(y_t, y_t) + \phi_1(y_t, y_t)\\
&\leq t\,(F_1(y_t, y) + \phi_1(y_t, y)) + (1-t)(F_1(y_t, q) + \phi_1(y_t, q))\\
&\leq t\,(F_1(y_t, y) + \phi_1(y_t, y)) + (1-t)(F_1(q, y_t) + \phi_1(q, y_t))\\
&\leq F_1(y_t, y) + \phi_1(y_t, y).
\end{aligned}$$

This implies that

$$F_1(y_t, y) + \phi_1(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, and by using assumption together with the upper hemicontinuity of ϕ_1, we obtain

$$F_1(q, y) + \phi_1(q, y) \geq 0, \quad \forall y \in C.$$

This implies that $q \in \mathrm{GEP}(F_1, \phi_1)$.

We next show that $Aq \in \mathrm{GEP}(F_2, \phi_2)$. Since A is a bounded linear operator, $Aw_n \to Aq$. Thus, from (34) we have

$$T_{r_n}^{(F_2,\phi_2)} Aw_n \to Aq. \tag{42}$$

By the definition of $T_{r_n}^{(F_2,\phi_2)} Aw_n$, we have

$$F_2\left(T_{r_n}^{(F_2,\phi_2)} Aw_n, y\right) + \phi_2\left(T_{r_n}^{(F_2,\phi_2)} Aw_n, y\right) + \frac{1}{r_n}\left\langle y - T_{r_n}^{(F_2,\phi_2)} Aw_n, T_{r_n}^{(F_2,\phi_2)} Aw_n - Aw_n\right\rangle \geq 0, \quad \forall y \in Q.$$

Since F_2 and ϕ_2 are upper semicontinuous in the first argument, it follows from (42) that,

$$F_2(Aq, y) + \phi_2(Aq, y) \geq 0, \quad \forall y \in Q.$$

This implies that $Aq \in \mathrm{GEP}(F_2, \phi_2)$. Hence, $q \in \mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2)$.

Step 7: Finally, we show that $q = P_\Omega x_i$.
We know that $x_n = P_{C_n} x_1$ and $\Omega \subset C_n$, then it follows that $\langle x_1 - x_n, x_n - p\rangle \geq 0$ for all $p \in \Omega$. Hence, we have $\langle x_1 - q, q - p\rangle \geq 0$ for all $p \in \Omega$. This implies that $q = P_\Omega x_1$.

Consequently, we can conclude by steps 1–8 that $\{x_n\}$ converges strongly to $q = P_\Omega x_1$ as required. □

If $\phi_1 = \phi_2 = 0$ in (2)–(3), then the SGEP reduces to the SEP. Hence, from Theorem 3.1, we obtain the following consequent result for approximating a common element of the set of solutions of SEP and the set of common fixed points of a countable family of nonexpansive multivalued mappings.

Corollary 3.2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of nonexpansive multivalued mappings of C into $CB(C)$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying* Assumption 2.9. *Let F_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{SEP}(F_1, F_2) \neq \emptyset$ and $S_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{F_1}\left(I - \gamma_n A^*\left(I - T_{r_n}^{F_2}\right)A\right)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}y_{n,i}, \quad y_{n,i} \in S_i u_n, \\ C_{n+1} = \left\{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\left\langle x_n - p, x_{n-1} - x_n\right\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{43}$$

$$\gamma_n = \begin{cases} \dfrac{\tau_n\|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} & \textit{if } Aw_n \neq T_{r_n}^{F_2}Aw_n, \\ \gamma & \textit{otherwise } (\gamma \textit{ being any nonnegative real number}), \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1$, $\{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0, 1)$, *such that* $\sum_{i=0}^{n} \alpha_{n,i} = 1$, *and* $\{r_n\} \subset (0, \infty)$. *Suppose that the following conditions hold:*
(*C1*) $\liminf_{n\to\infty} r_n > 0$,
(*C2*) *the limits* $\lim_{n\to\infty}\alpha_{n,i} \in (0, 1)$ *exist for all* $i \geq 0$.
Then, the sequence $\{x_n\}$ generated by (43), *converges strongly to* $P_\Omega x_1$.

By the properties of the best approximation operator, we obtain the following consequent result.

Corollary 3.3. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $\{S_i\}$ be a countable family of multivalued mappings of C into $P(C)$ such that P_{S_i} is nonexpansive. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$, $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying* Assumption 2.9. *Let ϕ_1, ϕ_2 be monotone, ϕ_1 be upper hemicontinuous, and F_2 and ϕ_2 be upper semicontinuous in the first argument. Assume that $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{SGEP}(F_1, \phi_1, F_2, \phi_2) \neq \emptyset$. Let $x_0, x_1 \in C$ with $C_1 = C$, and let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = T_{r_n}^{(F_1,\phi_1)}\left(I - \gamma_n A^*\left(I - T_{r_n}^{(F_2,\phi_2)}\right)A\right)w_n, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}y_{n,i}, \quad y_{n,i} \in P_{S_i}u_n, \\ C_{n+1} = \left\{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases} \tag{44}$$

$$\gamma_n = \begin{cases} \dfrac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \textit{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \textit{otherwise } (\gamma \textit{ being any nonnegative real number}), \end{cases}$$

where $0 < a \leq \tau_n \leq b < 1$, $\{\theta_n\} \subset \mathbb{R}$, $\{\alpha_{n,i}\} \subset (0, 1)$, *such that* $\sum_{i=0}^{n} \alpha_{n,i} = 1$, *and* $\{r_n\} \subset (0, \infty)$. *Suppose that the following conditions hold:*
(*C1*) $\liminf_{n\to\infty} r_n > 0$,
(*C2*) *the limits* $\lim_{n\to\infty}\alpha_{n,i} \in (0, 1)$ *exist for all* $i \geq 0$.
Then the sequence $\{x_n\}$ generated by (44), *converges strongly to* $P_\Omega x_1$.

Proof. Since P_{S_i} satisfies the common endpoint condition and $F(S_i) = F(P_{S_i})$ for each $i \in \mathbb{N}$, then the result follows from Theorem 3.1. □

4 Applications

In this section, we apply our results to approximate solutions of some important optimization problems.

4.1 Split mixed variational inequality and fixed point problems

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $B : H \to H$ be a single-valued mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction. The mixed variational inequality problem (MVIP) is defined as follows:

$$\text{Find } x^* \in C \text{ such that } \langle y - x^*, Bx^* \rangle + \phi(x^*, y) \geq 0, \quad \forall y \in C. \tag{45}$$

We denote the set of solutions of MVIP by $\mathrm{MVI}(C, B, \phi)$. If we take $\phi = 0$ in (45), then the MVIP reduces to the VIP, which is to find a point $x^* \in C$ such that $\langle y - x^*, Bx^* \rangle \geq 0$, $\forall y \in C$. The solution set of the VIP is denoted by $\mathrm{VI}(C, B)$. Variational inequality was first introduced independently by Fichera [48] and Stampacchia [49]. The VIP is a useful mathematical model that unifies many important concepts in applied mathematics, such as necessary optimality conditions, complementarity problems, network EPs, and systems of nonlinear equations (see [3,50,51]). Several methods have been proposed and analyzed for solving VIP and related OPs, see [5,37,52,53] and references therein.

Here, we apply our result to study the following SMVIP:

$$\text{Find } x^* \in \bigcap_{i=1}^{\infty} F(S_i) \text{ such that } \langle x - x^*, B_1 x^* \rangle + \phi_1(x^*, x) \geq 0, \quad \forall x \in C \tag{46}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle y - y^*, B_2 y^* \rangle + \phi_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{47}$$

where C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $\{S_i\}$ is a countable family of nonexpansive multivalued mappings of C into $CB(C)$, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : C \to H_1$, $B_2 : Q \to H_2$ are monotone mappings, and $\phi_1 : C \times C \to \mathbb{R}$, $\phi_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumptions (A5)–(A7). Moreover, ϕ_1, ϕ_2 are monotone with ϕ_1 being upper hemicontinuous and ϕ_2 upper semicontinuous in the first argument. We denote the solution set of problems (46)–(47) by Ω and assume that $\Omega \neq \emptyset$. By taking $F_j(x, y) := \langle y - x, B_j x \rangle$, $j = 1, 2$, then the SMVIP (46)–(47) becomes the problem of finding a solution of the SGEP (2)–(3), which is also a solution of the countable family of nonexpansive multivalued mappings $\{S_i\}$. In addition, all the conditions of Theorem 3.1 are satisfied. Hence, Theorem 3.1 provides a strong convergence theorem for approximating a common solution of SMVIP and fixed point of a countable family of nonexpansive multivalued mappings.

4.2 Split minimization and fixed point problems

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator, then the SMP is defined as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*) \leq f(x), \quad \forall x \in C \tag{48}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } g(y^*) \leq g(y), \quad \forall y \in Q. \tag{49}$$

We denote the solution set of SMP (48)–(49) by Φ and assume that $\Phi \neq \emptyset$. For some recent results on iterative algorithms for solving MP, see [54,55] and references therein. Let $F_1(x, y) := f(y) - f(x)$ for all $x, y \in C$ and $F_2(u, v) := f(v) - f(u)$ for all $u, v \in Q$, and taking $\phi_1 = \phi_2 = 0$ in the SGEP (2)–(3). Then, $F_1(x, y)$ and $F_2(u, v)$ satisfy Assumptions (A1)–(A4) provided f and g are convex and lower semi-continuous on C and Q, respectively. Clearly, ϕ_1 and ϕ_2 satisfy Assumptions (A5)–(A7). Therefore, from Theorem 3.1, we obtain a strong convergence theorem for approximating a common solution of SMP and fixed point problem for a countable family of nonexpansive multivalued mappings in real Hilbert spaces.

5 Numerical experiments

In this section, we present some numerical experiments to illustrate the performance of our algorithm as well as comparing it with Algorithm 9 in the literature. All numerical computations were carried out using Matlab version R2019(b).

We define the sequences $\{\alpha_{n,i}\}$ as follows for each $i \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$:

$$\alpha_{n,i} = \begin{cases} \dfrac{1}{b^{i+1}}\left(\dfrac{n}{n+1}\right), & n > i, \\ 1 - \dfrac{n}{n+1}\left(\sum_{k=1}^{n} \dfrac{1}{b^k}\right), & n = i, \\ 0, & n < i, \end{cases} \tag{50}$$

where $b > 1$.

Example 5.1. Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = [0, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{3}$ for all $x \in H_1$. Then, we have that $A^*y = \frac{y}{3}$ for all $y \in H_2$. For $x \in C$, $i \in \mathbb{N}$, we define the multivalued mappings $S_i : C \to CB(C)$ as follows:

$$S_i(x) = \left[0, \frac{x}{10i}\right], \quad \forall i \in \mathbb{N}. \tag{51}$$

It can easily be checked that S_i is nonexpansive for all $i \in \mathbb{N}$, $S_i(0) = \{0\}$, and $\bigcap_{i=1}^{\infty} F(S_i) = \{0\}$. We define the bifunctions $F_1, \phi_1 : C \times C \to \mathbb{R}$ by $F_1(x, y) = y^2 + 3xy - 4x^2$ and $\phi_1(x, y) = y^2 - x^2$ for $x, y \in C$, and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ by $F_2(w, v) = 2v^2 + wv - 3w^2$ and $\phi_2(w, v) = w - v$ for $w, v \in Q$. Choose $r_n = \frac{n-3}{n+2}$, $\theta_n = 0.8$, and $\tau_n = 0.7$. It can easily be verified that all the conditions of Theorem 3.1 are satisfied with $\Omega = \{0\}$. Now, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $u \in C$ such that for all $z \in C$

$$\begin{aligned} 0 &\leq F_1(u, z) + \phi_1(u, z) + \frac{1}{r}\langle z - u, u - x\rangle \\ &= 2z^2 + 3uz - 5u^2 + \frac{1}{r}\langle z - u, u - x\rangle \\ &\Leftrightarrow \\ 0 &\leq 2rz^2 + 3ruz - 5ru^2 + (z - u)(u - x) \\ &= 2rz^2 + 3ruz - 5ru^2 + uz - xz - u^2 + ux \\ &= 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux). \end{aligned}$$

Let $h(z) = 2rz^2 + (3ru + u - x)z + (-5ru^2 - u^2 + ux)$. Then, $h(z)$ is a quadratic function of z with coefficients $a = 2r$, $b = 3ru + u - x$, and $c = -5ru^2 - u^2 + ux$. We determine the discriminant Δ of $h(z)$ as follows:

$$\begin{aligned}\Delta &= (3ru + u - x)^2 - 4(2r)(-5ru^2 - u^2 + ux) \\ &= 49r^2u^2 + 14ru^2 - 14rux + u^2 - 2ux + x^2 \\ &= ((7r + 1)u - x)^2.\end{aligned} \tag{52}$$

By Lemma 2.10, $T_r^{(F_1,\phi_1)}$ is single-valued. Hence, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Therefore, from (52), we have that $u = \frac{x}{7r+1}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{x}{7r+1}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. Find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{w \in Q : F_2(w, d) + \phi_2(w, d) + \frac{1}{s}\left\langle d - w, w - v\right\rangle \geq 0, \quad \forall d \in Q\right\}.$$

By following similar procedure as above, we obtain $w = \frac{v+s}{5s+1}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{v+s}{5s+1}$. We take $y_{n,i} = \frac{u_n}{10i}$ for all $i \in \mathbb{N}$. Then, Algorithm (15) becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = \dfrac{w_n}{7r_n + 1} - \gamma_n \dfrac{15w_n r_n + 2w_n - 3r_n}{9(7r_n + 1)(5r_n + 1)}, \\ z_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}\dfrac{u_n}{10i}, \\ C_{n+1} = \left\{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\left\langle x_n - p, x_{n-1} - x_n\right\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \end{cases}$$

where

$$\gamma_n = \begin{cases} \dfrac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})\,Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})\,Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases}$$

In this example, we set the parameter b on $\{\alpha_{n,i}\}$ in (50) to be $b = 50$, and we choose different initial values as follows:

Case Ia: $x_0 = \frac{11}{2}$, $x_1 = \frac{2}{5}$;
Case Ib: $x_0 = 8$, $x_1 = 1$;
Case Ic: $x_0 = 5$, $x_1 = \frac{7}{10}$;
Case Id: $x_0 = 6$, $x_1 = \frac{4}{5}$.

We compare the performance of our Algorithm (15) with Algorithm (9). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

Example 5.2. Let $H_1 = H_2 = L_2([0, 1])$ with the inner product defined as

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\,\mathrm{d}t, \quad \forall x, y \in L_2([0, 1]).$$

Let

$$C := \{x \in H_1 : \langle a, x\rangle = d\},$$

where $a = 2t^2$ and $d \geq 0$. Here, we have

$$P_C(x) = x + \frac{d - \langle a, x\rangle}{\|a\|^2}a.$$

Also, let

$$Q := \{x \in H_2 : \langle c, x\rangle \leq e\},$$

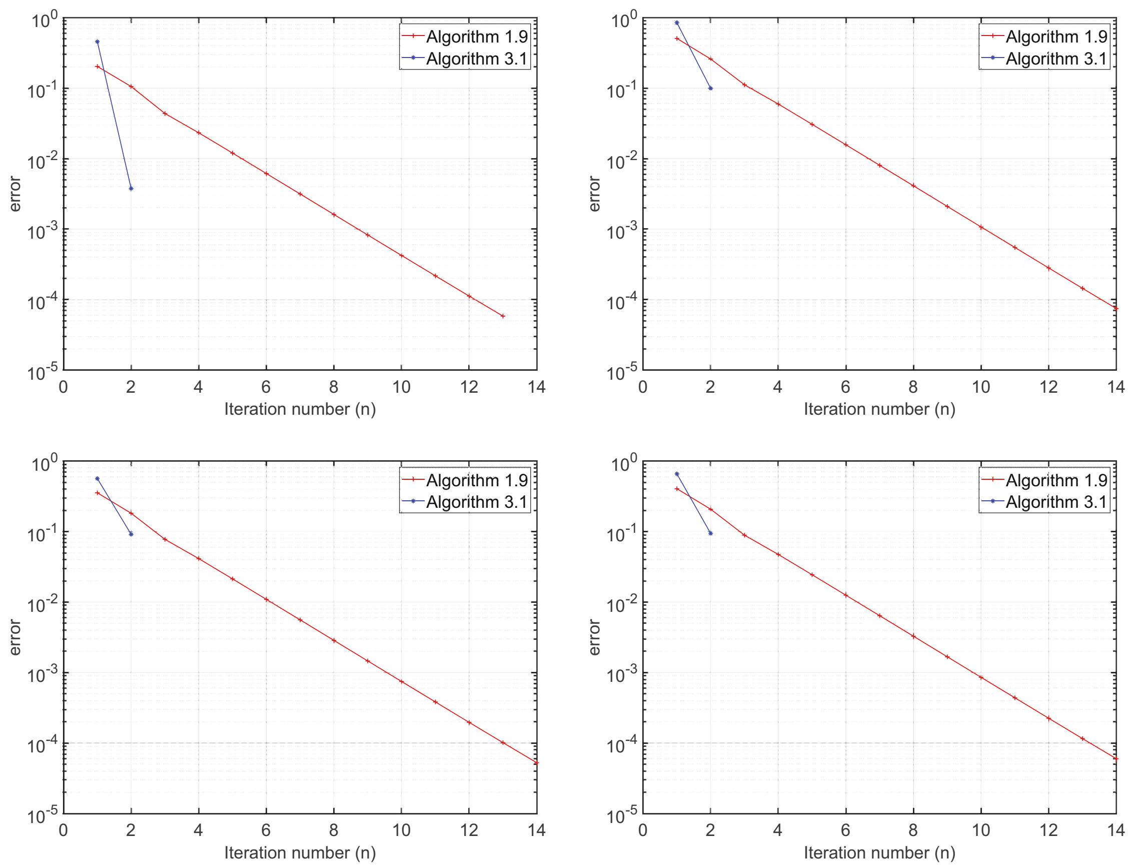

Figure 1: Top left: Case Ia; top right: Case Ib; bottom left: Case Ic; and bottom right: Case Id.

Table 1: Numerical results for Example 5.1

		Alg. 9	**Alg. 15**
Case Ia	CPU time (s)	2.1794	0.1722
	No of iter.	13	3
Case Ib	CPU time (s)	2.2136	0.1514
	No. of iter.	14	3
Case Ic	CPU time (s)	2.2338	0.1517
	No of iter.	14	3
Case Id	CPU time (s)	2.1757	0.1495
	No of iter.	14	3

where $c = \frac{t}{3}$ and $e = 1$, we get

$$P_Q(x) = x + \max\left\{0, \frac{e - \langle c, x\rangle}{\|c\|^2} c\right\}.$$

We define $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ by $F_1(x, y) = \langle L_1 x, y - x\rangle$ and $F_2(x, y) = \langle L_2 x, y - x\rangle$, where $L_1 x(t) = \frac{x(t)}{2}$ and $L_2 x(t) = \frac{x(t)}{5}$. It can easily be verified that F_1 and F_2 satisfy Conditions (A1)–(A4). Also, take $\phi_1 = \phi_2 = 0$. Moreover, let $A : L_2([0, 1]) \to L_2([0, 1])$ be defined by $Ax(t) = \frac{x(t)}{2}$ and $A^* y(t) = \frac{y(t)}{2}$. Then, A is a bounded linear operator. We consider the case for which the countable family of nonexpansive multivalued mappings $\{S_i\}$ are singled-valued. Define a countable family of nonexpansive mappings $S_i : L^2([0,1]) \to L^2([0,1])$ by

$$(S_i x)(t) = \int_0^1 t^i x(s)\,\mathrm{d}s \quad \text{for all } t \in [0,1].$$

Observe that S_i is nonexpansive for each $i \in \mathbb{N}$. Choose $\theta_n = 0.9$, $\tau_n = 0.8$, $r_n = \frac{n}{n+1}$. It can easily be checked that all the conditions on the control sequences in Theorem 3.1 are satisfied. Next, we compute $T_r^{(F_1,\phi_1)}(x)$. We find $z \in C$ such that for all $y \in C$

$$\begin{aligned} & F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\left\langle y - z, z - x\right\rangle \geq 0 \\ \Leftrightarrow\ & \left\langle \frac{z}{2}, y - z\right\rangle + \frac{1}{r}\left\langle y - z, z - x\right\rangle \geq 0 \\ \Leftrightarrow\ & \frac{z}{2}(y - z) + \frac{1}{r}(y - z)(z - x) \geq 0 \\ \Leftrightarrow\ & (y - z)[rz + 2(z - x)] \geq 0 \\ \Leftrightarrow\ & (y - z)[(r + 2)z - 2x] \geq 0. \end{aligned} \tag{53}$$

According to Lemma 2.10,

$$T_r^{(F_1,\phi_1)}(x) = \left\{z \in C : F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}$$

is single-valued for all $x \in H_1$. Hence, from (53) we have that $z = \frac{2x}{r+2}$. This implies that $T_r^{(F_1,\phi_1)}(x) = \frac{2x}{r+2}$. Similarly, we compute $T_r^{(F_2,\phi_2)}(v)$. We find $w \in Q$ such that for all $d \in Q$

$$T_s^{(F_2,\phi_2)}(v) = \left\{w \in Q : F_2(w,d) + \phi_2(w,d) + \frac{1}{s}\langle d - w, w - v\rangle \geq 0, \quad \forall d \in Q\right\}.$$

Following similar procedure as above, we obtain $w = \frac{5v}{s+5}$. This implies that $T_s^{(F_2,\phi_2)}(v) = \frac{5v}{s+5}$. Then, Algorithm (15) becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = \dfrac{2w_n}{r_n + 2} - \gamma_n \dfrac{2r_n + 5}{2(r_n + 5)(r_n + 2)} w_n, \\ z_n = \alpha_{n,0} u_n + \sum_{i=1}^n \alpha_{n,i} S_i u_n, \\ C_{n+1} = \left\{p \in C_n : \|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle + \theta_n^2\|x_{n-1} - x_n\|^2\right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases}$$

where

$$\gamma_n = \begin{cases} \dfrac{\tau_n\|(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2}{\|A^*(I - T_{r_n}^{(F_2,\phi_2)})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{(F_2,\phi_2)}Aw_n, \\ \gamma & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases}$$

Here, we set the parameter b on $\{\alpha_{n,i}\}$ in (50) to be $b = 2$, and we choose different initial values as follows:

Case Ia: $x_0 = t^3$, $x_1 = t^2 + t^4$;

Case Ib: $x_0 = t^2 + t^6 + t^8$, $x_1 = t^3$;

Case Ic: $x_0 = t^5 + t^9 + t^{11}$, $x_1 = t^5$;

Case Id: $x_0 = t + t^2 + t^4 + t^6$, $x_1 = t^2 + t^7$.

We compare the performance of our Algorithm (15) with Algorithm (9). The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-4}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

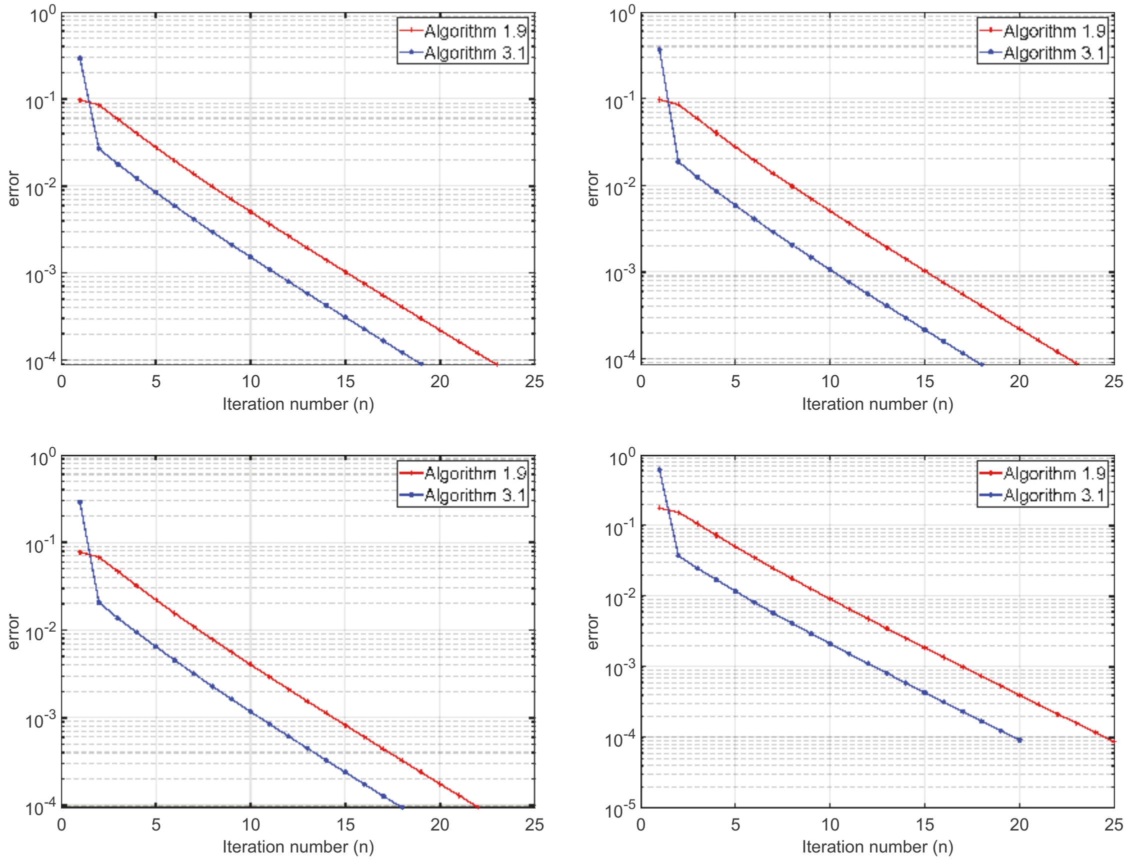

Figure 2: Top left: Case Ia; top right: Case Ib; bottom left: Case Ic; and bottom right: Case Id.

Table 2: Numerical results for Example 5.2

		Alg. 9	**Alg. 15**
Case Ia	CPU time (s)	2.2241	1.3724
	No. of iter.	23	19
Case Ib	CPU time (s)	2.2247	1.2772
	No. of iter.	23	18
Case Ic	CPU time (s)	2.1359	1.3056
	No of iter.	22	18
Case Id	CPU time (s)	2.3458	1.4506
	No of iter.	25	20

6 Conclusion

In this article, we proposed a new inertial shrinking projection algorithm with self-adaptive step size for approximating a common solution of SGMEP and FPP for a countable family of nonexpansive multivalued mappings. We proved strong convergence results for the considered problems without a prior knowledge of the operator norm. Finally, we applied our results to solve some other important OPs and presented some numerical experiments to demonstrate the efficiency of our proposed method in comparison with other existing methods. Our results extend and improve several existing results in this direction in the current literature.

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