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Infima of Recursively Enumerable Truth Table Degrees

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Introduction Let r be a reducibility between sets of natural numbers. It is natural to study structural properties of the partial order of r-degrees, and for varying values of r such studies have formed one of the main thrusts of classical recursion theory. Because of the importance of recursively enumerable (r.e.) sets, particular attention has focused on the study of the r.e. r-degrees (i.e., those r-degrees which contain an r.e. set). For most (but not all) values of r, both the r-degrees as a whole and the r.e. r-degrees form upper semi-lattices but not lattices; that is, every pair of r-degrees in the structure has a supremum, but some pairs do not have an infimum (inf). For this reason, much attention has been paid to constructing sets whose r-degrees have some specified behavior with respect to the inf(imum) operation.

The basic internal problem is the existence of substructures with specified infima, i.e., the lattice embedding question. For some reducibilities this problem has been essentially solved. Thus, for example, the sublattices of the r.e. m- and wtt-degrees are exactly the countable distributive ones by Lachlan ([7] and [8]) and Stob ([15]). On the other hand, every recursively presented lattice is embeddable in the r.e. tt-degrees by Fejer and Shore ([4]). The situation for the r.e. T-degrees is, however, quite complex. All countable distributive lattices are embeddable ([10] and [16]) as are some ([8]) but not all ([9]) finite nondistributive ones. The general problem for T-degrees remains open. Ambos-Spies and Lerman ([1]) present the current state of affairs.

Another basic question concerns the way the r.e. r-degrees sit inside all the r-degrees. In particular, one would want to know what is the relationship, if any, between the inf of $\mathbf{a}_0$ and $\mathbf{a}_1$ considered as elements of the structure of the r.e. r-degrees and the inf of $\mathbf{a}_0$ and $\mathbf{a}_1$ considered as elements of the structure of all r-degrees. A priori, there are five possibilities if $\mathbf{a}_0,\mathbf{a}_1$ are r.e. r-degrees:

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1. $\mathbf{a}_0,\mathbf{a}_1$ have the same inf in both structures
2. $\mathbf{a}_0,\mathbf{a}_1$ have infs in both structures, but they are different
3. $\mathbf{a}_0,\mathbf{a}_1$ have an inf in the structure of the r.e. r-degrees, but not in the structure of all r-degrees
4. $\mathbf{a}_0,\mathbf{a}_1$ have an inf in the structure of all r-degrees, but not in the structure of the r.e. r-degrees
5. $\mathbf{a}_0,\mathbf{a}_1$ do not have an inf in either structure.

Based on the definitions alone, all we can say is that if both infs exist, then the one in the r.e. r-degrees is $\leq$ the one in all the r-degrees.

An early result of Lachlan ([6], Lemma 18) showed that for $r =$ Turing (T) reducibility

(∗) two r.e. r-degrees have an inf among all the r-degrees iff they have an inf among the r.e. r-degrees, and if these infs exist, they are the same.

(so only the first and fifth possibilities given above can be realized for $r = T$). A slight modification of Lachlan's argument shows that (∗) holds for $r =$ weak truth table. (See [12] for definitions of the reducibilities mentioned, and [11] for a survey of results about reducibilities which are stronger than T-reducibility.) For $r =$ many-one or one-one, we have that if $\mathbf{a},\mathbf{b}$ are r-degrees, $\mathbf{a} \leq \mathbf{b}$, and $\mathbf{b}$ is r.e., then $\mathbf{a}$ is r.e. Thus (∗) also holds for these reducibilities. Of the most commonly studied reducibilities this leaves only truth table (tt) reducibility. No obvious modification of Lachlan's proof works for tt-reducibility. In [11] we find as Problem 17 the question of whether or not (∗) holds for $r = tt$ as well. In this paper we answer Odifreddi's question negatively, in fact, we give a complete answer to this question by showing that any of the five possibilities listed above can be achieved for tt-reducibility. (Of course the fact that possibilities 1 or 5 can hold is not surprising. It is the other three that are interesting.)

It is quite easy to see that possibility 1 can hold. For instance, we can take $\mathbf{a}_0,\mathbf{a}_1$ to be comparable r.e. degrees. For a less trivial example, let A_0 and A_1 be r.e. sets whose T-degrees form a minimal pair (i.e., have inf $\mathbf{0}$). (Such A_0, A_1 exist by [6] or [17].) Let $\mathbf{a}_0,\mathbf{a}_1$ be the tt-degrees of A_0,A_1. Then $\mathbf{a}_0,\mathbf{a}_1$ are incomparable and have inf $\mathbf{0}$ among all tt-degrees and among the r.e. tt-degrees.

Results of Degtev ([2], [3]) and Kobzev ([5]) show that possibility 5 can hold for two r.e. tt-degrees. (See [11], Theorem 6.11.) Their argument is indirect. In Section 4 we indicate how a direct construction can be given using our methods.

In the next three sections we show that each of the remaining possibilities can be achieved. We give the first construction in detail and then describe the others by noting where they are different from the first. Although the constructions are similar, each has its subtleties.

We complete the introduction by giving the definitions and notation which we will use.

Our notation is for the most part standard (see e.g. [13]). By set we mean set of natural numbers and by number we mean natural number. We identify a set with its characteristic function. A *string* is an element of $2^{<\omega}$. If $\sigma \in 2^n$, then the *length* of σ, denoted $|\sigma|$, is n. We write $\{e\}$ for the eth Turing reduction.

A *truth table* is a function from 2^n to 2 for some nonnegative number n. If α: $2^n \to 2$ then the *length* of α, denoted $|\alpha|$, is n. Truth tables are finite objects and we let $\{\alpha_n\}_{n\in\omega}$ be some effective enumeration of all truth tables such that $|\alpha_n| \leq n$.

If A is a set (which we confuse with its characteristic function) and α a truth table, then we say that A *satisfies* α if $\alpha(A \upharpoonright |\alpha|) = 1$ in which case we write $\alpha^A = 1$. Otherwise we say $\alpha^A = 0$ and similarly for finite strings σ of length $\geq |\alpha|$.

If A and B are sets, we say that A is truth table (tt) reducible to B ($A \leq_{tt} B$) if for some recursive function f, for every x, $x \in A \leftrightarrow \alpha^B_{f(x)} = 1$. Then $\leq_{tt}$ is reflexive and transitive and we define as usual an equivalence relation $\equiv_{tt}$ by $A \equiv_{tt} B$ if $A \leq_{tt} B$ and $B \leq_{tt} A$. We write $\deg_{tt}(A)$ for the $\equiv_{tt}$ equivalence class of A, i.e., for the tt-degree of A. By a result of Nerode (see [12], p. 143), $A \leq_{tt} B$ iff $A = \{e\}^B$ for some e such that $\{e\}^X$ is total for every X.

If $\{e\}(x)\downarrow$, we write $[e](x)$ for $\alpha_{\{e\}(x)}$. For a number e and set A we define a partial function $[e]^A$ by

$$[e]^A(x) = \begin{Bmatrix} ([e](x))^A & \text{if } \{e\}(x)\downarrow \\ \uparrow & \text{otherwise} \end{Bmatrix}.$$

Then $A \leq_{tt} B$ iff $(\exists e)(A = [e]^B)$.

If σ is a string, we define a partial function $[e]^\sigma$ by

$$[e]^\sigma(x) = \begin{Bmatrix} ([e](x))^\sigma & \text{if } \{e\}(x)\downarrow \text{ and } |\sigma| \geq |[e](x)| \\ \uparrow & \text{otherwise} \end{Bmatrix}.$$

We also define $[e]^A_s$ by

$$[e]^A_s(x) = \begin{Bmatrix} ([e](x))^A & \text{if } \{e\}_s(x)\downarrow \\ \uparrow & \textit{otherwise} \end{Bmatrix}.$$

$[e]^\sigma_s$ is defined similarly.

We adopt the usual conventions, so that if $\{e\}_s(x)\downarrow$, then

$$|[e](x)| \leq \{e\}_s(x) < s.$$

If A is a finite set, we write $\#A$ for the cardinality of A. We let $\langle .,. \rangle$ be a recursive pairing function such that $x,y \leq \langle x,y \rangle$ and we write $A^{[n]}$ for $\{\langle n,x \rangle : \langle n,x \rangle \in A\}$ and $A^{[\leq n]}$ for $\bigcup_{n' \leq n} A^{[n']}$.

We write ω for the set of natural numbers. If ϕ and ψ are partial funtions, $\phi(x) \neq \psi(x)$ means that $\phi(x)$ and $\psi(x)$ both converge and have different values; $\neg(\phi(x) = \psi(x))$ means that either $\phi(x)$ or $\psi(x)$ diverges, or else $\phi(x) \neq \psi(x)$.

1 Two r.e. tt-degrees with different infima in the r.e. tt-degrees and in all the tt-degrees

Theorem 1.1 *There are r.e. tt-degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *form a minimal pair (i.e., have inf* $\mathbf{0}$*) among the r.e. tt-degrees, but have a nonzero inf among all the tt-degrees.*

Proof: We build A_0,A_1 r.e. and C not of r.e. *tt*-degree so that if $\mathbf{a}_0 = \deg_{tt}(A_0)$, $\mathbf{a}_1 = \deg_{tt}(A_1)$ and $\mathbf{c} = \deg_{tt}(C)$, then $\mathbf{a}_0,\mathbf{a}_1$ have inf $\mathbf{c}$ among all the *tt*-degrees, but $\mathbf{a}_0,\mathbf{a}_1$ have inf $\mathbf{0}$ in the r.e. *tt*-degrees.

We have three types of requirements, where e and i run over the integers:

Type I: $C \neq \{e\}$
Type II: $W_e = [i]^C \to W_e$ is recursive
Type III: $[e]^{A_0} = [e]^{A_1} = Y \to Y \leq_{tt} C$.

In addition, we will ensure that $C \leq_{tt} A_i$ for $i = 0$ or 1. Suppose that we meet all the requirements and meet the conditions of the previous sentence. By a now-standard argument due to Posner, if we meet all the Type III requirements and in addition ensure that $C \leq_{tt} A_i$, then $\mathbf{a}_0,\mathbf{a}_1$ have inf $\mathbf{c}$ among all *tt*-degrees. Thus by the Type II requirements, the only r.e. sets *tt*-reducible to both A_0 and A_1 are the recursive ones so $\mathbf{a}_0,\mathbf{a}_1$ have inf $\mathbf{0}$ among the r.e. *tt*-degrees. The Type I requirements ensure that $\mathbf{c} \neq \mathbf{0}$.

Let $\{R_n\}_{n\in\omega}$ be an effective listing of the requirements. We say that $R_{n'}$ has higher priority than R_n if $n' < n$.

In order to make $C \leq_{tt} A_i$, $i = 0$ and 1, we ensure during the construction that

$$x \in C_s \text{ iff } \#A_{i,s}^{[x]} \text{ is odd}$$

and

$$\langle x,y\rangle \in A_{i,s},\ y' < y \to \langle x,y'\rangle \in A_{i,s}.$$

In addition, we put on restraints so that R_n cannot make any changes in $A_i^{[n']}$ with $n' < n$. Since we will have a finite injury construction, these restraints will ensure that for each x, $A_i^{[x]}$ is finite. The conditions given so far are enough to ensure that $A_i \leq_T C$; however, to get $A_i \leq_{tt} C$ we need more, namely a recursive f such that $\langle x,y\rangle \in A_i \to y \leq f(x)$. One might hope that the existence of such an f would follow immediately from the way we try to meet the requirements R_n, but the situation is more complicated than that because of the Type III requirements. We will return to this point later.

We meet Type I requirements by the usual Friedberg-Muchnik technique; i.e., we pick a witness x and keep it out of C either until we see $\{e\}_s(x) = 0$ or permanently. If we see $\{e\}_s(x) = 0$, then we put x into C.

Our strategy for meeting Type II requirements might be described as "do your best to make $W_e \neq [i]^C$; if you fail then W_e is recursive". In more detail, for each x, we wait for a stage $s + 1$ such that $\{i\}_s(x)\downarrow$. Then we look for two strings σ_0,σ_1, both suitable to be initial segments of C such that $[i]^{\sigma_0}(x) = 0$, $[i]^{\sigma_1}(x) = 1$. If such strings exist, we begin an attack on the requirement by making sure that $\sigma_1 \subseteq C_{s+1}$. If at some later stage t we see $x \in W_{e,t}$, then we change C so that $\sigma_0 \subseteq C_t$. In this way, we ensure that $W_e(x) \neq [i]^C(x)$. In order to maintain the strategy to get $C \leq_{tt} A_i$ discussed above, each time a number x is put into or taken out of C, we must put into A_i, $i = 0$ and 1, the least number in $\omega^{[x]}$ which is not yet in A_i.

Suppose that $W_e = [i]^C$. Then we argue that W_e is recursive as follows (we ignore other requirements): Given x, find s such that $\{i\}_s(x)\downarrow$. At stage $s + 1$ we do not begin an attack (else $W_e \neq [i]^C$). Thus for all suitable σ, $[i]^\sigma(x)$

gives the same answer. Since $C \restriction |[i](x)|$ is one of the "suitable" σ's which was considered, it follows that this common answer for $[i]^{\sigma}(x)$ is $[i]^{C}(x) = W_e(x)$. Thus we have a way of computing W_e recursively.

To meet Type III requirements, we again try to make the antecedant $([e]^{A_0} = [e]^{A_1})$ false. If we never get an opportunity to do this, then $[e]^{A_0} \leq_{tt} C$. In more detail, given x, we wait for an s such that $\{e\}_s(x)\downarrow$. At stage $s+1$, we consider all pairs B_0, B_1 of finite sets which are suitable to be $A_{0,s+1}$ and $A_{1,s+1}$. (Since A_0, A_1, are to be r.e., suitability entails, among other things, that $A_{i,s} \subseteq B_i$, $i = 0,1$.) For each such pair with the additional property that for each y, $\#B_0^{[y]}, \#B_1^{[y]}$ have the same parity, we see if $[e]^{B_0}(x) = [e]^{B_1}(x)$. If not, then we begin an attack by making $A_{i,s+1} = B_i$, $i = 0,1$, and restraining numbers $< |[e](x)|$ from entering either A_i. We also adjust C to maintain our strategy to make $A_i \leq_{tt} C$. (This is why we need the condition on $\#B_i^{[y]}$.) If the restraints are not violated, we have thus ensured that $[e]^{A_0}(x) \neq [e]^{A_1}(x)$.

Now suppose that $[e]^{A_0} = [e]^{A_1} = Y$. We argue that $Y \leq_{tt} C$ as follows (we again ignore other requirements): Given x, find s such that $\{e\}_s(x)\downarrow$. Since no attack is made at stage $s+1$, it must be that for any two suitable B_0, B_1 with $\#B_0^{[y]}, \#B_1^{[y]}$ of the same parity for every y, $[e]^{B_0}(x) = [e]^{B_1}(x)$. We can now give a tt-reduction. Given σ, a potential initial segment of C of the proper length, find suitable B_0, B_1, if any, such that for all $x < |\sigma|$, $\#B_0^{[x]}$ is odd iff $\sigma(x) = 1$ and give $[e]^{B_0}(x) = [e]^{B_1}(x)$ as answer. As long as large enough initial segments of A_0, A_1 are among the "suitable" B_0, B_1, this tt-reduction, when applied to C gives the answer $[e]^{A_0}(x) = [e]^{A_1}(x) = Y(x)$, so $Y \leq_{tt} C$.

We now discuss what constraints must be put on the action taken for sake of Type III requirements to ensure the existence of the recursive function f needed to make $A_i \leq_{tt} C$. If a given Type III requirement R_n is allowed to add numbers $\langle x,y\rangle$ into A_i with no bound on y, then at the end of the construction there is no reason why we should have the function f we desire. Thus we must try another approach. We calculate a recursive function f such that the restriction $\langle x,y\rangle \in A_i \rightarrow y \leq f(x)$ still leaves enough room to act for all Type I and II requirements which might want to put numbers into $A_i^{[x]}$. Now we might consider restricting Type III R_n by insisting that if such an R_n puts a number $\langle x,y\rangle$ into A_i, then $y \leq f(x)$. However, this still leaves such R_n too much leeway. For, if a Type III R_n puts all numbers $\langle x,y\rangle$ with $y \leq f(x)$ into A_i for some x, then no higher priority R_n' of Type II can put x into C or take x out of C, because all possible numbers for coding such a change are already in A_i. If this happens often enough, then a given Type II requirement may never be met because of the actions of lower priority Type III requirements.

To prevent this from happening, we define a function $f'(x,n)$ and add the restriction on a Type III R_n that for B_i to be suitable, if $\langle x,y\rangle \in B_i - A_{i,s}$, then $y \leq f'(x,n)$. We make $f'(x,n)$ sufficiently smaller than $f(x)$ so that even if a Type III R_n puts all numbers $\langle x,y\rangle$ with $y \leq f'(x,n)$ into A_i, for some x, there are still enough y with $f'(x,n) < y \leq f(x)$ so that higher priority $R_{n'}$'s can change $C(x)$ if they want to. Since $f'(x,n) < f(x)$, this means that a given Type III R_n does not always consider initial segments of the final A_0, A_1 to be suitable. This may seem to be a problem, but the key observation is that $f'(x,n)$ can be defined so that if s is such that no requirement $R_{n'}$ with $n' < n$ receives attention at a stage $\geq s$ and $\langle x,y\rangle$ enters A_i after stage s, then $y \leq f'(x,n)$.

Thus, once higher priority requirements stop acting, a given Type III requirement does consider relevant initial segments of A_0, A_1 to be suitable in spite of the restrictions put on R_n, and this is enough to make the $Y \leq_{tt} C$ argument from above go through.

It turns out by a careful counting argument that $f(x) = 2x + 2$ and $f'(x,n) = 2(x - n)$ work.

We may summarize a part of this discussion by saying that we make $C \leq_{tt} A_i$ by ensuring

(1.1) $x \in C$ iff $\#\{y: y \leq 2x + 2 \wedge \langle x,y \rangle \in A_i\}$ is odd.

We now give the definitions and construction. From time to time in the construction, a requirement may "receive attention". In particular, it may begin an attack through some number x. (At most one attack exists for a given requirement at a given time.) At the same time, the requirement puts on a restraint. The restraint is said to have the same priority as the requirement. The attack remains in effect until some higher priority requirement receives attention and thus cancels attacks of lower priority requirements. When an attack on a Type II requirement is begun, an "alternate string" is assigned to the attack. Aside from restraints put on when an attack is made, each requirement puts on an initial restraint at stage 0. Restraints are not canceled.

At the end of stage s, we let $r(n,s)$ be the largest restraint put on by R_n by the end of stage s and we define

$$R(n,s) = \max\{r(n',s): n' < n\}.$$

If R_n receives attention at stage s and x enters A_0 or A_1 at stage s, then we say that R_n puts x into A_0 or A_1 at stage s.

We now define the phrase "R_n requires attention at stage $s + 1$" according to the type of R_n.

(1.2) *R_n is $C \neq \{e\}$ (Type I):*
- (a) R_n is not under attack at stage $s + 1$, or
- (b) R_n is under attack through x and
  - (i) $\{e\}_{s+1}(x) = 0$, and
  - (ii) $x \notin C_s$.

(1.3) *R_n is $W_e = [i]^C \to W_e$ is recursive (Type II):*
- (a) R_n is not under attack at stage $s + 1$ and there are an x and strings σ_0, σ_1 such that
  - (i) $\{i\}_s(x)\downarrow$
  - (ii) $|\sigma_0| = |\sigma_1| = |[i](x)|$
  - (iii) $(\forall y < R(n,s))\,[y < |\sigma_0| \to \sigma_0(y) = \sigma_1(y) = C_s(y)]$
  - (iv) $[i]^{\sigma_0}(x) = 0,\ [i]^{\sigma_1}(x) = 1$

or
- (b) R_n is under attack through x and
  - (i) $x \in W_{e,s+1}$, and
  - (ii) $[i]_s^{C_s}(x) = 1$.

(1.4) *R_n is $[e]^{A_0} = [e]^{A_1} = Y \to Y \leq_{tt} C$ (Type III):* R_n is not under attack at stage $s + 1$ and there are a number x and finite sets B_0, B_1 such that
- (i) $x \geq R(n,s)$
- (ii) $\{e\}_s(x)\downarrow$

(iii) $A_{i,s} \subseteq B_i$, $i = 0,1$
(iv) for $i = 0$ or 1, $[w \in B_i - A_{i,s} \to w \in \omega^{[y]}$ for some y, $R(n,s) \leq y < |[e](x)|]$
(v) $\langle y,z\rangle \in B_i \to (\forall z' < z)(\langle y,z'\rangle \in B_i)$, for $i = 0$ and 1
(vi) $\langle y,z\rangle \in B_i \wedge y \geq R(n,s) \to z \leq 2(y - n)$, for $i = 0$ and 1
(vii) $(\forall y < |[e](x)|)(\#B_0^{[y]}, \#B_1^{[y]}$ have the same parity), and
(viii) $[e]^{B_0}(x) \neq [e]^{B_1}(x)$.

Note that one can effectively determine at a given stage whether or not a given R_n of Type III requires attention: by (1.4ii) there are only finitely many x to consider and by (1.4iv) and (1.4vi) for each such x there are only finitely many B_0, B_1 to consider.

We now construct A_0, A_1, C. We write $A_{0,s}, A_{1,s}, C_s$ for those numbers in the given set at the end of stage s. A number once put into A_0 or A_1 is never removed. A number may be put into and taken out of C repeatedly. We show later that for each x there is a stage s such that x is neither put into nor taken out of C after stage s. This ensures that C is a well-defined Δ_0^2 set.

Construction:

Stage $s = 0$. For each n, enumerate a restraint of priority n equal to $n + 1$.

Stage $s + 1$. Find the least n such that R_n requires attention at stage $s + 1$ and carry out the appropriate action below depending on the type of R_n. We say that R_n receives attention at stage $s + 1$. (Such n exists since at stage $s + 1$ almost all requirements of Type I require attention via (1.2a).)

R_n *is* $C \neq \{e\}$:

(1.2a) holds: Begin an attack on R_n through x where x is the least number $\geq R(n,s)$ which is not in C_s and such that $\{e\}_{s+1}(x)\uparrow$. Enumerate a restraint of priority n equal to $\max\{x + 1, s + 1\}$.

(1.2b) holds: Put x into C_{s+1}. For $i = 0$ and 1, find the least z such that $\langle x,z\rangle \notin A_{i,s}$ and put $\langle x,z\rangle$ into $A_{i,s+1}$. Enumerate a restraint of priority n equal to $s + 1$.

R_n *is* $W_e = [i]^C \to W_e$ *is recursive*:

(1.3a) holds: Take the least x and then the first strings σ_0, σ_1 (in some effective enumeration of pairs of strings) which make (1.3a) hold. Begin an attack on R_n through x. Change $C_s \upharpoonright |\sigma_1|$ as needed so that $\sigma_1 \subseteq C_{s+1}$. Assign σ_0 to be the alternate string for the attack. Enumerate a restraint of priority n equal to $\max\{|\sigma_0|, s + 1\}$.

For each y such that $C_s(y) \neq C_{s+1}(y)$ and $i = 0$ or 1, find the least z such that $\langle y,z\rangle \notin A_{i,s}$ and put $\langle y,z\rangle$ into $A_{i,s+1}$.

(1.3b) holds: Modify $C_s \upharpoonright |\sigma|$, where σ is the alternate string for the attack, so that $\sigma \subseteq C_{s+1}$. Enumerate a restraint of priority n equal to $s + 1$. Proceed as in the second paragraph under "(1.3a) holds.".

R_n *is* $[e]^{A_0} = [e]^{A_1} = Y \to Y \leq_{tt} C$:

Take the least x and then the least B_0, B_1 (in some effective enumeration of pairs of finite sets) which make (1.4) true. Make $A_{i,s+1} = B_i$, $i = 0$ and 1. (By

(1.4iii) this involves only adding elements to A_i.) Enumerate a restraint of priority n equal to $\max\{|[e](x)|, s+1\}$.

For each $y < |[e](x)|$, put y into C or take y out of C, if necessary, to ensure that $y \in C_{s+1}$ iff $\#A_{0,s+1}^{[y]}$ is odd.

We give some properties of the construction.

(1.5) For each s, $R(0,s) \leq R(1,s) \leq R(2,s) \leq \ldots$.
For each n, $R(n,0) \leq R(n,1) \leq \ldots$.

(1.6) If no $R_{n'}$ with $n' < n$ receives attention at stage $s+1$, then $R(n,s+1) = R(n,s)$.

(1.7) For all $x,s, x \in C_s$ iff $\#A_{0,s}^{[x]}$ is odd.

(1.8) For all x,s, $\#A_{0,s}^{[x]}$, $\#A_{1,s}^{[x]}$ have the same parity.

(1.9) For all x,y,s and $i = 0$ or 1,

$$\langle x,y\rangle \in A_{i,s} \to (\forall y' < y)(\langle x,y'\rangle \in A_{i,s}).$$

(1.10) If R_n receives attention at stage $s+1$ and $C_{s+1}(x) \neq C_s(x)$ or $A_{i,s+1}(\langle x,y\rangle) \neq A_{i,s}(\langle x,y\rangle)$, then $R(n,s) = R(n,s+1) \leq x < R(n+1, s+1)$.

Properties (1.5)–(1.9) are easy to verify. Property (1.10) is only slightly harder. Suppose that (1.10) holds for all $s' < s$ and that R_n receives attention at stage $s+1$ and $C_{s+1}(x) \neq C_s(x)$ or $A_{i,s+1}(\langle x,y\rangle) \neq A_{i,s}(\langle x,y\rangle)$. We consider the case where R_n is of Type II and (1.3b) holds; the other cases are similar but easier. The fact that $R(n,s) = R(n,s+1)$ is immediate. To see that $x < R(n+1, s+1)$, note that $x < |\sigma|$ where σ is the alternate string and when the attack began at stage $t+1 \leq s$, a restraint of priority $n \geq |\sigma|$ was enumerated. Finally, to see that $R(n,s) \leq x$, it suffices to show that if $z < R(n,s)$, then $C_s(z) = C_{s+1}(z)$. If $z \geq |\sigma|$, then this is immediate, so suppose that $z < |\sigma|$. Then $C_{s+1}(z) = \sigma(z)$. Since the attack on R_n was not canceled before stage $s+1$, no $R_{n'}$ with $n' < n$ received attention at a stage u with $t+1 \leq u \leq s$, so by (1.6), $R(n,u) = R(n,t)$ for all u, $t \leq u \leq s$. Thus $z < R(n,s) = R(n,t)$, so by (1.3a.iii), $\sigma(z) = C_t(z)$. Now, applying the induction hypothesis to u with $t \leq u < s$ and using (1.5), we have that $C_u \upharpoonright R(n,u) = C_{u+1} \upharpoonright R(n,u)$. But $z < R(n,s) = R(n,u)$, so $C_u(z) = C_{u+1}(z)$. Thus $C_t(z) = C_s(z)$ and so finally $C_{s+1}(z) = \sigma(z) = C_t(z) = C_s(z)$ as desired.

Lemma 1.2 *Suppose that R_n is a requirement of Type I or II, R_n receives attention at stage $t+1$, $s > t$, and no requirement $R_{n'}$ with $n' < n$ receives attention at a stage u, $t+1 \leq u \leq s+1$. Then there is at most one u, $t+1 < u \leq s+1$ such that R_n receives attention at stage u. Furthermore, if R_n is $C \neq \{e\}$ and the attack at stage $t+1$ is through x, then $\neg(C_{s+1}(x) = \{e\}_{s+1}(x))$, while if R_n is $W_e = [i]^C \to W_e$ is recursive, and the attack at stage $t+1$ is through x, then $W_{e,s+1}(x) \neq [i]_{s+1}^{C_{s+1}}(x)$.*

Proof: First note that the attack on R_n in progress at the end of stage $t+1$ is still in progress at the end of stage $s+1$. Say that this attack is through x. Suppose that R_n is $C \neq \{e\}$. We consider two cases. First suppose that R_n receives attention at some stage $u+1$ with $t+1 \leq u+1 \leq s+1$ such that (1.2b) holds for $u+1$. Then x is put into C_{u+1} and one easily sees that x stays in C until at least the end of stage $s+1$. Thus R_n does not require attention at any stage v,

$u + 1 < v \leq s + 1$, so R_n does not receive attention at any such stage v, and R_n receives attention only once at a stage after $t + 1$ and at or before $s + 1$. Also $\{e\}_{u+1}(x) = 0$, so $\{e\}_{s+1}(x) = 0 \neq 1 = C_{s+1}(x)$. If the first case does not hold, then (1.2a) must hold at stage $t + 1$ and R_n does not receive attention at any stage u, $t + 1 < u \leq s + 1$. Also $x \notin C_t$ and one easily sees that x is not put into C until at least after stage $s + 1$, so $C_s(x) = C_{s+1}(x) = 0$. If $\{e\}_{s+1}(x) = 0$, then R_n would receive attention at stage $s + 1$, contradicting our assumption, so $\neg(C_{s+1}(x) = \{e\}_{s+1}(x))$.

Now suppose that R_n is $W_e = [i]^C \to W_e$ is recursive. We again consider two cases. First suppose that R_n receives attention at some stage $u + 1$ with $t + 1 \leq u + 1 \leq s + 1$ such that (1.3b) holds for $u + 1$. Then the alternate string σ for the attack is $\subseteq C_{u+1}$ and one easily sees that $\sigma \subseteq C_v$ for all v with $u + 1 \leq v \leq s + 1$. By definition of σ, this means that $[i]^{C_v}(x) = 0$ for all v, $u + 1 \leq v \leq s + 1$, and thus R_n does not receive attention at any stage v with $u + 1 < v \leq s + 1$. Since (1.3a) does not hold any stage v with $t + 1 < v \leq s + 1$, it follows that R_n receives attention at most once after stage $t + 1$ and at or before stage $s + 1$. Furthermore, $x \in W_{e,u+1}$ so $x \in W_{e,s+1}$ and $W_{e,s+1}(x) = 1 \neq 0 = [i]^{C_{s+1}}(x)$. If the first case does not hold, then (1.3a) holds at stage $t + 1$ and R_n does not receive attention at any stage u with $t + 1 < u \leq s + 1$. Furthermore, $[i]^{C_{s+1}}(x) = 1$ and C does not change below $|[i](x)|$ by the end of stage $s + 1$ so $[i]^{C_s}(x) = [i]^{C_{s+1}}(x) = 1$. If $x \in W_{e,s+1}$, then R_n would receive attention at stage $s + 1$, contradicting our assumption. Thus $W_{e,s+1}(x) = 0 \neq 1 = [i]^{C_{s+1}}(x)$.

Lemma 1.3 *For $i = 0$ or 1 and all $\langle x,y\rangle, \langle x,y\rangle \in A_i$ only if $y \leq 2(x + 1)$. In fact, for each fixed x, if $\langle x,y\rangle \in A_{i,s}$ ($i = 0$ or 1), $-1 \leq n \leq x$, and no requirement of priority $\leq n$ has put a number into $A_0^{[x]}$ or $A_1^{[x]}$ by the end of stage s, then $y \leq 2(x - n)$. The basic fact is then just the case $n = -1$ as there are no requirements of priority <0.*

Proof: Fix x. We use backwards induction on n. Suppose $n = x$. At stage 0, a restraint of priority n equal to $n + 1 = x + 1$ is put on. Thus by (1.10) no $R_{n'}$ with $n' > n$ can put a number into $A_i^{[x]}$ ($i = 0$ or 1) at any stage. Hence, under the hypotheses, $A_{i,s}^{[x]} = \emptyset$ ($i = 0,1$) so the lemma holds vacuously.

Now suppose that the result holds for some n with $0 \leq n \leq x$. We show that the result holds for $n - 1$. Hence we suppose that no requirement of priority $\leq n - 1$ has put a number into $A_0^{[x]}$ or $A_1^{[x]}$ by the end of stage s and $\langle x,y\rangle \in A_{i,s}$. If in fact no requirement of priority $\leq n$ has put a number into $A_0^{[x]}$ or $A_1^{[x]}$ by the end of stage s then by induction hypothesis, $y \leq 2(x - n) < 2(x - (n - 1))$ so we are done. Thus suppose that R_n has put a number into $A_0^{[x]}$ or $A_1^{[x]}$ by the end of stage s. Say that this first happens at stage $t + 1 \leq s$. Now we may apply the induction hypothesis to stage t to conclude that if $\langle x,y\rangle \in A_{i,t}$, then $y \leq 2(x - n) < 2(x - (n - 1))$. Furthermore, by (1.10), $x < R(n + 1, t + 1)$. Hence, at no stage $\geq t + 1$ can a requirement $R_{n'}$ with $n' > n$ put a number into $A_i^{[x]}$. Thus we may assume that $\langle x,y\rangle \in A_{i,s}$ entered A_i at a stage $\geq t + 1$ and hence was put in by R_n. If R_n is of Type III, then by (1.4iv) and (1.4vi), $y \leq 2(x - n)$.

Suppose that R_n is of Type I or II and that $\langle x,y\rangle$ was put into A_i at stage $u + 1$, $t + 1 \leq u + 1 \leq s$. Now since R_n put a number into $A_0^{[x]}$ or $A_1^{[x]}$ at stage $t + 1$, it follows from the usual conventions (e.g., if $\{e\}_s(x)\downarrow$ then $x < s$) that

$x < t + 1$. If some $R_{n'}$ with $n' < n$ received attention at a stage v, $t + 1 \leq v \leq u + 1$, then $R(n,u + 1) \geq R(n,v) \geq v \geq t + 1 > x$ and hence R_n could not put a number into $A_i^{[x]}$ at stage $u + 1$. Thus at no stage v, $t + 1 \leq v \leq u + 1$ does an $R_{n'}$ with $n' < n$ receive attention. By Lemma 1.2, stage $u + 1$ is at most the second stage at or after stage $t + 1$ at which R_n has received attention. No requirement other than R_n can put a number into $A_i^{[x]}$ at a stage $\geq t + 1$ and $\leq s$. As already noted, if $\langle x,z\rangle \in A_{i,t}$, then $z \leq 2(x - n)$. Thus $y \leq 2(x - n) + 2 = 2(x - (n - 1))$ as desired.

Lemma 1.4 *For each x, $\lim_s C_s(x)$ exists (so C is a well-defined set) and $C \leq_{tt} A_0, A_1$.*

Proof: By Lemma 1.3 with $n = -1$, if $\langle x,y\rangle \in A_i$, then $y \leq 2x + 2$. If $C_s(x) \neq C_{s+1}(x)$ then by construction a new element is enumerated into $A_0^{[x]}$ and $A_1^{[x]}$ at stage $s + 1$. Thus, for a fixed x, there are at most $2x + 3$ stages s such that $C_s(x) \neq C_{s+1}(x)$, so $\lim_s C_s(x)$ exists. Also it follows from (1.7), (1.8), and the facts just shown that $x \in C$ iff $\#\{y: y \leq 2x + 2 \ \&\ \langle x,y\rangle \in A_i\}$ is odd. Thus $C \leq_{tt} A_i$.

Lemma 1.5 *For each n, R_n is met and receives attention only finitely often.*

Proof: Suppose that the result holds for all $n' < n$. Then let s_0 be such that for no $s \geq s_0$ does an $R_{n'}$ with $n' < n$ receive attention at stage s. If $s \geq s_0$ and R_n requires attention at stage s, then R_n receives attention at stage s; also, no attack on R_n is cancelled at a stage $\geq s_0$. By (1.6) if $s \geq s_0$ then $R(n,s) = R(n,s_0)$. Let us call this common value r_0.

Suppose that R_n is $C \neq \{e\}$. Because of (1.2a), there is an attack on R_n which is never cancelled. Say the attack is through x and is made at stage $t + 1$. It follows from Lemma 1.2 that R_n receives attention at most once after stage $t + 1$ and that for all $s > t$, $\neg(C_{s+1}(x) = \{e\}_{s+1}(x))$. Thus R_n receives attention only finitely often and is met.

Suppose that R_n is $W_e = [i]^c \rightarrow W_e$ recursive. Lemma 1.2 implies that R_n receives attention at most twice after stage s_0, so R_n receives attention only finitely often. Suppose that $W_e = [i]^C$. If there were an attack on R_n which is never cancelled (say the attack is through x), then by Lemma 1.2, $W_e(x) \neq [i]^C(x)$. Thus every attack on R_n is later cancelled, so at no stage $\geq s_0$ is R_n under attack. Furthermore, it follows from (1.10) that $(\forall s \geq s_0)[C_s \upharpoonright r_0 = C \upharpoonright r_0]$. We must show that W_e is recursive. In fact, let $D = C \upharpoonright r_0$. We show that $W_e = [i]^D$ which certainly makes W_e recursive. Given x, let $\sigma = D \upharpoonright |[i](x)|$, $\tau = C \upharpoonright |[i](x)|$. Let $s \geq s_0$ be such that $\{i\}_s(x)\downarrow$. Then (1.3a i-iii) are satisfied with σ and τ in place of σ_0, σ_1. Since R_n is not under attack at the end of stage $s + 1$, $[i]^\sigma(x) = [i]^\tau(x)$. Thus $[i]^D(x) = [i]^\sigma(x) = [i]^\tau(x) = [i]^C(x) = W_e(x)$ as desired.

Finally, suppose that R_n is $[e]^{A_0} = [e]^{A_1} = Y \rightarrow Y \leq_{tt} C$. By construction, R_n will receive attention at most once after stage s_0 so R_n receives attention only finitely often. Suppose that $[e]^{A_0} = [e]^{A_1} = Y$. We must give a *tt*-reduction which gives Y when applied to C. First note that if an attack on R_n is never cancelled, then $[e]^{A_0} \neq [e]^{A_1}$. Thus every attack on R_n must be cancelled, so at no stage $\geq s_0$ is R_n under attack.

Given an $x \geq r_0$, we define a truth table of length $|[e](x)|$, i.e., given a σ of length $|[e](x)|$ we must effectively produce a 0 or 1 as answer. To do this, find a stage $s \geq s_0$ such that $\{e\}_s(x)\downarrow$. Then see if there are any finite sets B_0, B_1 satisfying (1.4 iii–vi) for x and in addition for $i = 0$ and 1,

(1.11) $(\forall y < |[e](x)|)[\sigma(y) = 1 \text{ iff } \#\{z: \langle y,z\rangle \in B_i\} \text{ is odd}]$.

If no such B_0, B_1 exist, then, arbitrarily, give 0 as an answer. If B_0, B_1 are such sets then since R_n is not under attack before or after stage $s+1$, $[e]^{B_0}(x) = [e]^{B_1}(x) = j$, say. Furthermore, if $\hat{B}_0$ and $\hat{B}_1$ also satisfy (1.4 iii–vi) and (1.11) then the common answer $[e]^{\hat{B}_0}(x) = [e]^{\hat{B}_1}(x)$ must also be j, or else we could attack R_n using B_0 and $\hat{B}_1$. We thus give j as the answer for our tt-reduction.

Now we must show the correctness of this tt-reduction; i.e., if $\sigma = C \upharpoonright |[e](x)|$, then the answer obtained above should be $Y(x)$. Let s be as above and let $B_i = A_{i,s} \cup A_i \upharpoonright \omega^{[<|[e](x)|]}$. Then B_0 and B_1 are finite. From previous work, it is not hard to see that B_0 and B_1 satisfy (1.4 iii–v) as well as (1.11) with the given σ. (For (1.4 iv) note that for $i = 0$ or 1, $A_{i,s}^{[<r_0]} = A_i^{[<r_0]}$.) We show that B_0, B_1 satisfy (1.4 vi). Suppose that $y \geq r_0$ and $\langle y,z\rangle \in A_i$. Then no requirement $R_{n'}$ with $n' < n$ ever puts a number into $A_i^{[y]}$ since if this happened, a restraint of priority $<n$ which was $>y$ would be enumerated, contradicting $R(n,s) = r_0$ $\forall s \geq s_0$. If R_n put a number into $A_i^{[y]}$, say at stage s_1, then the corresponding attack must later by cancelled by an $R_{n'}$ with $n' < n$ say at stage s_2. Then at stage s_2 a restraint of priority $<n$ which is $\geq s_2 > s_1 \geq y \geq r_0$ is enumerated; again this is a contradiction. Thus no $R_{n'}$ with $n' \leq n$ ever puts a number into $A_i^{[y]}$. Hence, by Lemma 1.3, $z \leq 2(y-n)$. Since $B_i \subseteq A_i$, (1.4 vi) holds. Thus our tt-reduction, applied to σ, gives as answer $[e]^{B_0}(x) = [e]^{B_1}(x)$. But $B_0 \upharpoonright |[e](x)| = A_0 \upharpoonright |[e](x)|$. Thus the answer produced by the tt-reduction is $[e]^{A_0}(x) = Y(x)$, as desired.

2 *Two r.e. tt-degrees with an infimum among all the tt-degrees, but no infimum among the r.e. tt-degrees*

Theorem 2.1 *There are r.e. tt-degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *have no infimum among the r.e. tt-degrees but have an infimum among all the tt-degrees.*

Proof: We build A_0, A_1 r.e., and C so that if $\mathbf{a}_0 = \deg_{tt}(A_0)$, $\mathbf{a}_1 = \deg_{tt}(A_1)$, $\mathbf{c} = \deg_{tt}(C)$, then $\mathbf{a}_0, \mathbf{a}_1$ have inf $\mathbf{c}$ among all the tt-degrees, but $\mathbf{a}_0, \mathbf{a}_1$ have no inf in the r.e. tt-degrees.

We have three types of requirements where e, i, m run over the integers:

Type I: $C^{[m+1]} \neq [e]^{C^{[\leq m]}}$
Type II: $W_e = [i]^C \rightarrow (\exists r)(W_e \leq_{tt} C^{[\leq r]})$
Type III: $[e]^{A_0} = [e]^{A_1} = Y \rightarrow Y \leq_{tt} C$.

In addition, we ensure that $C \leq_{tt} A_i$, $i = 0,1$, and that, for all n, $C^{[n]}$ is r.e. Suppose that we meet all the requirements and meet the conditions of the previous sentence. Then it is immediate that $\mathbf{a}_0, \mathbf{a}_1$ have inf $\mathbf{c}$ among all the tt-degrees. Suppose that $\mathbf{a}_0, \mathbf{a}_1$ have inf $\mathbf{d}$ in the r.e. tt-degrees, say $W_e \in \mathbf{d}$. Then $W_e \leq_{tt} C$, say $W_e = [i]^C$. Then by a Type II requirement, $(\exists r)(W_e \leq_{tt} C^{[\leq r]})$. So by the Type I requirements $W_e \leq_{tt} C^{[\leq r]} <_{tt} C^{[\leq r+1]} \leq_{tt} C \leq_{tt} A_0, A_1$ and

$C^{[\leq r+1]}$ is r.e. But this contradicts the assumption that $\mathbf{a}_0,\mathbf{a}_1$ have inf $\mathbf{d}$ among the r.e. degrees.

Our strategy to make $C \leq_{tt} A_i$ is the same as in Theorem 1.1. Our strategy to make $C^{[n]}$ r.e. is to ensure that for some stage s (which depends on n), no number is removed from $C^{[n]}$ after stage s. We accomplish this by specifying that no requirement $R_{n'}$ with $n' \geq n$ may remove a number from $C^{[n]}$. Since the $R_{n'}$ with $n' < n$ act only finitely often, this suffices.

We use a Friedberg-Muchnik type strategy to meet the Type I requirements; i.e., we choose an $x \in \omega^{[m+1]}$ as a witness and keep x out of C until we see a computation $[e]^{C_s^{[\leq m]}}(x) = 0$. Then we put x into C and preserve the computation.

To meet a requirement R_n of Type II we follow a strategy similar to that for Type II requirements in Theorem 1.1; however, because of the strategy to make each $C^{[m]}$ r.e., when we consider strings σ_0, σ_1 as in (1.3a), we must add conditions saying that σ_0 and σ_1 must agree on $\omega^{[\leq n]}$ and cannot remove any number from $C^{[\leq n]}$. Because of these new restrictions, if $W_e = [i]^C$, then we get $W_e \leq_{tt} C^{[\leq n]}$ rather then W_e recursive.

The strategy to meet a Type III requirement is like that for Type III requirements in Theorem 1.1, with a slight change to accommodate the strategy to make each $C^{[m]}$ r.e.

The construction is quite like that of Theorem 1.1. Here we give only the changes. The definition of "requires attention" for a Type I requirement is the same as (1.2) except that (1.2bi) is replaced by

$$(2.1) \qquad [e]_{s+1}^{C_s^{[\leq m]}}(x) = 0.$$

The definition of "requires attention" for a Type II requirement is as in (1.3) except that we add to (1.3a)

$$(2.2) \quad \text{(i) } (\forall y \in \omega^{[\leq n]})(y < |\sigma_0| \to \sigma_0(y) = \sigma_1(y))$$
$$\text{(ii) } (\forall y \in \omega^{[\leq n]})(y < |\sigma_0| \wedge y \in C_s \to \sigma_0(y) = 1).$$

The definition of "requires attention" for a Type III requirement is as in (1.4) with the additional clause

$$(2.3) \quad (\forall y \in \omega^{[\leq n]})(y < |[e](x)| \wedge y \in C_s \to \# B_0^{[y]} \text{ is odd}).$$

The construction is as in Theorem 1.1 with the addition that if the requirement $C^{[m+1]} \neq [e]^{C^{[\leq m]}}$ receives attention and (1.2a) holds, then the number x chosen must be in $\omega^{[m+1]}$.

Properties (1.5)-(1.10) hold for this construction. In addition we have

(2.4) If $x \in C_s \cap \omega^{[\leq n]}$ and no $R_{n'}$ with $n' < n$ receives attention at stage $s + 1$, then $x \in C_{s+1}$.

To see (2.4), suppose that $x \in C_s \cap \omega^{[\leq n]}$ and $R_{n'}$ with $n' \geq n$ receives attention at stage $s + 1$. Then $x \in C_{s+1}$ is easily seen unless $R_{n'}$ is of Type II and (1.3b) holds at stage $s + 1$, so suppose that this is the case. If $x \geq |\sigma|$ where σ is the alternate string for the attack, then $x \in C_{s+1}$, so suppose that $x < |\sigma|$. Let $t + 1$ be the stage at which the attack is begun and let σ_0, σ_1 be as in (1.3a) and (2.2). Then $C_{t+1}(x) = \sigma_1(x)$, and by (2.2i) and the fact that $n' \geq n$, $\sigma_0(x) = \sigma_1(x)$. Now for no u with $t + 1 \leq u \leq s + 1$ does a requirement of higher priority than $R_{n'}$ receive attention at stage u (or else the attack on $R_{n'}$ would be cancelled). Thus, using (1.10), it follows by induction on u that for $t+1 \le u \le s+1$, $C_u(x) = C_{t+1}(x)$, so $C_{s+1}(x) = C_s(x) = 1$.

Lemma 2.2 *Suppose that R_n is a requirement of Type I or II, R_n receives attention at stage $t+1$, $s > t$, and no requirement $R_{n'}$ with $n' < n$ receives attention at a stage u, $t+1 \le u \le s+1$. Then there is at most one u, $t+1 < u \le s+1$, such that R_n receives attention at stage u and if R_n is $C^{[m+1]} \ne [e]^{C^{[\le m]}}$ and the attack at stage $t+1$ is through x, then $\neg(C^{[m+1]}_{s+1}(x) = [e]^{C_s^{[\le m]}}_{s+1}(x))$, while if R_n is $W_e = [i]^C \rightarrow (\exists r)(W_e \le_{tt} C^{[\le r]})$ and the attack at stage $t+1$ is through x, then $W_{e,s+1}(x) \ne [i]^{C_{s+1}}_{s+1}(x)$.*

Proof: The attack on R_n in progress at the end of stage $t+1$ is still in progress at the end of stage $s+1$; say the attack is through x. Suppose that R_n is $C^{[m+1]} \ne [e]^{C^{[\le m]}}$. We consider two cases. First suppose that R_n receives attention at some stage $u+1$ with $t+1 \le u+1 \le s+1$ such that (1.2b) as modified by (2.1) holds for $u+1$. Then $x \in C_{u+1}$ and x stays in C until at least the end of stage $s+1$. Thus R_n does not require attention at any stage v, $u+1 < v \le s+1$, so R_n does not receive attention at any such stage v, and R_n receives attention at most once at a stage after $t+1$ and at or before $s+1$. Also, $[e]^{C_{u+1}^{[\le m]}}_{u+1}(x) = 0$ (since $x \notin \omega^{[\le m]}$) and $C_{u+1} \restriction |[e](x)| = C_{s+1} \restriction |[e](x)|$, so $[e]^{C_{s+1}^{[\le m]}}_{s+1}(x) = [e]^{C_{u+1}^{[\le m]}}_{u+1}(x) = 0 \ne 1 = C^{[m+1]}_{s+1}(x)$. If the first case does not hold, then (1.2a) must hold at stage $t+1$ and R_n does not receive attention at any stage u, $t+1 < u \le s+1$. Also, $x \notin C_t$ and one easily sees that x is not put into C until at least after stage $s+1$, so $C_s(x) = C_{s+1}(x) = 0$. If $[e]^{C_s^{[\le m]}}_{s+1}(x) = 0$, then R_n would receive attention at stage $s+1$, contradicting our assumption, so

$$\neg(C^{[m+1]}_{s+1}(x) = [e]^{C_s^{[\le m]}}_{s+1}(x)).$$

If R_n is Type II, then the proof is as in Lemma 1.2.

Lemmas 1.3 and 1.4 hold for this construction as well, with proofs unchanged.

Lemma 2.3 *For each n, R_n is met and receives attention only finitely often.*

Proof: The proof is similar to that of Lemma 1.5. In particular, we use the notation and definitions of the first paragraph of that proof.

If R_n is Type I, then the result follows from Lemma 2.2 just as the corresponding fact in the proof of Lemma 1.5 follows from Lemma 1.2.

If R_n is $W_e = [i]^C \rightarrow (\exists r)(W_e \le_{tt} C^{[\le r]})$, then Lemma 2.2 implies that R_n receives attention at most twice after stage s_0, so R_n receives attention only finitely often. Suppose that $W_e = [i]^C$. If there were an attack on R_n which is never cancelled, then by Lemma 2.2, $W_e \ne [i]^C$. Thus every attack on R_n is later cancelled, so at no stage $\ge s_0$ is R_n under attack. Furthermore, it follows from (1.10) that $(\forall s \ge s_0)[C_s \restriction r_0 = C \restriction r_0]$. We will show that $W_e \le_{tt} C^{[\le n]}$. In fact, let $D = C^{[\le n]} \cup C \restriction r_0$. We show that $W_e = [i]^D$, which suffices. Given x, let $\sigma = D \restriction |[i](x)|$, $\tau = C \restriction |[i](x)|$. Let $s \ge s_0$ be such that $\{i\}_s(x)\downarrow$. Then (1.3a i–iii) and (2.2) are satisfied with σ, τ in place of σ_0, σ_1. ((2.2ii) holds because, by (2.4), $y \in C_s^{[\le n]} \rightarrow y \in C$.) Since R_n is not under attack after stage

$s+1$, $[i]^\sigma(x) = [i]^\tau(x)$. Thus $[i]^D(x) = [i]^\sigma(x) = [i]^\tau\,(x) = [i]^C(x) = W_e(x)$ as desired.

If R_n is of Type III, then the argument is similar to the corresponding argument in the proof of Lemma 1.5. The only differences are: (a) In defining the *tt*-reduction, we ask that the sets B_0, B_1 satisfy (2.3) as well as (1.4iii–vi) and (1.11). For any such $B_0, B_1, [e]^{B_0}(x) = [e]^{B_1}(x)$ and we give this result as answer. (b) In showing the correctness of the *tt*-reduction, when we define B_0, B_1, we must show that they satisfy (2.3) as well as (1.4iii–vi) and (1.11). But (2.3) follows from (1.11) since by (2.4) if $y \in C_s^{[\leq n]}$, then $y \in C$, so $\sigma(y) = 1$.

Lemma 2.4 *For all n, $C^{[n]}$ is r.e.*

Proof: By Lemma 2.3, take s_0 so that at no stage $s \geq s_0$ does an $R_{n'}$ with $n' < n$ receive attention. Then by (2.4)

$$C^{[n]} = \{x \in \omega^{[n]} \colon (\exists s \geq s_0)\,[x \in C_s]\}$$

so $C^{[n]}$ is r.e.

3 *Two r.e. tt-degrees with an infimum among the r.e. tt-degrees, but no infimum among all tt-degrees*

Theorem 3.1 *There are r.e. tt-degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a}_0$ and $\mathbf{a}_1$ form a minimal pair (i.e., have inf $\mathbf{0}$) among the r.e. tt-degrees, but have no inf among all the tt-degrees.*

Proof: We build A_0, A_1 r.e. and C so that if $\mathbf{a}_0 = \deg_{tt}(A_0)$, and $\mathbf{a}_1 = \deg_{tt}(A_1)$ then $\mathbf{a}_0, \mathbf{a}_1$ are as desired.

We have three types of requirements, where e, i, m run over the integers:

Type I: $C^{[m+1]} \neq [e]^{C^{[\leq m]}}$
Type II: $W_e = [i]^C \rightarrow W_e$ recursive
Type III: $[e]^{A_0} = [e]^{A_1} = Y \rightarrow (\exists r)(Y \leq_{tt} C^{[\leq r]})$.

In addition, we ensure that for each n, $C^{[n]} \leq_{tt} A_i$, for $i = 0,1$. Suppose that we meet all the requirements and meet the condition of the previous sentence. If $W_e \leq_{tt} A_0, A_1$, then by a Type III requirement, for some r, $W_e \leq_{tt} C^{[\leq r]} \leq_{tt} C$, so by a Type II requirement, W_e is recursive and hence $\mathbf{a}_0, \mathbf{a}_1$ have inf $\mathbf{0}$ among the r.e. degrees. Suppose that $\mathbf{a}_0, \mathbf{a}_1$ have inf $\deg_{tt}(D)$ among all the *tt*-degrees. Then by a Type III requirement, for some r, $D \leq_{tt} C^{[\leq r]}$, so by a Type I requirement $D \leq_{tt} C^{[\leq r]} <_{tt} C^{[\leq r+1]} \leq_{tt} A_i$, $i = 0,1$, contradicting the assumption that $\mathbf{a}_0, \mathbf{a}_1$ have inf $\deg_{tt}(D)$.

Our strategy to make $C^{[n]} \leq_{tt} A_i$ is to arrange that

$$(3.1)\quad \begin{cases} \text{for each } n \text{ there is some } M \text{ such that} \\ (\forall x \geq M)(x \in \omega^{[\leq n]} \rightarrow x \in C \text{ iff } \#\{y\colon y \leq 2x+2 \wedge \langle x,y\rangle \in A_i\} \text{ is} \\ \text{odd}). \end{cases}$$

Our strategies for Type I and II requirements are as in Theorems 2.1 and 1.1 respectively. Our strategy for Type III requirements is similar to that of Theorem 1.1, but if R_n is Type III then in trying to make $[e]^{A_0} \neq [e]^{A_1}$, R_n needs to respect the coding strategy of (3.1) only for those $x \in \omega^{[\leq n]}$. Thus if we get

$[e]^{A_0} = [e]^{A_1} = Y$, we can show that $Y \leq_{tt} C^{[\leq n]}$ rather than just $Y \leq_{tt} C$. The tradeoff here is that we now only have $C^{[n]} \leq_{tt} A_i$ for each n and $i = 0,1$ via (3.1) rather than $C \leq_{tt} A_i$ via (1.1).

The construction is similar to those of Theorems 1.1 and 2.1. The definition of "requires attention" for a Type I requirement is as in Theorem 2.1. The definition of "requires attention" for a Type II requirement is as in (1.3). For a Type III requirement, the definition is as given by (1.4) except that (1.4vii) is replaced by

(3.2) $\begin{cases} (\forall y)\,[R(n,s) \leq y < |[e]\,(x)| \wedge y \in \omega^{[\leq n]} \rightarrow \#B_0^{[y]}, \#B_1^{[y]} \text{ have the same} \\ \text{parity}]. \end{cases}$

The construction is as in Theorem 1.1 except that Type I requirements are handled as in Theorem 2.1.

Properties (1.5)–(1.7), (1.9), (1.10) hold for this construction. In place of (1.8), we have the following property:

(3.3) $\begin{cases} \text{For all } y,s,n,y \in \omega^{[n]},\ y \geq R(n,s) \rightarrow \#A_{0,s}^{[y]}, \#A_{1,s}^{[y]} \text{ have the same} \\ \text{parity.} \end{cases}$

We verify (3.3) by induction on s. For $s = 0$, $\#A_{i,s}^{[y]} = 0$ for $i = 0,1$, so the result holds. Suppose that the result holds for s and that $y \in \omega^{[n]}$, $y \geq R(n,s+1)$. By (1.5), $R(n,s) \leq R(n,s+1) \leq y$, so by induction hypothesis, $\#A_{0,s}^{[y]}, \#A_{1,s}^{[y]}$ have the same parity. Let $R_{n'}$ receive attention at stage $s+1$. If $R_{n'}$ is Type I or II, then by construction either $A_0^{[y]}, A_1^{[y]}$ are unchanged at stage $s+1$ or else one element is added to each set, so the desired result holds for $s+1$. If $R_{n'}$ is Type III and $n' < n$, then $A_0^{[y]}, A_1^{[y]}$ do not change at stage $s+1$ (since by (1.10), if $A_0^{[w]}$ or $A_1^{[w]}$ changes at stage $s+1$, then $w \leq R(n'+1, s+1) \leq R(n,s+1) \leq y$), so the result holds. If $n \leq n'$ and $y < R(n',s)$ or $y \geq |[e]\,(x)|$ (where e,x,B_0,B_1 are as in (1.4)), then by (1.4iv) $A_0^{[y]}, A_1^{[y]}$ do not change at stage $s+1$. If $R(n',s) \leq y < |[e]\,(x)|$, then by (3.2) and the fact that $y \in \omega^{[n]} \subseteq \omega^{[\leq n']}$, the desired result holds for $s+1$.

Lemma 2.2, with $W_e = [i]^C \rightarrow (\exists r)(W_e \leq_{tt} C^{[\leq r]})$ replaced by $W_e = [i]^C \rightarrow W_e$ is recursive, and Lemma 1.3 hold for this construction.

Lemma 3.2 *For each x, $\lim_s C_s(x)$ exists (so C is a well-defined set).*

Proof: By Lemma 1.3 with $n = -1$, if $\langle x,y \rangle \in A_i$, then $y \leq 2x+2$. If $C_s(x) \neq C_{s+1}(x)$, then by construction, a new element is enumerated into $A_0^{[x]}$ at stage $s+1$. Thus, for a fixed x, there are at most $2x+3$ stages s such that $C_s(x) \neq C_{s+1}(x)$, so $\lim_s C_s(x)$ exists.

Lemma 3.3 *Each R_n receives attention only finitely often and is met.*

Proof: The proof is similar to that of Lemma 1.5 and we take preliminary definitions and notations from that proof.

If R_n is of Type I, then the result follows as in Lemma 2.3. If R_n is of Type II, then the result follows as in Lemma 1.5. If R_n is of Type III, say $[e]^{A_0} = [e]^{A_1} = Y \rightarrow (\exists r)(Y \leq_{tt} C^{[\leq r]})$, then the argument is similar to the corresponding argument in the proof of Lemma 1.5. The only differences are: (a) If $[e]^{A_0} = [e]^{A_1} = Y$, then we show that $Y \leq_{tt} C^{[\leq n]}$ rather than C; (b) In defining the tt-reduction, we ask that the sets B_0, B_1 satisfy (1.4iii–vi) and in addition

(3.4) $(\forall y)(r_0 \leq y < |[e](x)| \wedge y \in \omega^{[\leq n]} \rightarrow \sigma(y) = 1 \text{ iff } \#B_i^{[y]} \text{ is odd})$.

For any such B_0, B_1, $[e]^{B_0}(x) = [e]^{B_1}(x)$ and we give this result as answer. (c) In showing the correctness of the tt-reduction, given $\sigma = C^{[\leq n]} \upharpoonright |[e](x)|$, we define B_0, B_1 as before and they satisfy (1.4iii–vi). In addition, using (1.7), (3.3), and the fact that each $A_i^{[y]}$ is finite, we see that B_0, B_1 satisfy (3.4) as well. Thus the tt-reduction, applied to σ, gives $[e]^{B_0}(x) = [e]^{A_0}(x) = Y(x)$ as answer.

Lemma 3.4 *For each n and $i = 0,1$, $C^{[n]} \leq_{tt} A_i$.*

Proof: By Lemma 3.3, fix s_0 so that no $R_{n'}$ with $n' < n$ receives attention at a stage $\geq s_0$. Then $(\forall s \geq s_0)[R(n,s) = R(n,s_0)]$. Call this final value r_0. By (1.7) and (3.3), if $y \geq r_0$, $y \in \omega^{[n]}$, $s \geq s_0$, and $i = 0$ or 1 then $y \in C_s$ iff $\#A_{i,s}^{[y]}$ is odd.

Taking s large enough and remembering that by Lemma 1.3 with $n = -1$, if $\langle y,z \rangle \in A_i$, then $z \leq 2(y+1)$, we get that $(\forall y \geq r_0, y \in \omega^{[n]})(y \in C \leftrightarrow \#\{z : z \leq 2y + 2 \wedge \langle y,z \rangle \in A_i\}$ is odd). Thus $C^{[n]} \leq_{tt} A_0, A_1$.

4 *Two r.e. tt-degrees with no infimum in either structure*

Theorem 4.1 *There are r.e. tt-degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *have an infimum neither among the r.e. tt-degrees nor among all the tt-degrees.*

Proof: We build A_0, A_1, and C r.e. so that if $\mathbf{a}_0 = \deg_{tt}(A_0)$, $\mathbf{a}_1 = \deg_{tt}(A_1)$, then $\mathbf{a}_0, \mathbf{a}_1$ are as desired.

We have two types of requirements, where e, m run over the integers:

Type I: $C^{[m+1]} \neq [e]^{C^{[\leq m]}}$

Type III: $[e]^{A_0} = [e]^{A_1} = Y \rightarrow (\exists r)(Y \leq_{tt} C^{[\leq r]})$.

In addition, we ensure that for each n, $C^{[n]} \leq_{tt} A_i$, for $i = 0,1$. Suppose that we meet all the requirements and meet the condition of the previous sentence. If $Y \leq_{tt} A_0, A_1$, then by a Type III requirement, $Y \leq_{tt} C^{[\leq r]}$ for some r. But by a Type I requirement, $C^{[\leq r]} <_{tt} C^{[\leq r+1]}$ and $C^{[\leq r+1]} \leq_{tt} A_0, A_1$. Furthermore, since C is r.e., $C^{[\leq r+1]}$ is r.e. Thus $\mathbf{a}_0, \mathbf{a}_1$ have an infimum neither among the r.e. tt-degrees, nor among all the tt-degrees.

Our strategies to make $C^{[n]} \leq_{tt} A_i$ as well as to meet the Type I requirements are the same as in the proof of Theorem 3.1. Because we no longer have Type II requirements, we can make C r.e. if we modify the Theorem 3.1 strategy for meeting Type III requirements by insisting that action for Type III requirements can only add elements to C, not remove them.

More formally, the definition of "requires attention" for a Type III requirement is the same as in Theorem 3.1 with the addition of

(4.1) $(\forall y < |[e](x)|)(y \in C_s \rightarrow \#B_0^{[y]} \text{ is odd})$.

The rest of the construction is the same as that in Theorem 3.1. Because of (4.1), when a Type III requirement receives attention, no numbers are removed from C, so C is r.e.

The proof that the construction works is similar to the proof in Theorem 3.1 and is omitted.

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