Inference on a Distribution Function from Ranked Set Samples

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I. The Setting

In some situations, ranking observations is much easier than obtaining their precise values ...

Consider $n \cdot k$ independent random variables

$$X_{ij}, \quad 1 \le i \le n,\ 1 \le j \le k,$$

with unknown continuous c.d.f. F.

Instead of complete i-th sample $(X_{ij})_{j=1}^{k}$ observe only

- one of its elements, X_i, and
- its rank $R_i \in \{1, 2, \ldots, k\}$ within $(X_{ij})_{j=1}^{k}$.

Example 1: Ranked set sampling (RSS)
(McIntyre 1952)

For $1 \leq i \leq n$ let

$$X_{i:1} \ < \ X_{i:2} \ < \ \cdots \ < \ X_{i:k}$$

be the order statistics of $(X_{ij})_{j=1}^{k}$.

- Pick a rank $R_i \in \{1, 2, \ldots, k\}$
- Obtain precise value $X_i := X_{i:R_i}$.

Special case: $k = n$ and $R_i = i$.

Example 2: Judgement post-stratification (JPS)
(MacEachern et al. 2004)

- $X_i := X_{i1}$,
- $R_i := \sum_{j=1}^{k} 1_{[x_{ij} \leq x_{i1}]}$.

General setting:

Independent observations (X_1, R_1), (X_2, R_2), $\ldots$, (X_n, R_n).

Conditional on $R_i = r$,

$$X_i \sim F_r(x) := B_r(F(x))$$

with

$$B_r(p) := \sum_{i=r}^{k} \binom{k}{i} p^i (1-p)^{k-i} = \int_0^p \beta_r(u)\, du$$

$$\beta_r(u) := k \binom{k-1}{r-1} u^{r-1} (1-u)^{k-r}$$

(c.d.f. and density of $\mathrm{Beta}(r, k+1-r)$).

Basic fact

$$\frac{1}{k}\sum_{r=1}^{k} \mathrm{Beta}(r, k+1-r) \;=\; \mathrm{Unif}[0,1],$$

whence

$$\frac{1}{k}\sum_{r=1}^{k} F_r \;=\; F.$$

II. Estimation paradigms

Paradigm 0: Naive empirical distribution function

$$\widehat{F}(x) \;:=\; \frac{1}{n}\sum_{i=1}^{n} 1_{[X_i \le x]}.$$

In JPS setting: Okay but inefficient.

In unbalanced RSS settings: Potentially severely biased.

Paradigm 1: Stratification (Stokes and Sager 1988)

Groupwise empirical c.d.f.s:

$$\widehat{F}_r(x) \;:=\; \frac{1}{N_r} \sum_{i\,:\,R_i=r} 1_{[X_i \le x]} \quad \text{with} \quad N_r := \#\{i : R_i = r\}.$$

Stratified estimator

$$\widehat{F}^{\mathrm{S}} \;:=\; \frac{1}{k} \sum_{r=1}^{k} \widehat{F}_r.$$

Potential problems with small/zero numbers N_r.

Paradigm 2: Likelihood (Kvam and Samaniego 1994)

$$\widehat{F}^{\mathrm{L}}(x) \;:=\; \underset{p\in[0,1]}{\arg\max}\, L(x,p)$$

with (conditional) log-likelihood function

$$\begin{aligned} L(x,p) \;&:=\; \sum_{i=1}^{n}\left[1_{[X_i\le x]}\log B_{R_i}(p) + 1_{[X_i>x]}\log(1-B_{R_i}(p))\right] \\ &=\; \sum_{r=1}^{k} N_r\left[\widehat{F}_r(x)\log B_r(p) + (1-\widehat{F}_r(x))\log(1-B_r(p))\right] \end{aligned}$$

Paradigm 3: Moments

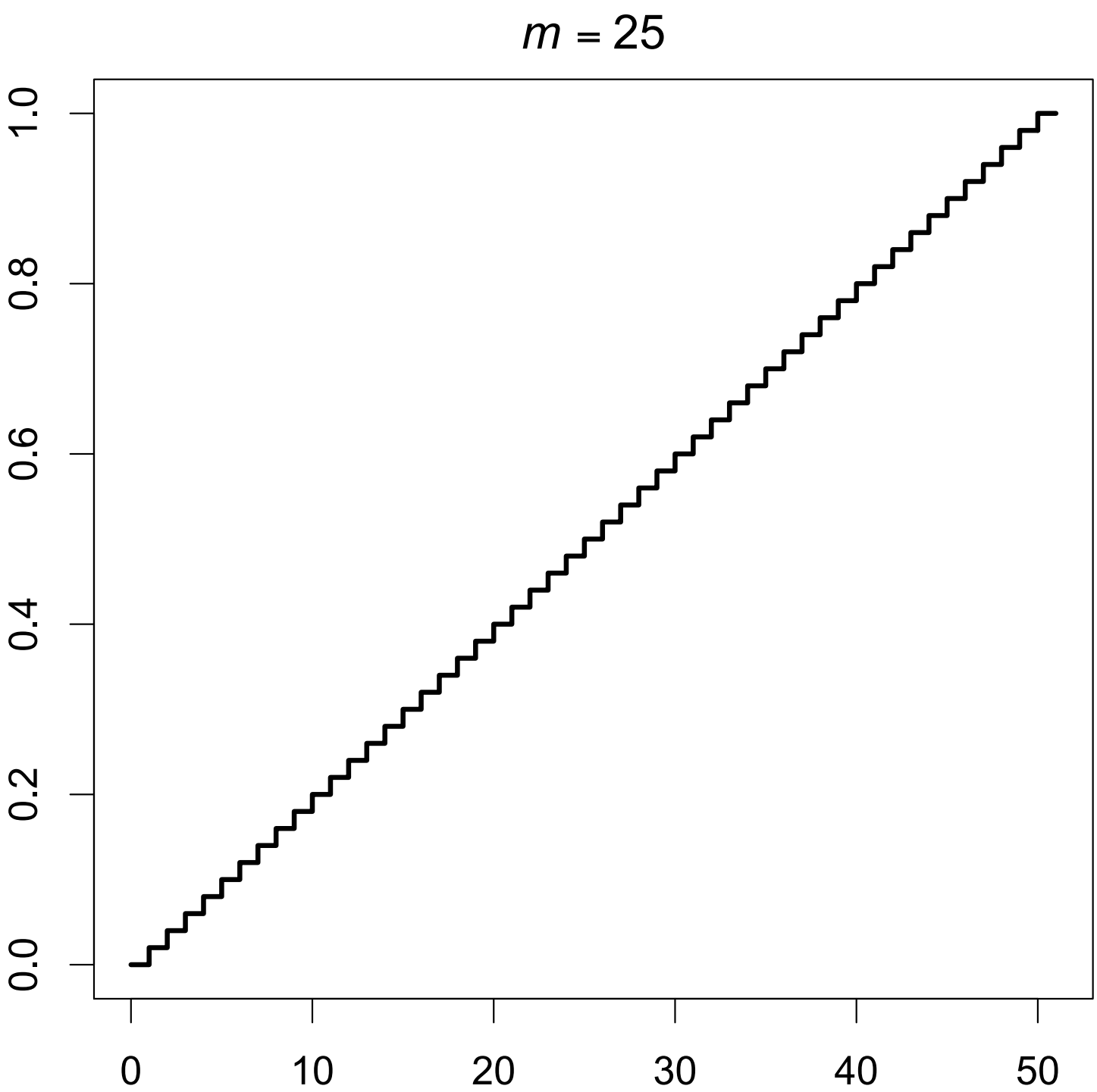
$$\mathbb{E}\big(\widehat{F}(x) \,\big|\, \mathbf{R}\big) \;=\; \sum_{r=1}^{k} \frac{N_r}{n} B_r(F(x)).$$

$$\widehat{F}(x) \;\stackrel{!}{=}\; \sum_{r=1}^{k} \frac{N_r}{n} B_r(\widehat{F}^{\mathrm{M}}(x)).$$

Paradigms 2-3 deal with small/zero numbers N_r properly.

Estimator $\widehat{F}^{\mathrm{M}}$ when

$$k = 2 \quad \text{and} \quad \mathbf{N} = (m, 50 - m)^{\top}$$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0.8
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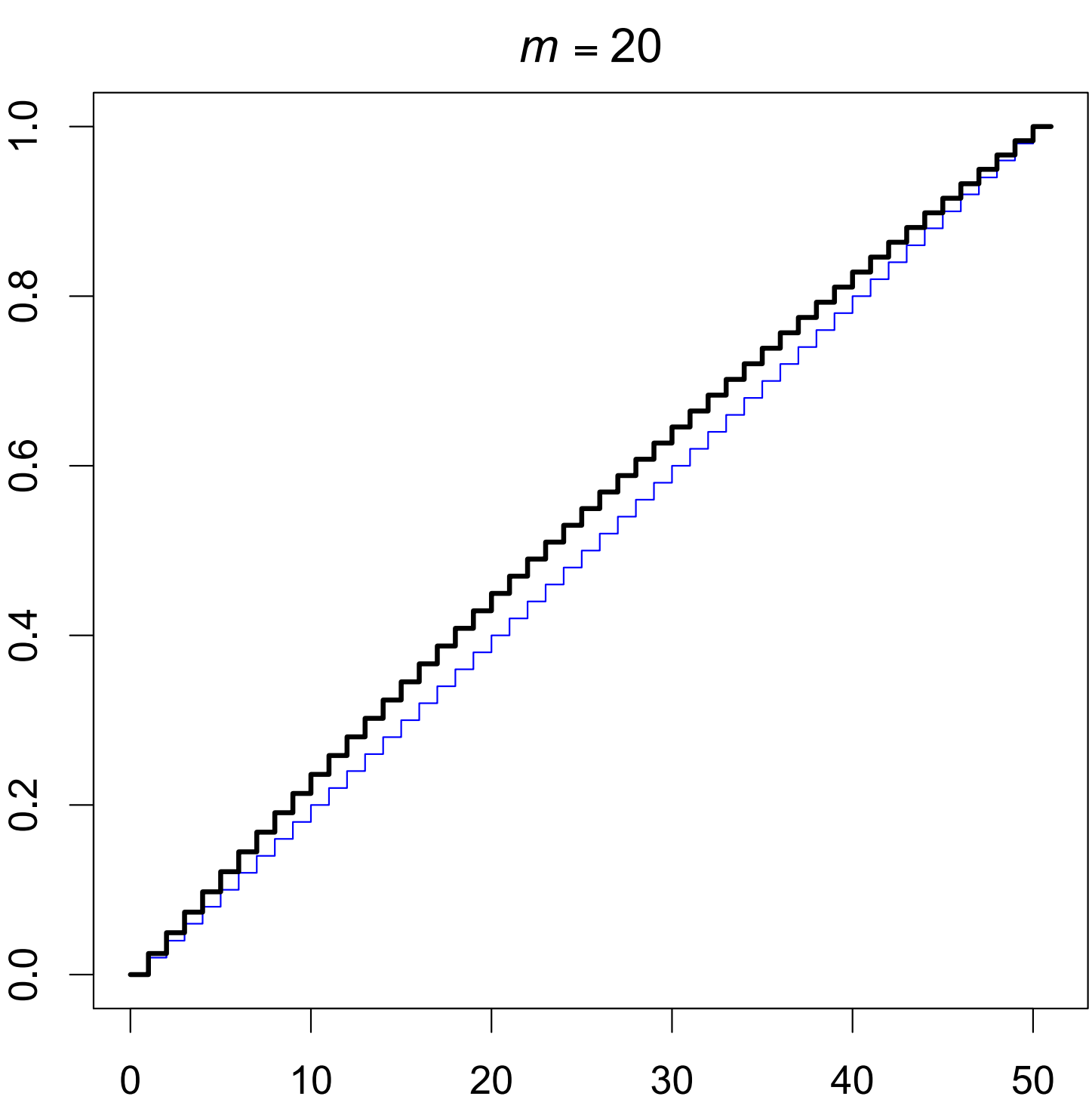
0
10
20
30
40
50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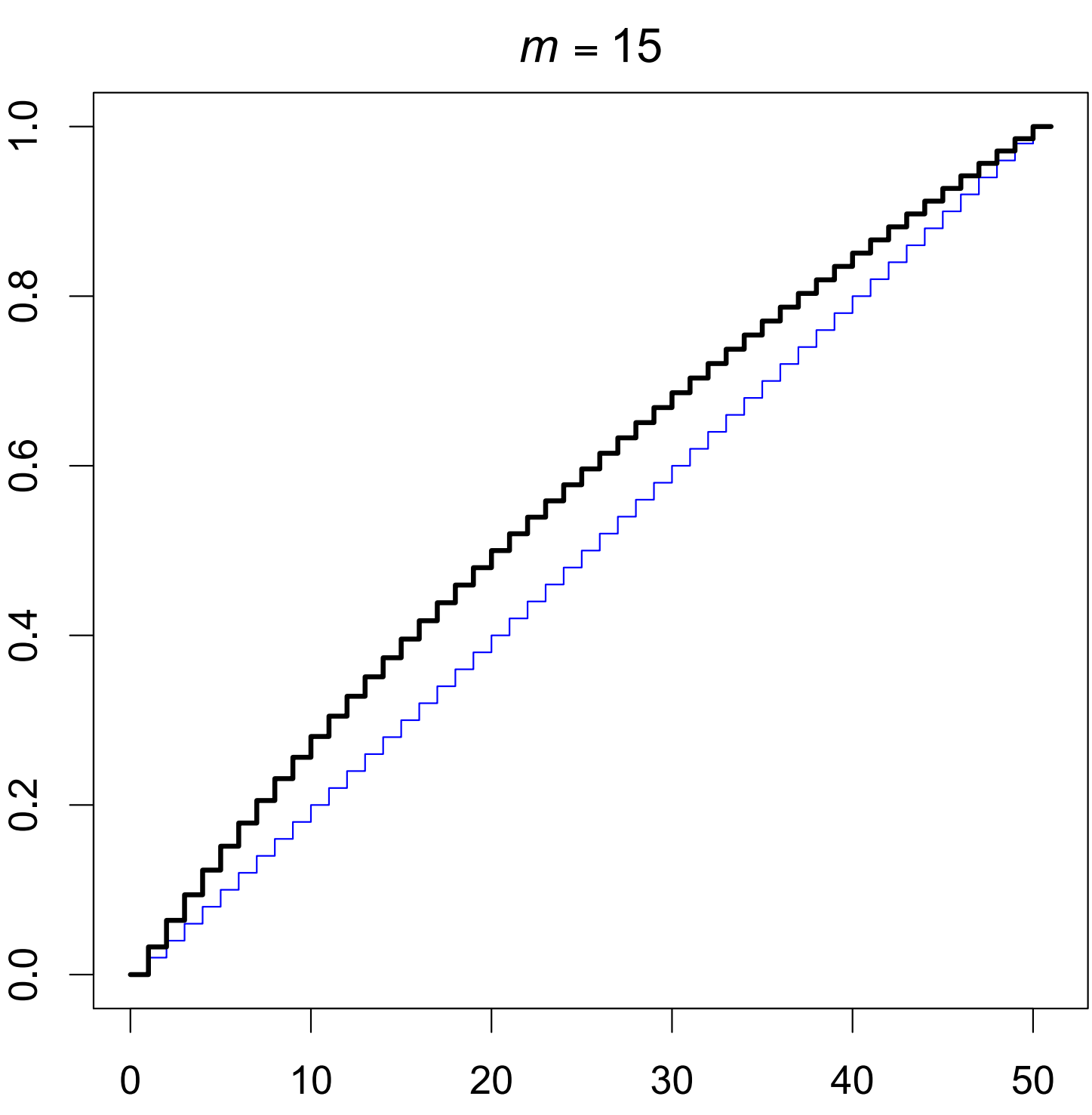
0.8
1.0
0
10
20
30
40
50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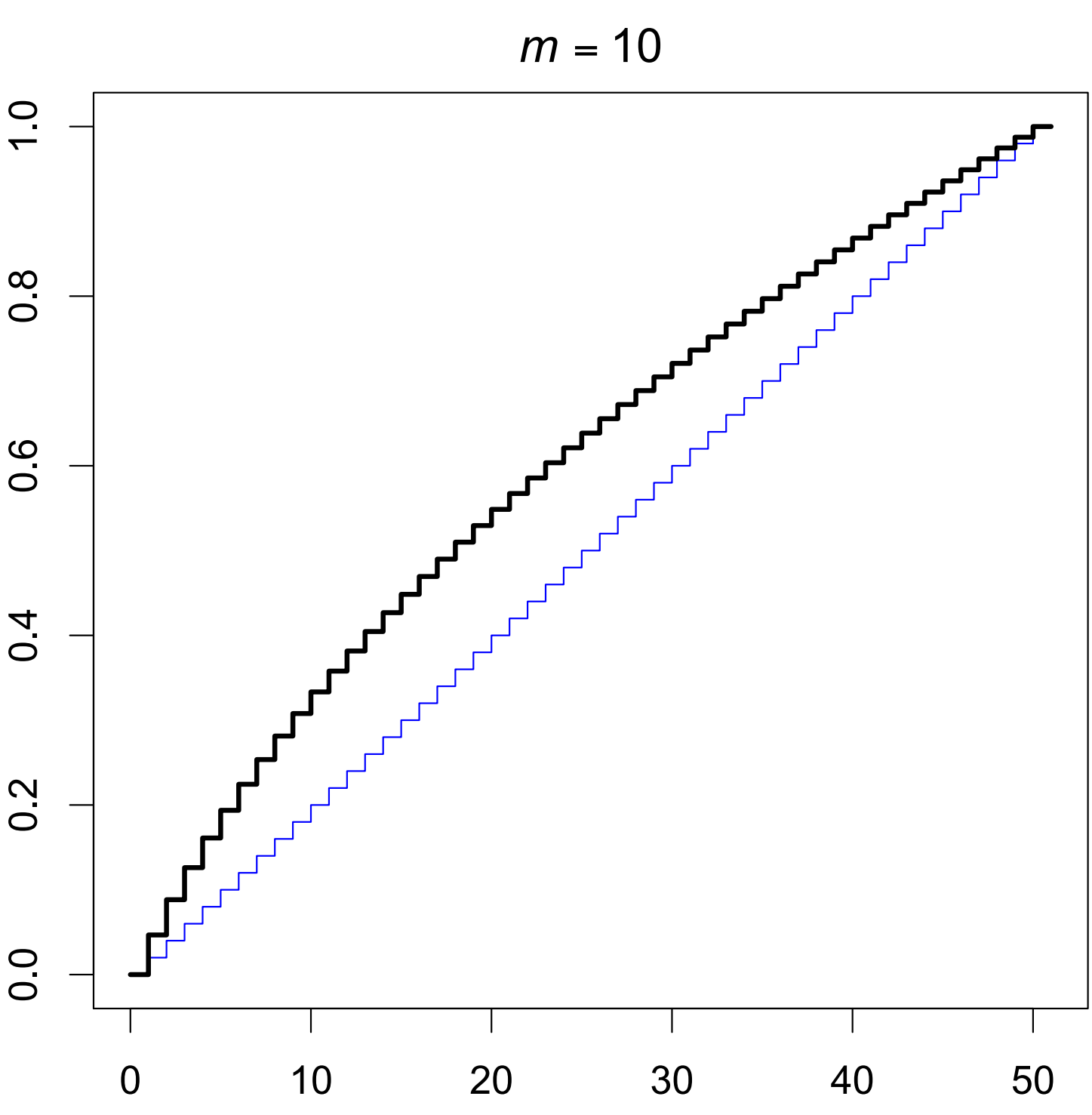
0.4
0.6
0.8
1.0
0
10
20
30
40
50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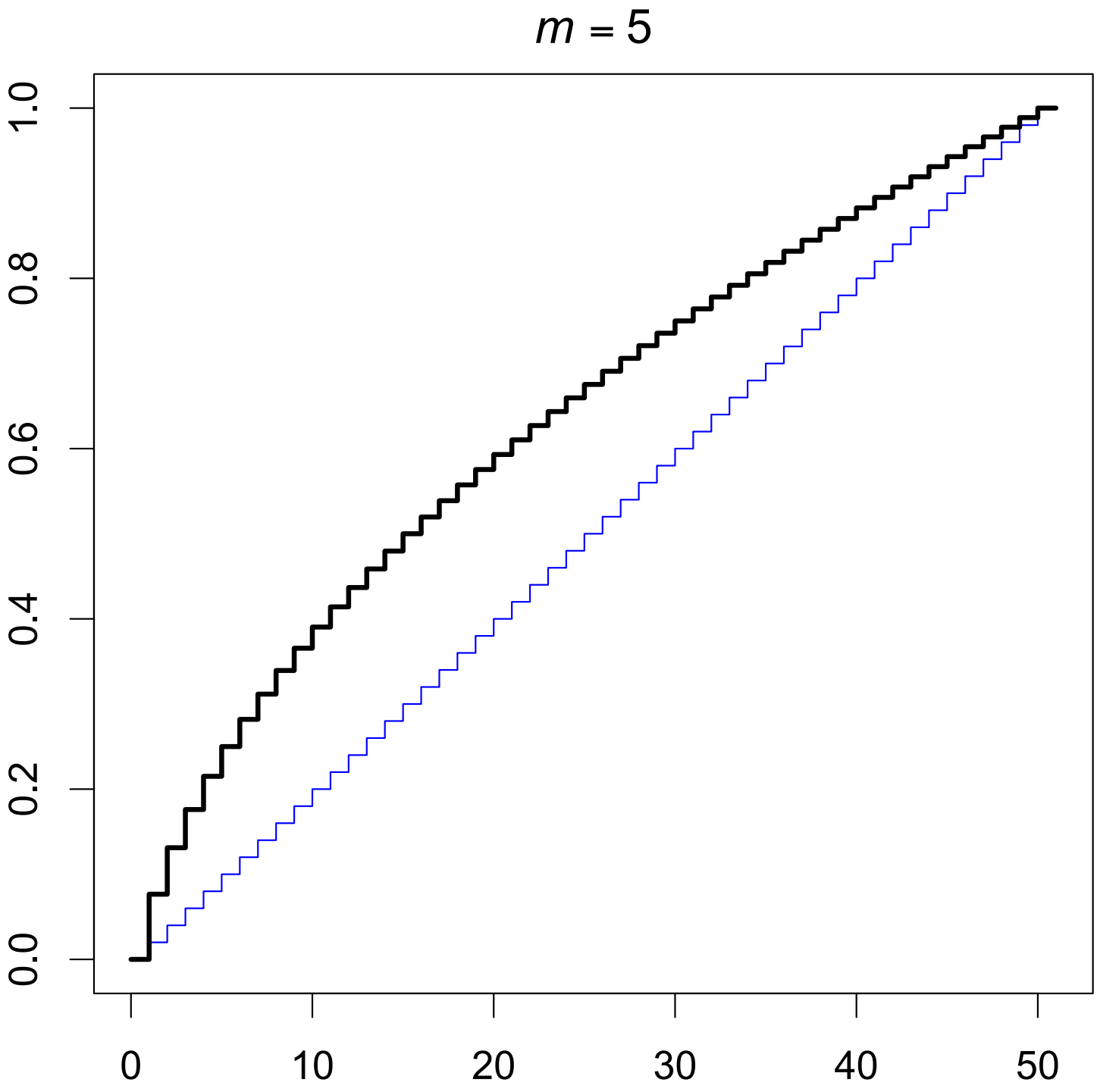
1.0
0
10
20
30
40
50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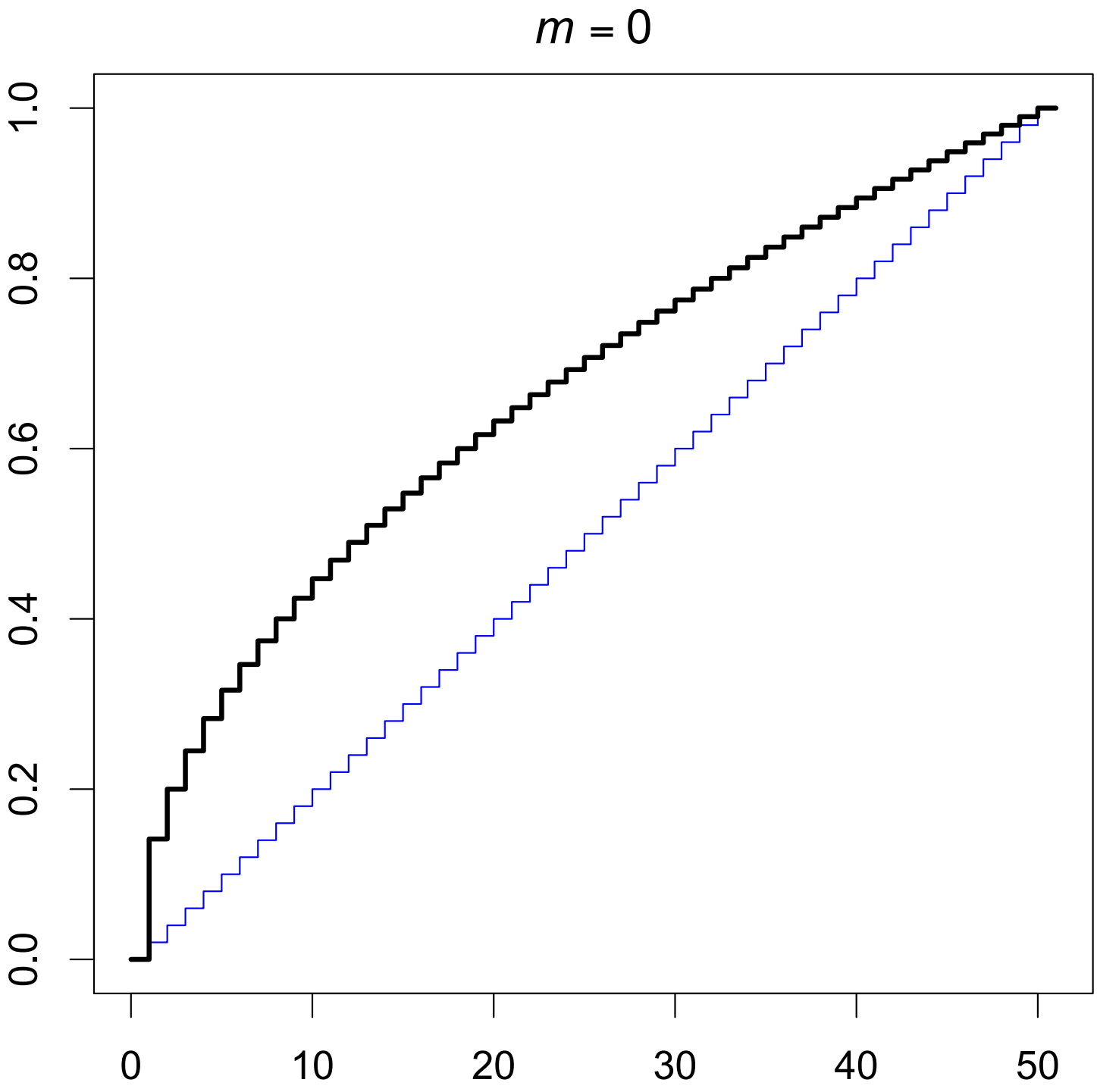
1.0
0
10
20
30
40
50


$m = 0$
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50

III. Asymptotics

Refine and generalize results by
Stokes and Sager (1988) and Huang (1997)

Condition on **R** and assume that

$$\frac{N_r}{n} \;\to\; \pi_r \quad \text{for } 1 \le r \le k$$

where

$$\begin{cases} \pi_1, \pi_2, \ldots, \pi_k > 0 & \text{for } \widehat{F}^{\mathrm{S}}, \\ \pi_1, \pi_k > 0 & \text{for } \widehat{F}^{\mathrm{M}}, \widehat{F}^{\mathrm{L}}. \end{cases}$$

Without loss of generality let

$$F \mathrel{\widehat{=}} \mathrm{Unif}[0, 1].$$

Empirical processes:

$$\mathbb{V}_r(t) \;:=\; \sqrt{N_r}\big(\widehat{F}_r(B_r^{-1}(t)) - t\big), \quad t \in [0,1],$$

(Standard empirical process of sample of size N_r from $\mathrm{Unif}[0,1]$.)

Linear expansion: For $\mathrm{Z} = \mathrm{S}, \mathrm{L}, \mathrm{M}$,

$$\sqrt{n}\,(\widehat{F}^{\mathrm{Z}} - F) \;\approx\; \sum_{r=1}^{k} \gamma_r^{\mathrm{Z}}\; \mathbb{V}_r \circ B_r$$

with certain weight functions γ_1^{Z}, γ_2^{Z}, $\ldots$, γ_k^{Z}.

Weights for $\widehat{F}^{\mathrm{S}}$

$$\begin{aligned}
\sqrt{n}(\widehat{F}^{\mathrm{S}} - F) &= \sum_{r=1}^{k} \frac{\sqrt{n}\,(\widehat{F}_r - B_r)}{k} \\
&= \sum_{r=1}^{k} \frac{\sqrt{n}\,\mathbb{V}_r \circ B_r}{k\sqrt{N_r}} \\
&\approx \sum_{r=1}^{k} \gamma_r^{\mathrm{S}}\ \mathbb{V}_r \circ B_r
\end{aligned}$$

with

$$\gamma_r^{\mathrm{S}} := \frac{1}{k\sqrt{\pi_r}}.$$

Weights for $\widehat{F}^{\mathrm{M}}$

$$
\begin{aligned}
0 \stackrel{!}{=} & \; n\widehat{F}(t) - \sum_{r=1}^{k} N_r B_r(p) \\
= & \sum_{r=1}^{k} N_r \big(\widehat{F}_r(t) - B_r(t) + B_r(t) - B_r(p)\big) \\
\approx & \sum_{r=1}^{k} N_r \big(\widehat{F}_r(t) - B_r(t) - \beta_r(t)(p-t)\big) \\
\approx & \sqrt{n} \sum_{r=1}^{k} \sqrt{\pi_r}\, \mathbb{V}_r(B_r(t)) - n(p-t) \sum_{r=1}^{k} \pi_r \beta_r(t).
\end{aligned}
$$

$$0 \overset{!}{\approx} \sum_{r=1}^{k} \sqrt{\pi_r}\, \mathbb{V}_r(B_r(t)) - \sqrt{n}\,(p-t) \sum_{r=1}^{k} \pi_r \beta_r(t).$$

Hence we expect that

$$\sqrt{n}\,(\widehat{F}^{\mathrm{M}} - F) \approx \sum_{r=1}^{k} \gamma_r^{\mathrm{M}}\, \mathbb{V}_r \circ B_r$$

with

$$\gamma_r^{\mathrm{M}} := \frac{\sqrt{\pi_r}}{\sum_{s=1}^{k} \pi_s \beta_s}.$$

Weights for $\widehat{F}^{\mathrm{L}}$

With

$$w_r \;:=\; \frac{\beta_r}{B_r(1-B_r)}$$

we write

$$\begin{aligned}
0 \;\stackrel{!}{=}\; \frac{\partial}{\partial p} L(t,p) \;&=\; \sum_{r=1}^{k} N_r w_r(p)\big[\widehat{F}_r(t) - B_r(p)\big] \\
&\approx\; \sum_{r=1}^{k} N_r w_r(t)\big(\widehat{F}_r(t) - B_r(t) - \beta_r(t)(p-t)\big) \\
&\approx\; \sqrt{n}\sum_{r=1}^{k} \sqrt{\pi_r} w_r(t)\,\mathbb{V}_r(B_r(t)) - n(p-t)\sum_{r=1}^{k} \pi_r w_r(t)\beta_r(t).
\end{aligned}$$

$$0 \overset{!}{\approx} \sum_{r=1}^{k} \sqrt{\pi_r} w_r(t) \, \mathbb{V}_r(B_r(t)) - \sqrt{n}\,(p-t) \sum_{r=1}^{k} \pi_r w_r(t) \beta_r(t).$$

Hence we expect that

$$\sqrt{n}\,(\widehat{F}^{\mathrm{L}} - F) \;\approx\; \sum_{r=1}^{k} \gamma_r^{\mathrm{L}} \; \mathbb{V}_r \circ B_r$$

with

$$\gamma_r^{\mathrm{L}} \;:=\; \frac{\sqrt{\pi_r} w_r}{\sum_{s=1}^{k} \pi_s w_s \beta_s}.$$

Theorem 1 (Linear expansions). For $\mathrm{Z} = \mathrm{S}, \mathrm{L}, \mathrm{M}$ let

$$\mathbb{V}^{\mathrm{Z}} := \sum_{r=1}^{k} \gamma_r^{\mathrm{Z}} \, \mathbb{V}_r \circ B_r.$$

For any $\delta \in [0, 1/2)$,

$$\sup_{t \in (0,1)} \frac{\left| \sqrt{n} \left(\widehat{F}^{\mathrm{Z}}(t) - t \right) - \mathbb{V}^{\mathrm{Z}}(t) \right|}{t^{\delta}(1-t)^{\delta}} \to_p 0 \quad \text{as } n \to \infty.$$

$$\sup_{t \in (0,c) \cup (1-c,1)} \frac{\sqrt{n} \left| \mathbb{V}^{\mathrm{Z}}(t) \right|}{t^{\delta}(1-t)^{\delta}} \to_p 0 \quad \text{as } n \to \infty, c \downarrow 0.$$

All three estimators asymptotically equivalent in the tail regions!

Theorem 2 (Tail behaviour). Define

$$\mathbb{V}^{(\ell)} := \frac{\mathbb{V}_1 \circ B_1}{k\sqrt{N_1/n}} \quad \text{and} \quad \mathbb{V}^{(r)} := \frac{\mathbb{V}_k \circ B_k}{k\sqrt{N_k/n}}.$$

For $\mathrm{Z} = \mathrm{S}, \mathrm{L}, \mathrm{M}$ and any fixed $\kappa \in [1/2, 1)$,

$$\sup_{t\in(0,c)} \frac{\left|\sqrt{n}\,(\widehat{F}^{\mathrm{Z}}(t) - t) - \mathbb{V}^{(\ell)}(t)\right|}{t^{\kappa}} \to_p 0$$

$$\sup_{t\in(1-c,1)} \frac{\left|\sqrt{n}\,(\widehat{F}^{\mathrm{Z}}(t) - t) - \mathbb{V}^{(r)}(t)\right|}{(1-t)^{\kappa}} \to_p 0$$

as $n \to \infty$ and $c \downarrow 0$.

Theorem 3 (Limiting distributions). For $\mathrm{Z} = \mathrm{S}, \mathrm{L}, \mathrm{M}$,

$$\left(\sqrt{n}(\widehat{F}^{\mathrm{Z}}(t) - t)\right)_{t\in[0,1]} \;\to_{\mathcal{L}}\; \mathbb{G}^{\mathrm{Z}} \;=\; \sum_{r=1}^{k} \gamma_r^{\mathrm{Z}}\; \mathbb{G}_r \circ B_r$$

with independent Brownian bridges $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_k$.

The covariance function K^{Z} of $\mathbb{G}^{\mathrm{Z}}$ equals

$$K^{\mathrm{Z}}(s,t) \;=\; \sum_{r=1}^{k} \gamma_r^{\mathrm{Z}}(s)\gamma_r^{\mathrm{Z}}(t)\; K\big(B_r(s), B_r(t)\big)$$

with

$$K(s,t) \;:=\; \min\{s,t\} - st.$$

In the balanced case, i.e. $\pi_1 = \cdots = \pi_k = 1/k$,

$$\mathbb{G}^{\mathrm{S}} \;\equiv\; \mathbb{G}^{\mathrm{M}}.$$

Theorem 4 (Asymptotic relative efficiencies). For any $t \in (0,1)$,

$$K^{\mathrm{L}}(t,t) \;\le\; K^{\mathrm{S}}(t,t) \quad \text{(equality for at most one } t\text{)}$$
$$K^{\mathrm{L}}(t,t) \;\le\; K^{\mathrm{M}}(t,t) \quad \text{(equality iff } k = 2 \text{ and } t = 1/2\text{)}.$$

On the other hand,

$$\sup_{\pi} \frac{K^{\mathrm{S}}(t,t)}{K^{\mathrm{L}}(t,t)} \;=\; \infty$$
$$\sup_{\pi} \frac{K^{\mathrm{M}}(t,t)}{K^{\mathrm{L}}(t,t)} \;=\; \frac{\rho(t) + \rho(t)^{-1} + 2}{4} \;<\; \infty$$

with

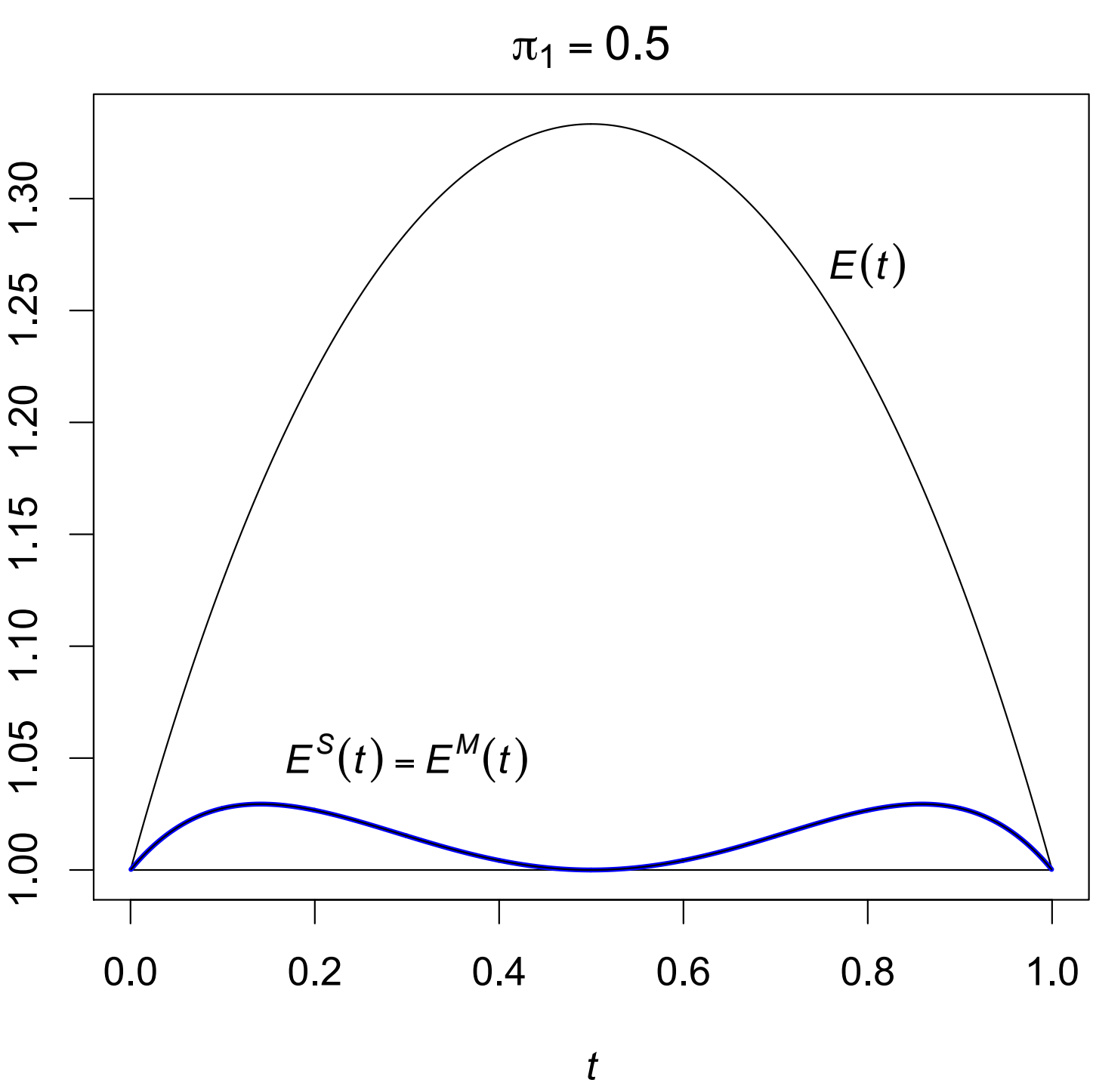
$$\rho(t) \;:=\; \max_r w_r(t) \big/ \min_r w_r(t).$$

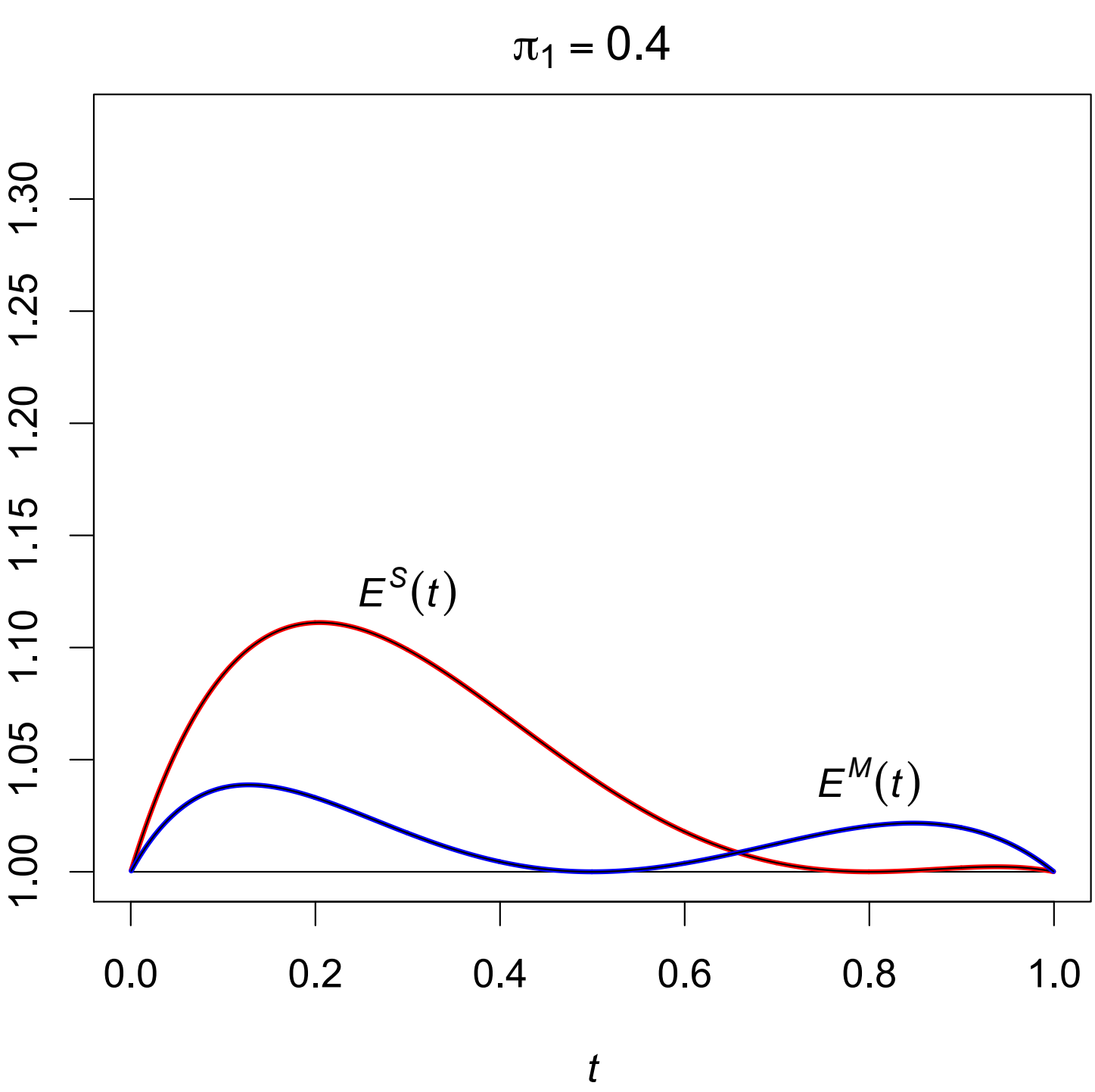
The case $k = 2$

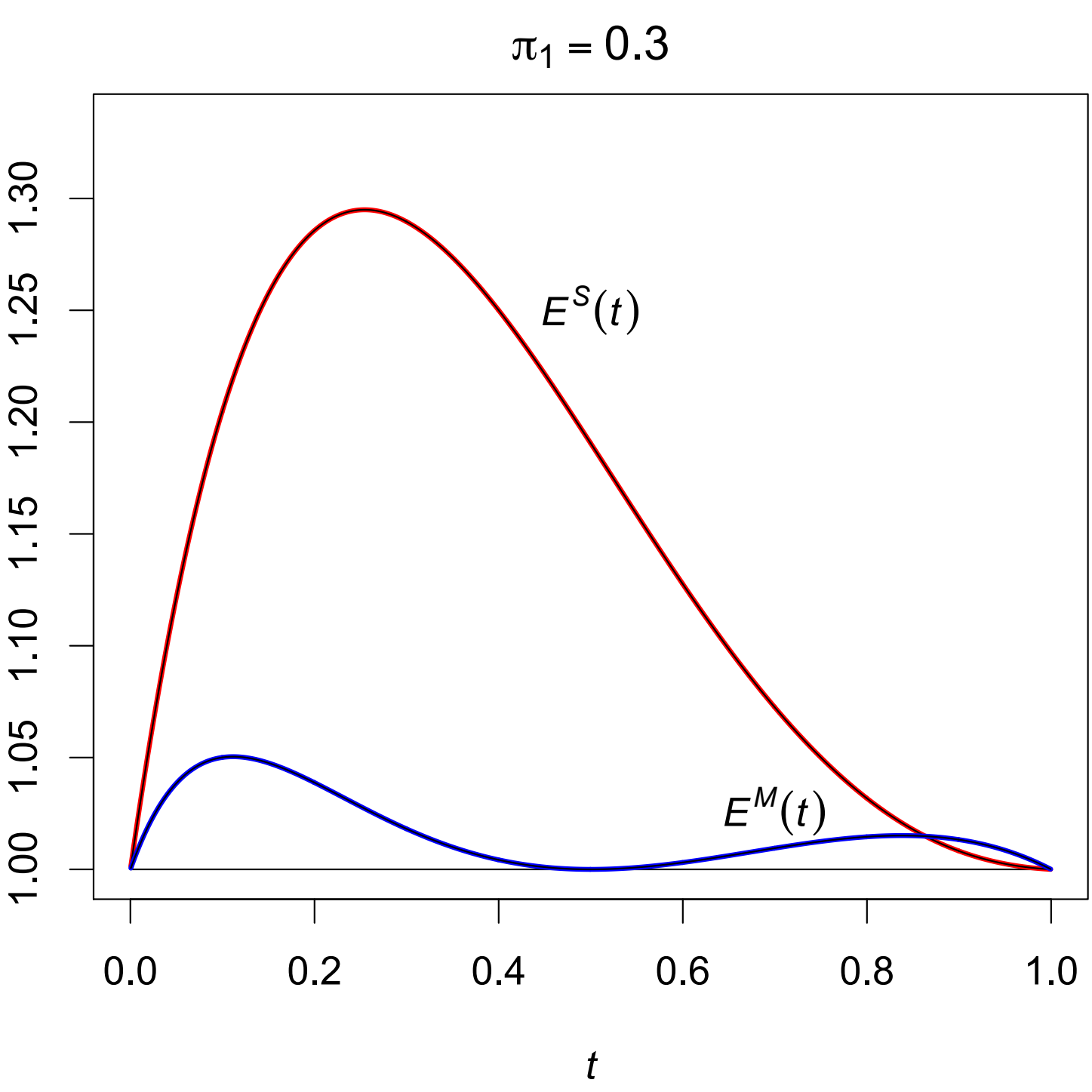
$$\frac{K^{\mathrm{M}}(t,t)}{K^{\mathrm{L}}(t,t)} \leq \frac{1}{1-(2t-1)^2/9} \leq 1.125.$$

Plots of asymptotic relative (in)efficiencies

$$(0,1) \ni t \mapsto \begin{cases} E^{\mathrm{S}}(t) & := & K^{\mathrm{S}}(t,t)/K^{\mathrm{L}}(t,t) \\ E^{\mathrm{M}}(t) & := & K^{\mathrm{M}}(t,t)/K^{\mathrm{L}}(t,t) \\ E(t) & := & K(t,t)/K^{\mathrm{L}}(t,t) \end{cases}$$


$\pi_1 = 0.5$
$E(t)$
$E^S(t) = E^M(t)$
1.00
1.05
1.10
1.15
1.20
1.25
1.30
0.0
0.2
0.4
0.6
0.8
1.0
t


$\pi_1 = 0.4$
1.00
1.05
1.10
1.15
1.20
1.25
1.30
$E^S(t)$
$E^M(t)$
0.0
0.2
0.4
0.6
0.8
1.0
t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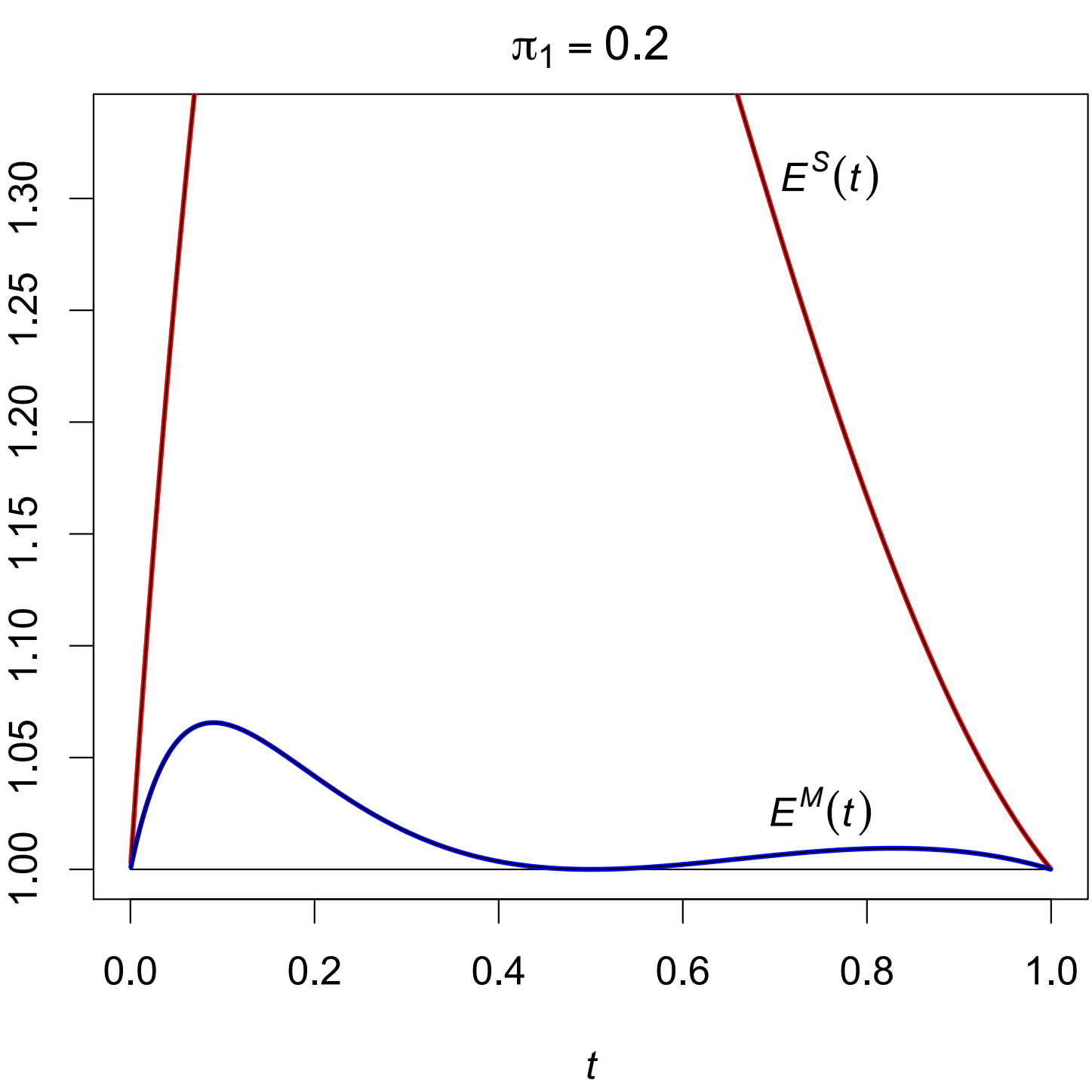
$E^M(t)$
1.00
1.05
1.10
1.15
1.20
1.25
1.30
0.0
0.2
0.4
0.6
0.8
1.0
t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$E^M(t)$
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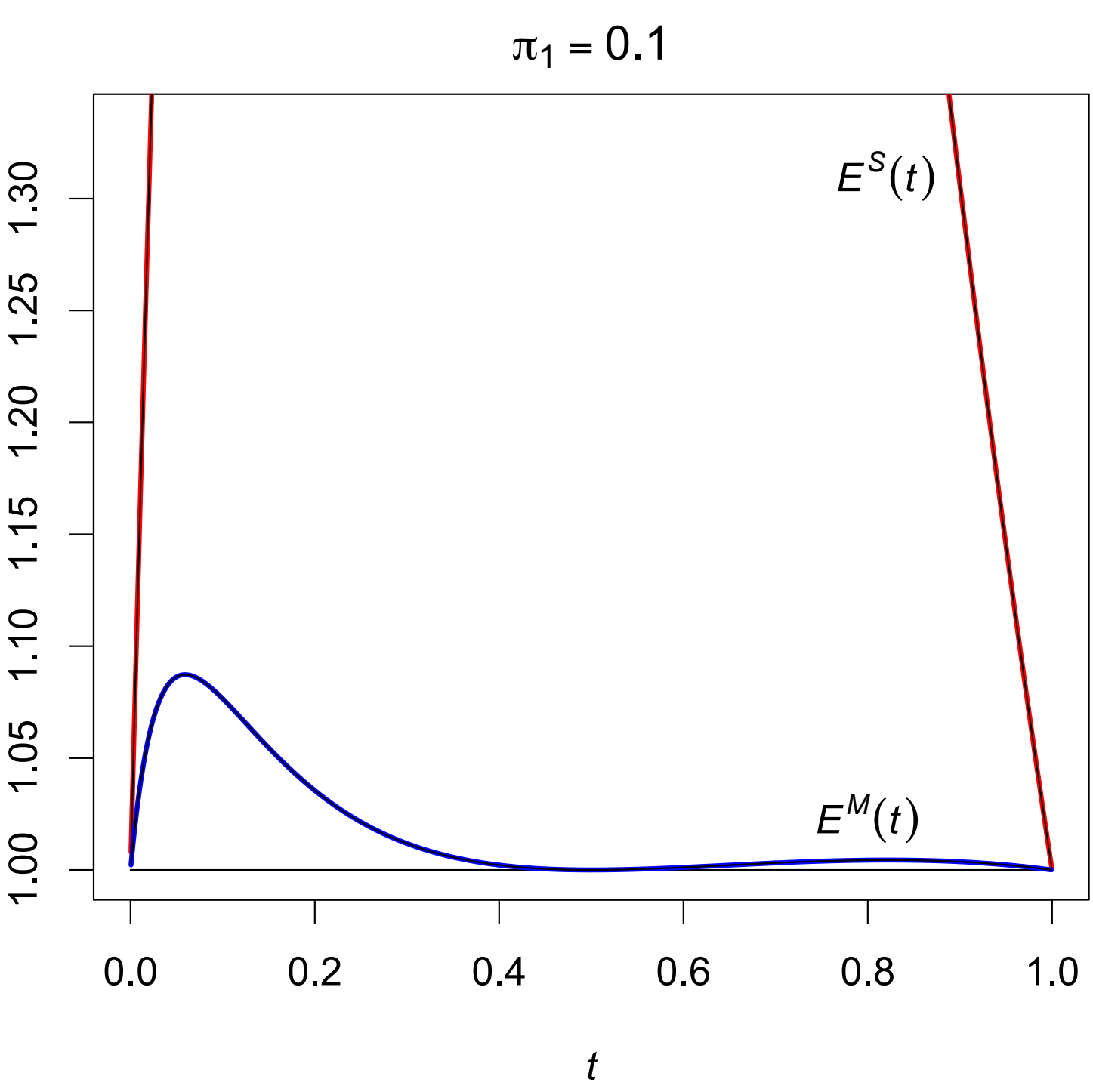
1.05
1.10
1.15
1.20
1.25
1.30
0.0
0.2
0.4
0.6
0.8
1.0
t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$E^S(t)$
$E^M(t)$
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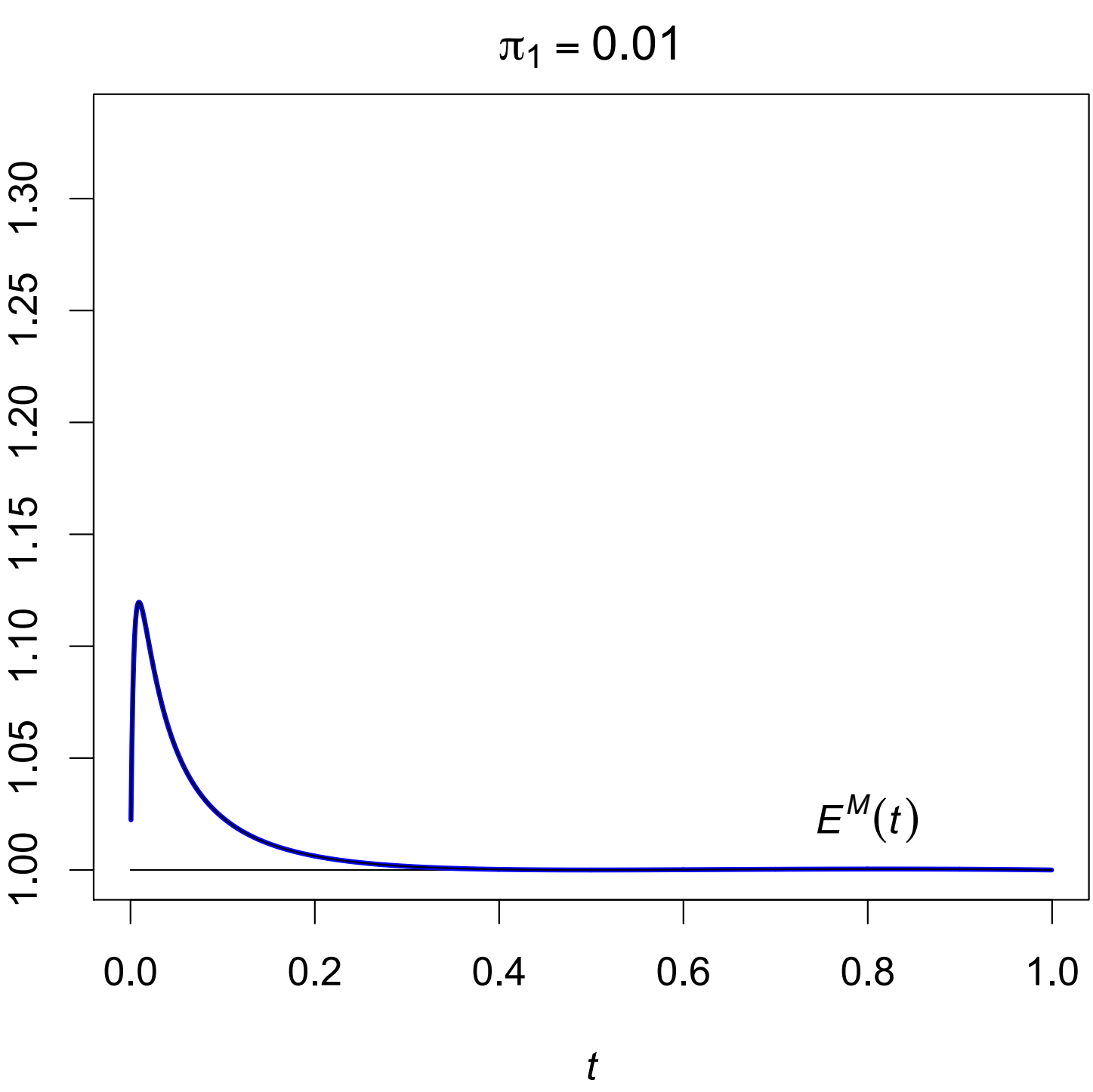
1.05
1.10
1.15
1.20
1.25
1.30
0.0
0.2
0.4
0.6
0.8
1.0
t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1.05
1.10
1.15
1.20
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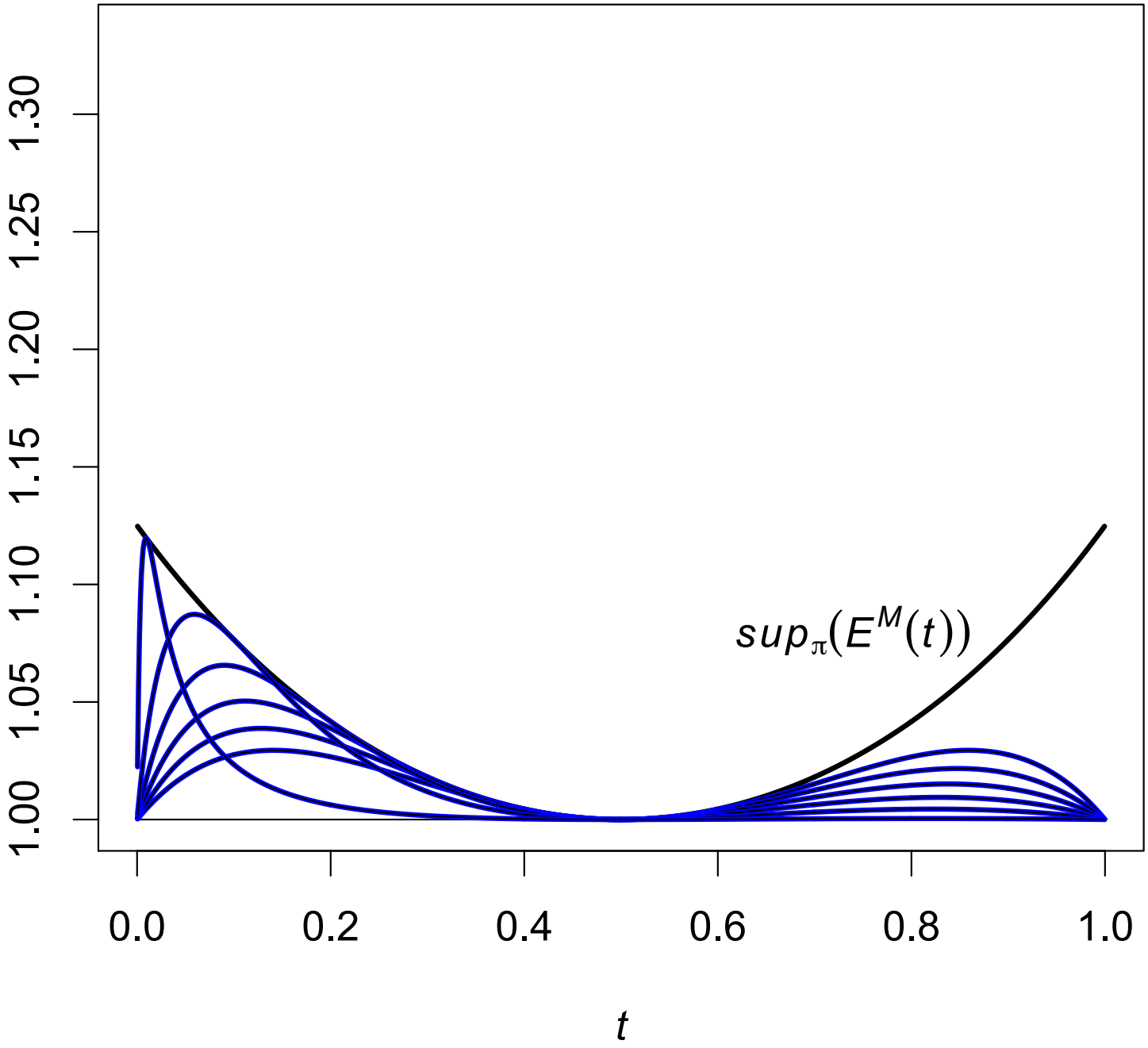
0.0
0.2
0.4
0.6
0.8
1.0
t
$E^M(t)$


1.00
1.05
1.10
1.15
1.20
1.25
1.30
0.0
0.2
0.4
0.6
0.8
1.0
t
$sup_{\pi}(E^{M}(t))$

IV. Exact Inference for Finite Samples

Pointwise confidence intervals

Adaptation of a method by Terpstra and Miller (2006) . . .

If $F(x) = p \in [0, 1]$, then conditional on $\mathbf{N} = (N_r)_{r=1}^k$,

$$n\widehat{F}(x) \ \sim \ \sum_{r=1}^{k} Y_{r,p}$$

with

$$Y_{1,p}, Y_{2,p}, \ldots, Y_{k,p} \text{ independent,}$$
$$Y_{r,p} \ \sim \ \mathrm{Bin}(N_r, B_r(p)).$$

Let

$$G_{\mathbf{N},p}(y) \;:=\; \mathbb{P}\Big(\sum_{r=1}^{k} Y_{r,p} \le y\Big).$$

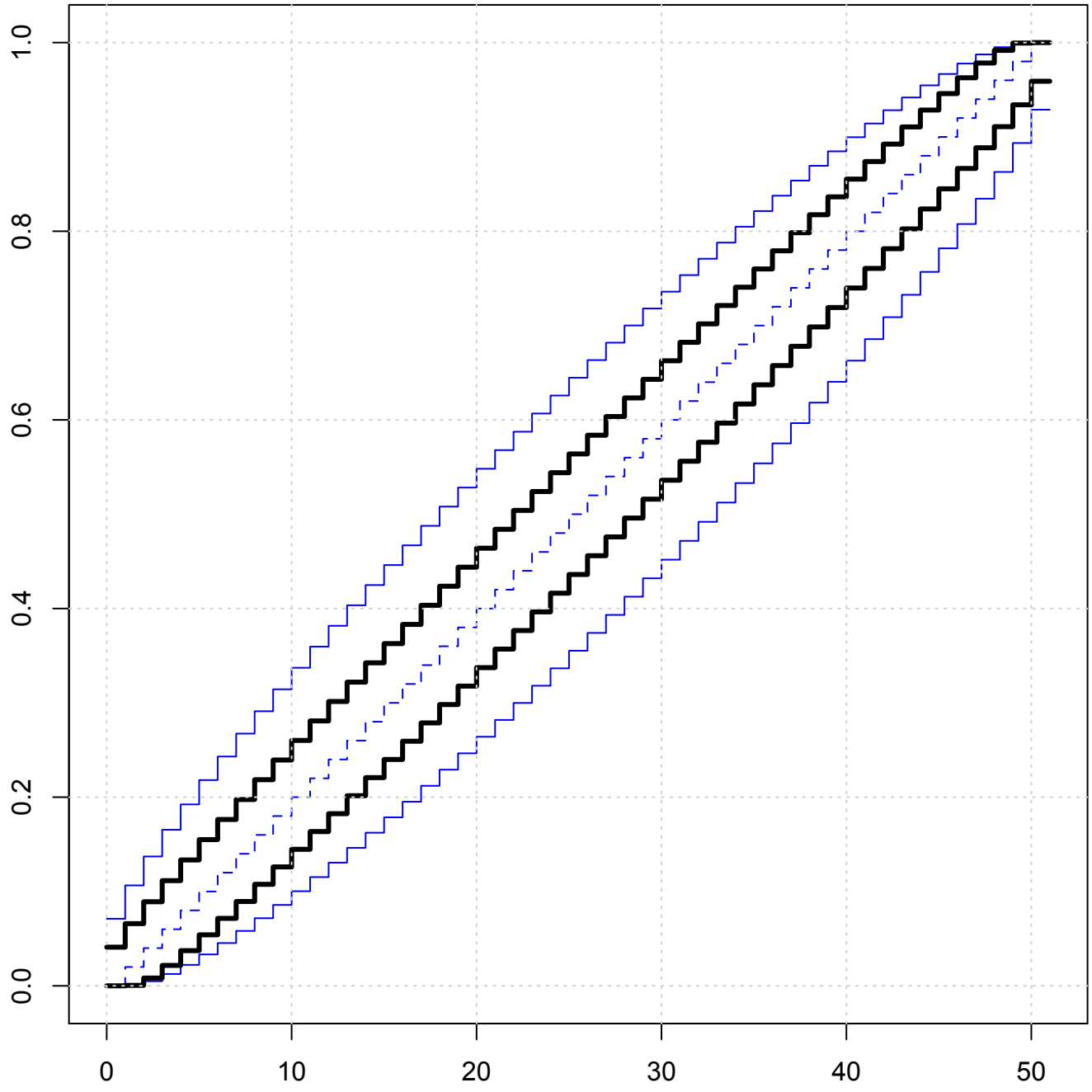
Exact p-values:

$$\begin{aligned}
&\text{Null hypothesis } \text{“}F(x) \ge p\text{”} \;:\quad G_{\mathbf{N},p}(n\widehat{F}(x)),\\
&\text{Null hypothesis } \text{“}F(x) \le p\text{”} \;:\quad 1 - G_{\mathbf{N},p}(n\widehat{F}(x) - 1).
\end{aligned}$$

Resulting one-sided confidence bounds:

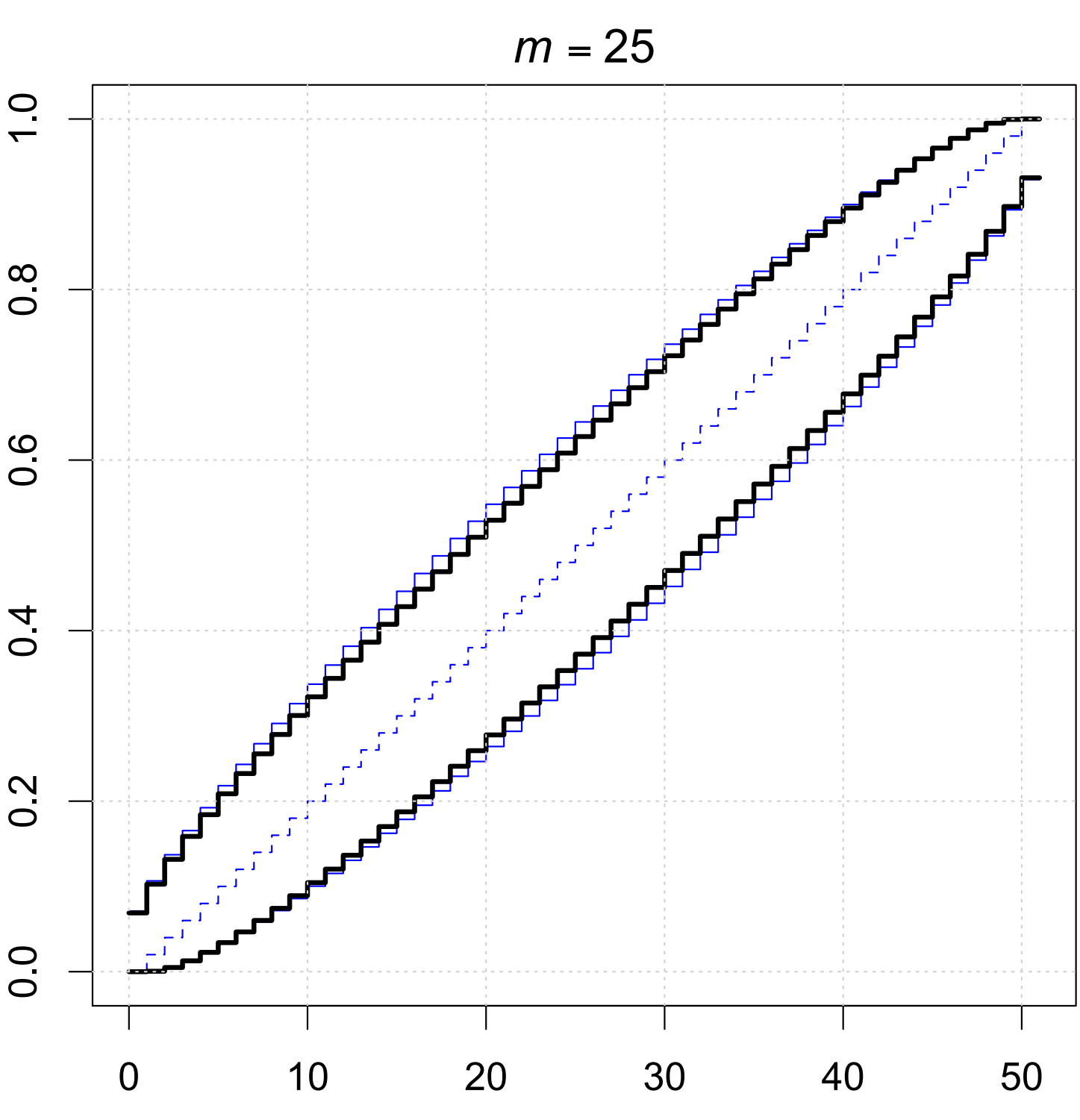
$$\begin{aligned}
\big\{p \in [0,1] : G_{\mathbf{N},p}(n\widehat{F}(x)) \ge \alpha\big\} &= \big[0, b_\alpha(\mathbf{N}, n\widehat{F}(x))\big],\\
\big\{p \in [0,1] : G_{\mathbf{N},p}(n\widehat{F}(x) - 1) \le 1 - \alpha\big\} &= \big[a_\alpha(\mathbf{N}, n\widehat{F}(x)), 1\big].
\end{aligned}$$

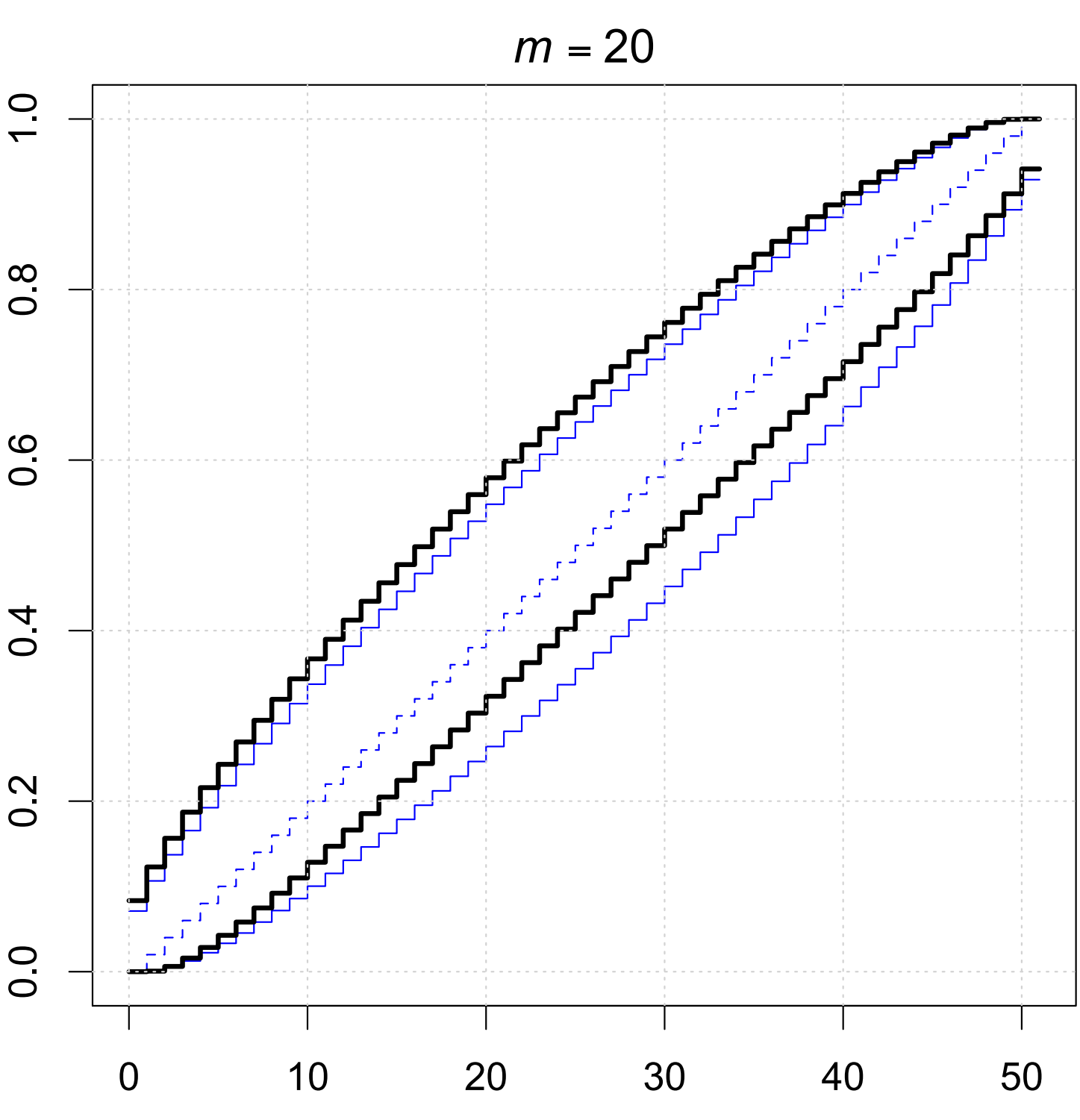
95%-confidence intervals in RSS setting, $n = k = 50$

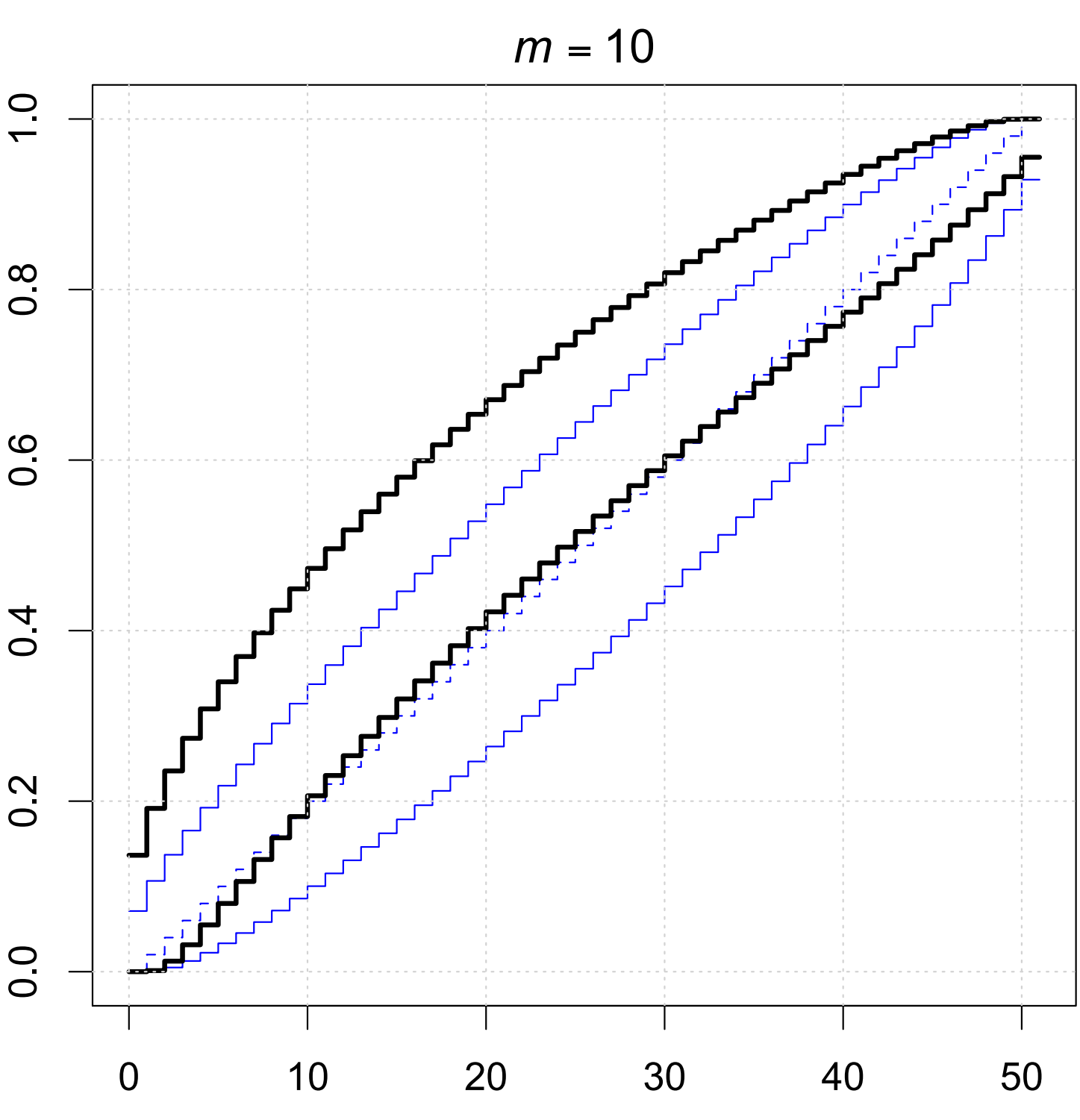


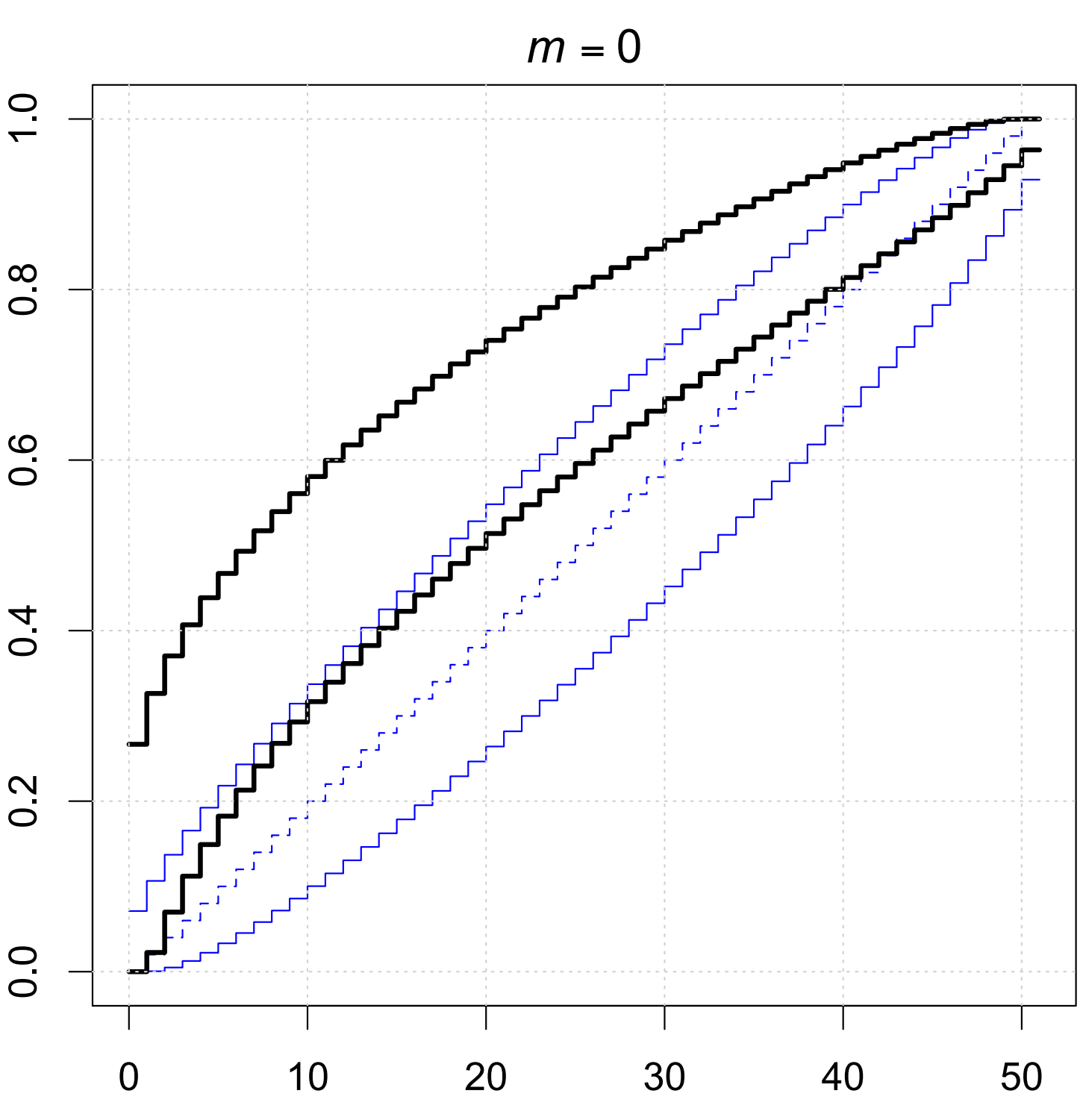
95%-confidence intervals when

$$k = 2 \quad \text{and} \quad \mathbf{N} = (m, 50 - m)^\top$$


$m = 25$
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50


m = 20
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50


$m = 10$
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50


$m = 0$
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50

Confidence bands

With confidence $1-\alpha$,

$$F(x) \;\in\; \left[\widehat{F}^{\mathrm{Z}}(x) \pm \kappa^{\mathrm{Z}}(\mathbf{N},\alpha)\right] \quad \text{for all } x \in \mathbb{R}$$

where

$$\kappa^{\mathrm{Z}}(\mathbf{N},\alpha) \;:=\; (1-\alpha)\text{-quantile of } \mathcal{L}\left(\sup_{t\in[0,1]} \left|\widehat{B}^{\mathrm{Z}}(t) - t\right| \;\middle|\; \mathbf{N} \right)$$

$$\widehat{B}^{\mathrm{Z}} \;:\quad \text{distributed as } \widehat{F}^{\mathrm{Z}} \text{ in case of } F \mathrel{\widehat{=}} \mathrm{Unif}[0,1]$$

Estimation of $\kappa^{\mathrm{Z}}(\mathbf{N},\alpha)$ via Monte Carlo simulations,
particularly easy for $\mathrm{Z} = \mathrm{M}$ ...

$\kappa^{\mathrm{M}}(m) := \kappa^{\mathrm{M}}((m, 50 - m), 5\%)$: