# INFINITARY HARMONIC NUMBERS 

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The infinitary divisors of a natural number $n$ are the products of its divisors of the form $p^{y, 2^{* *}}$, where $p^{y}$ is an exact prime-power divisor of $n$ and $\sum_{\alpha} y_{\alpha} 2^{\alpha}$ (where $y_{\alpha}=0$ or 1 ) is the binary representation of $y$. Infinitary harmonic numbers are those for which the infinitary divisors have integer harmonic mean. One of the results in this paper is that the number of infinitary harmonic numbers not exceeding $x$ is less than $2.2 x^{1 / 2} 2^{(1+e) \log x / \log \log x}$ for any $\varepsilon>0$ and $x>n_{0}(\varepsilon)$. A corollary is that the set of infinitary perfect numbers (numbers $n$ whose proper infinitary divisors sum to $n$ ) has density zero.

## 1. Introduction

Unless otherwise noted, in what follows lower-case letters will be used to denote natural numbers, with $p$ and $q$ always representing primes. If $\tau(n)$ and $\sigma(n)$ denote, respectively, the number and sum of the positive divisors of $n$, Ore [5] showed that the harmonic mean of the positive divisors of $n$ is given by $H(n)=n \tau(n) / \sigma(n)$. We say that $n$ is a harmonic number if $H(n)$ is an integer. It is easy to see that every perfect number is a harmonic number.

The unitary analogue of $H(n)$ was studied by Hagis and Lord [3]. Thus, if $\tau^{+}(n)$ and $\sigma^{+}(n)$ denote, respectively, the number and sum of the unitary divisors of $n$ (see Defimition 1, below), then the unitary harmonic mean of $n$ is given by $H^{*}(n)=n \tau^{*}(n) / \sigma^{*}(n)$, and $n$ is said to be a unitary harmonic number if $H^{*}(n)$ is an integer.

In [1], Cohen initiated the study of the infinitary divisors of a natural number. In the present paper we investigate $H_{\infty}(n)$, the harmonic mean of the infinitary divisors of $n$. Particular altention is paid to $I I I$, the set of natural numbers $n$ for which $H_{\infty}(n)$ is an integer.

## 2. Infinitary divisors

The following three definitions may be found in [1].
Definition 1: If $d \mid n, d$ is said to be a 0 -ary divisor of $n$. A divisor $d$ of $n$. is called a 1 -ary (or unitary) divisor of $n$ if the greatest common divisor of $d$ and $n / d$ is

[^0][^1]1. In general, if $k \geqslant 1$ then $d$ is called a $k$-ary divisor of $n$ (and we write $\left.d\right|_{k} n$ ) if $d \mid n$ and the greatest common $(k-1)$-ary divisor of $d$ and $n / d$ is 1 .

It is immediate that for any $n$ and $k,\left.1\right|_{k} n$ and $\left.n\right|_{k} n$. Also, $\left.p^{x}\right|_{k} p^{y}$ if and only if $\left.p^{y-x}\right|_{k} p^{y}$. If $\left.d\right|_{1} n$, we shall write $d \| n$.

Definition 2: We say $p^{x}$ is an infinitary divisor of $p^{y}$ (and we write $\left.p^{x}\right|_{\infty} p^{y}$ ) if $\left.p^{x}\right|_{y-1} p^{y}$.

In [1] it is proved that if $\left.p^{x}\right|_{y-1} p^{y}$ then $\left.p^{x}\right|_{k} p^{y}$ for $k \geqslant y-1$.
Definition 3: Suppose that $d \mid n$. We say that $d$ is an infinitary divisor of $n$ (and we write $\left.d\right|_{\infty} n$ ) if $p^{x} \| d$ implies that if $p^{y} \| n$ then $\left.p^{x}\right|_{\infty} p^{y}$. The only infinitary divisor of 1 is 1 .

Now let $\mathbf{P}$ be the set of all primes and let

$$
r=\left\{p^{2^{\alpha}} \mid p \in \mathbf{P} \& \alpha \in \mathbb{N}_{0}\right\}
$$

From the fundamental theorem of arithmetic and the fact that the binary representation of a natural number is unique, it follows that if $n>1$ then $n$ can be written in exactly one way (except for the order of the factors) as the product of distinct elements from $I$. We shall call each element of $I$ in this product an $I$-component of $n$.

Let the number of $I$-components of $n$ be denoted by $J(n)$. Then $J(1)=0$ and, if $y=\sum_{i=0}^{\infty} y_{i} 2^{i}$ where $y_{i}=0$ or $1, J\left(p^{y}\right)=\sum y_{i}$. It is obvious that $J$ is an additive function so that, if $n=\prod_{p^{y} \| n} p^{y}$, then $J(n)=\sum_{p^{y} \| n} J\left(p^{y}\right)$.

We shall say that $d$ is an $I$-divisor of $n$ if every $I$-component of $d$ is also an $I$-component of $n$. (Thus, if $n=2^{3} 3^{4} 5^{6}=2 \cdot 2^{2} \cdot 3^{4} \cdot 5^{2} \cdot 5^{4}$ then $2^{2} 5^{2}$ is an $I$-divisor of $n$ while $3^{2} 5^{4}$ is not.) If $\sigma_{I}(n)$ is the sum of the $I$-divisors of $n$, we see that $\sigma_{I}(1)=1$ and $\sigma_{I}\left(p^{y}\right)=\prod_{y_{i}=1}\left(1+{p^{2}}^{i^{i}}\right)$ if $y=\sum y_{i} 2^{i}$. It is obvious that $\sigma_{I}$ is a multiplicative function so that, if $n=\prod_{p^{y} \| n} p^{y}$ then $\sigma_{I}(n)=\prod_{p^{y} \| n} \prod_{y_{i}=1}\left(1+p^{2^{i}}\right)$. It follows that if $\tau_{I}(n)$ is the number of $I$-divisors of $n$ then $\tau_{I}(n)=\prod_{p^{y} \| n} 2^{J\left(p^{y}\right)}=2^{J(n)}$.

It is proved (implicitly) in the first four sections of [ 1$]$ that the set of infinitary divisors of $n$ is equal to the set of $l$-divisors of $n$. Therefore, if $\tau_{\infty}(n)$ and $\sigma_{\infty}(n)$ denote the number and sum, respectively, of the infinitary divisors of $n$, we have (see Theorem 13 in [ 1$]$ ):

Proposition 1. If $n=\prod_{p^{y} \| n} p^{y}$ and $y=\sum y_{i} 2^{i}$, then

$$
\tau_{\infty}(n)=\prod_{p^{y} \| n} 2^{J\left(p^{y}\right)}=2^{J(n)}
$$

where $J(n)=\sum_{p^{y} \|_{n}} J\left(p^{y}\right)=\sum_{p^{y} \|_{n}} \sum y_{i}$, and

$$
\sigma_{\infty}(n)=\prod_{p^{y} \|_{n}} \prod_{y_{i}=1}\left(1+p^{2^{i}}\right)
$$

It is easy to show that the infinitary harmonic mean of $n$ (the harmonic mean of the infinitary divisors of $n$ ) is given by

$$
\begin{equation*}
H_{\infty}(n)=\frac{n \tau_{\infty}(n)}{\sigma_{\infty}(n)}=2^{J(n)} \prod_{p^{y} \| n} \prod_{y_{i}=1} \frac{p^{2^{i}}}{1+p^{2^{i}}} \tag{1}
\end{equation*}
$$

We shall say that $n$ is an infinitary harmonic number if $H_{\infty}(n)$ is an integer and shall denote by $I H$ the set of these numbers. A computer search was made for the elements of $I H$ in the interval $\left[1,10^{6}\right]$ and 38 were found. They are listed in Table 1 below.

Cohen [1] has defined $n$ to be an infinitary perfect number if $\sigma_{\infty}(n)=2 n$ and has found fourteen such numbers. Since $J(n) \geqslant 1$ if $n>1$, the following result is immediate from (1).

Proposition 2. The set of infinitary perfect numbers is a subset of III.
3. Some elementary results concerning $H_{\infty}(n)$ and $I H$

Lemma 1. Let $J(n)=J$. Then, if $n>1$,

$$
\begin{equation*}
\frac{2^{J+1}}{J+2} \leqslant H_{\infty}(n)<2^{J} \tag{2}
\end{equation*}
$$

Proof: Since $x /(x+1)$ is monotonic increasing and bounded above by 1 for positive values of $\boldsymbol{x}$, it follows from (1) that

$$
2^{J}>H_{\infty}(n) \geqslant 2^{J} \frac{2}{3} \frac{3}{4} \cdots \frac{J+1}{J+2}=\frac{2^{J+1}}{J+2}
$$

Note. We have equality on the left in (2) if and only if $n=2$ or $n=2 \cdot 3$ or $n=2^{3} \cdot 3$ or $n=2^{3} \cdot 3 \cdot 5$. Also, using (2) it is clear that $H_{\infty}(n)=1$ if and only if $n=1$.

Lemma 2. Suppose that there are $s$ zeros in the binary representation of $y$. Then

$$
\frac{\tau\left(p^{y}\right)}{\tau_{\infty}\left(p^{y}\right)} \geqslant \frac{2^{s}+1}{2}
$$

Proof: Set $y=\sum_{i=0}^{t} y_{i} 2^{i}$ where $y_{t}=1$. Since $s$ values of $y_{i}$ are 0 ,

$$
y \geqslant 1+2+2^{2}+\cdots+2^{t-s-1}+2^{t}=2^{t}+2^{t-s}-1
$$

Therefore,

$$
\frac{\tau\left(p^{y}\right)}{\tau_{\infty}\left(p^{y}\right)}=\frac{y+1}{2^{\Sigma y_{i}}} \geqslant \frac{2^{t}+2^{t-t}}{2^{t+1-s}}=\frac{2^{s}+1}{2} .
$$

Theorem 1. For all $n, H^{*}(n) \leqslant H_{\infty}(n) \leqslant H(n)$. For $n>1$, equality holds on the left if and only if $p^{y} \| n$ implies $y=2^{\alpha}$, and on the right if and only if $p^{y} \| n$ implies $y=2^{\beta}-1$.

Proof: Since $H^{*}(1)=H_{\infty}(1)=H(1)$, we may suppose that $n>1$.
If $p^{y} \| n$ implies that $y=2^{\alpha}$ then, since $H^{*}\left({p^{2^{\alpha}}}^{\alpha}\right)=2{p^{2}}^{\alpha} /\left(1+{p^{2^{\alpha}}}^{\alpha}\right)=H_{\infty}\left(p^{2^{\alpha}}\right)$ and since $H^{*}$ and $H_{\infty}$ are each multiplicative, it follows that $H^{*}(n)=H_{\infty}(n)$.

Now suppose that $p^{y} \| n$ and $y \neq 2^{\alpha}$. Then $y=2^{a_{1}}+2^{a_{2}}+\cdots+2^{a_{u}}$, where $a_{1}>a_{2}>\cdots>a_{u} \geqslant 0$ and $u \geqslant 2$. It fullows that

$$
\begin{aligned}
\frac{H^{*}\left(p^{y}\right)}{H_{\infty}\left(p^{y}\right)} & =\frac{2 p^{y}}{1+p^{y}} \cdot \frac{\left(1+p^{2^{a_{1}}}\right) \ldots\left(1+p^{2^{a_{u}}}\right)}{2^{u} p^{y}}<\frac{1}{2^{u-1}} \cdot \frac{p^{y}+p^{y-1}+\cdots+p+1}{p^{y}} \\
& <\frac{1}{2^{u-1}}\left(1+\frac{1}{p}+\frac{1}{p^{2}}+\cdots\right) \leqslant \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^{2}}+\cdots\right)=1 .
\end{aligned}
$$

Therefore, $H^{\dagger}\left(p^{y}\right)<H_{\infty}\left(p^{y}\right)$, so that $I^{*}(n)<H_{\infty}(n)$.
If $p^{y} \| n$ implies that $y=2^{\beta}-1$ then, since $H\left(p^{2^{\beta}-1}\right)={p^{p^{\beta}-1} 2^{\beta}(p-1) /\left({p^{2}}^{\beta}-1\right) ~}_{x}$ $=H_{\infty}\left(p^{2^{\beta}-1}\right)$ and since $H$ and $H_{\infty}$ are each multiplicative, it follows that $H_{\infty}(n)=$ $H(n)$.

Now suppose that $p^{y} \| n$ and $y \neq 2^{\beta}-1$. We consider several cases.
Suppose first that, in Lemma 2, $s \geqslant 2$. Then

$$
\frac{H\left(p^{y}\right)}{H_{\infty}\left(p^{y}\right)}=\frac{\tau\left(p^{y}\right)}{\tau_{\infty}\left(p^{y}\right)} \cdot \frac{\sigma_{\infty}\left(p^{y}\right)}{\sigma\left(p^{y}\right)} \geqslant \frac{5}{2} \cdot \frac{\left(p^{y}+\jmath\right)(p-1)}{p^{y+1}-1}=1+\frac{3 p^{y+1}-5 p^{y}+5 p-3}{2\left(p^{y+1}-1\right)}>1
$$

since $p \geqslant 2$.
Now suppose that $y$ is odd. From Theorem 3 in $[1],\left.p\right|_{\infty} p^{y}$ and hence $\left.p^{y-1}\right|_{\infty} p^{y}$. Therefore, using Lemma 2 with $s \geqslant 1$,
$\frac{H\left(p^{y}\right)}{H_{\infty}\left(p^{y}\right)} \geqslant \frac{3}{2} \cdot \frac{\sigma_{\infty}\left(p^{y}\right)}{\sigma\left(p^{y}\right)} \geqslant \frac{3}{2} \cdot \frac{\left(p^{y}+p^{y-1}+1\right)(p-1)}{p^{y+1}-1}=1+\frac{p^{y+1}-3 p^{y-1}+3 p-1}{2\left(p^{y+1}-1\right)}>1$,
since $p \geqslant 2$.

One possibility remains: $s=1$ and $y$ is even. Then the binary representation of $y$ has the form 11...110, so that $y=2^{\gamma}-2$, where $\gamma \geqslant 2$.

If $\gamma=2$, then

$$
\frac{H\left(p^{y}\right)}{I_{\infty}\left(p^{y}\right)}=\frac{H\left(p^{2}\right)}{I_{\infty}\left(p^{2}\right)}=\frac{3}{2} \cdot \frac{p^{2}+1}{p^{2}+p+1}=1+\frac{(p-1)^{2}}{2\left(p^{2}+p+1\right)}>1
$$

since $p \geqslant 2$.
If $\gamma \geqslant 3$, then

$$
\begin{aligned}
\frac{H\left(p^{y}\right)}{I_{\infty}\left(p^{y}\right)} & =\frac{H\left(p^{2^{\gamma}-2}\right)}{H_{\infty}\left(p^{2^{\gamma}-2}\right)}=\frac{2^{\gamma}-1}{2^{\gamma-1}} \cdot \frac{\left(1+p^{2}\right)\left(1+p^{4}\right)\left(1+p^{8}\right) \ldots\left(1+p^{2^{\gamma-1}}\right)(p-1)}{p^{2^{\gamma-1}}-1} \\
& =\frac{2^{\gamma}-1}{2^{\gamma-1}} \cdot \frac{\left(1+p^{2}+p^{4}+p^{6}+\cdots+{p^{2}-2}^{\gamma}\right)(p-1)}{p^{2^{\gamma}-1}-1} \\
& =\frac{2^{\gamma}-1}{2^{\gamma-1}} \cdot \frac{\left(p^{2^{\gamma}}-1\right)(p-1)}{\left(p^{2}-1\right)\left(p^{2 \gamma-1}-1\right)}>\frac{2^{\gamma}-1}{2^{\gamma-1}} \cdot \frac{p}{p+1} \geqslant 2 \cdot \frac{2^{\gamma}-1}{2^{\gamma}} \cdot \frac{2}{3} \\
& =\frac{4}{3}\left(1-\frac{1}{2^{\gamma}}\right) \geqslant \frac{4}{3} \cdot \frac{7}{8}>1 .
\end{aligned}
$$

Therefore, $H\left(p^{y}\right)>H_{\infty}\left(p^{y}\right)$, and it follows that $H_{\infty}(n)<H(n)$. This completes 1.he proof of Theorem 1.

Since $2^{\alpha}=2^{\beta}-1$ if and only if $\alpha=0$ and $\beta=1$, it follows from Theorem 1 that $H^{+}(n)=H_{\infty}(n)=I I(n)$ if and only if $n$ is square-free (or $n=1$ ). Ore [5] proved that 6 is the only square-free harmonic number (he did not count 1 in this context; nor shall we), so the following result is immediate.

Corollary 1.1. The only square-free infinitary harmonic number is 6.
Since $2 \mid\left(1+{p^{2}}^{i}\right)$ if $p$ is odd, and since $4 \mid(1+p)$ if $p=4 m+3$, the next two results follow from (1) and the fact that $\left.p\right|_{\infty} p^{y}$ if and only if $y$ is odd (Theorem 3 in [1]).

Proposition 3. If $n$ is odd and $n \in I H$, then $H_{\infty}(n)$ is odd.
Proposition 4. If $n$ is odd, $n \in I H, p^{y} \| n$ and $p=4 m+3$, then $y$ is even.
Proposition 5. If $n \in I H,(p, n)=1$ and $\sigma_{\infty}\left(p^{y}\right) \mid \tau_{\infty}\left(p^{y}\right) H_{\infty}(n)$, then $p^{y} n \in I H$.

This follows from (1) and the fact that $H_{\infty}$ is multiplicative.
As an example of Proposition 5, $409500 \in I H$ and $H_{\infty}(409500)=30$; since $(29,409500)=1$ and $(1+29) \mid 2 \cdot 30$, it follows that $29 \cdot 409500 \in I H$. Other results
like Proposition 5, but where $p \mid n$, are easily obtained. For example, it may be shown that if $n \in I H, 3 \mid H_{\infty}(n)$ and $2^{2 a} \| n$, then $2 n \in I H$.

## 4. T'wo cardinality theorems

Theorem 2. If $S_{c}$ is the set of natural numbers $n$ such that $H_{\infty}(n)=c$, then $S_{c}$ is finite (or empty) for every real number $c$.

Proof: Since $2^{J+1} /(J+2) \geqslant J$, it follows from Lemma 1 that if $H_{\infty}(n)=c$ then the number of $I$-components of $n$ is bounded above by $c$. Assume that $S_{c}$ is infinite. Then $S_{c}$ must contain an infinite subset, say $S_{c m}$, each of whose elements has exactly $m I$-components. It fullows that an infinite sequence $n_{1}, n_{2}, n_{3}, \ldots$ of distinct integers exists with the following properties.
(i) $n_{i} \in S_{c m}$, so that $H_{\infty}\left(n_{i}\right)=c$ for $i=1,2,3, \ldots$.
(ii) $n_{i}=p_{1}^{2^{a_{1}}} \ldots p_{s-1}^{2^{a_{s-1}}} \cdot p_{i s}^{2^{a_{i s}}} \ldots p_{i m}^{2^{a_{i m}}}=P \cdot \prod_{j=s}^{m} p_{i j}^{2_{i j}}$, where $p_{1}^{2^{a_{1}}}<\cdots<$ $p_{s-1}^{2^{a_{s}}{ }^{1}}<p_{i s}^{2^{a_{i s}}}<\cdots<p_{i m}^{2^{a_{i m}}}$ for $i=1,2, \ldots$. (The $p$ s are primes which are not necessarily distinct; $P$ may be an empty product, but $s-1 \neq m$.)
(iii) $p_{i j}^{2^{a_{i j}}} \rightarrow \infty$ as $i \rightarrow \infty$ for $j=s, \ldots, m$.
(That is, each $n_{i}$ is composed of a fixed constant block of elements from $I$ and a variable block of elements from $I$ arranged monotonically within the block and such that each element of this variable block goes to infinity with i.)

From (i) and (ii) and (1) and the fact that $H_{\infty}$ is multiplicalive, we see that

$$
\frac{c}{H_{\infty}(P)}=\prod_{j=s}^{m} I_{\infty}\left(p_{i j}^{2^{a_{i j}}}\right)=2^{m-s+1} \cdot \prod_{j=s}^{m} \frac{p_{i j}^{2^{a_{i j}}}}{1+p_{i j}^{2_{i j}}}<2^{m-s+1}
$$

Therefore, there exists a fixed positive number $v$ such that $\prod_{j=s}^{m} H_{\infty}\left(p_{i j}^{2_{i j}}\right)=2^{m-s+1}-$ $v$. But, from (iii), it follows that $\lim _{i \rightarrow \infty} H_{\infty}\left(p_{i j}^{2^{a_{i j}}}\right)=2$ for $j=s, \ldots, m$. Therefore, for large $i$,

$$
\prod_{j=s}^{m} H_{\infty}\left(p_{i j}^{2_{i j}}\right)>2^{m-s+1}-v
$$

This contradiction completes the proof.
Theorem 3. There exist at most finitely many infinitary harmonic numbers with a specified number of $I$-components.

Proof: Consider the elements of $I I J$ with precisely $K I$-components. There are only finitely many integers between $2^{K+1} /(K+2)$ and $2^{K}$. From Theorem 2 , if $l$ is one of these integers then $S_{l}$ is finite (or empty).

COROLLARY 3.1. There is at most a finite number of infinitary perfect numbers with a specified number of $I$-components.

## 5. The distribution of the infinitary harmonic numbers

For each positive number $x$, we shall denote by $A(x)$ the number of integers $n$ such that $n \leqslant x$ and $n \in I H$.

Theorem 4. For any $\varepsilon>0$ and for all sufficiently large values of $x$,

$$
A(x)<2.2 x^{1 / 2} 2^{(1+\varepsilon) \log x / \log \log x} .
$$

Proof: A positive integer $m$ is powerful if $p \mid m$ implies that $p^{2} \mid m$. Every posilive integer can be written uniquely as a product $N_{P} N_{F}$, where ( $N_{P}, N_{F}$ ) =1, $N_{P}$ is powerful and $N_{F}$ is square-free. (We consider 1 to be both powerful and squarefree.) If $P(x)$ denotes the number of powerful numbers not exceeding $x$, it is proved in [2] that $P(x) \sim c x^{1 / 2}$, where $c=\zeta(3 / 2) / \zeta(3)=2.173 \ldots$ Therefore, $P(x)<2.2 x^{1 / 2}$ for all large values of $x$.

If $N_{P}$ is a (fixed) powerful number, let $g\left(N_{F}, x\right)$ denote the number of square-free numbers $N_{F}$ such that $\left(N_{P}, N_{F}\right)=1, N_{P} N_{F} \leqslant x$ and $N_{P} N_{F} \in I H$. If $G(x)=$ $\max \left\{g\left(N_{P}, x\right)\right\}$ for $N_{P} \leqslant x$, it follows that

$$
\begin{equation*}
A(x)<2.2 x^{1 / 2} G(x) \quad \text { for large } x . \tag{3}
\end{equation*}
$$

We now investigate the magnitude of $G(x)$. Let $N_{P}$ be a powerful number for which distinct square-free numbers $m_{1}, m_{2}, \ldots, m_{G(x)}$ exist such that $\left(N_{P}, m_{i}\right)=1$, $N_{P} m m_{i} \leqslant x$ and $N_{P} m_{i} \in I H$ for $i=1,2, \ldots, G(x)$. Then $I_{\infty}\left(N_{P} m_{i}\right)=H_{\infty}\left(N_{P}\right)$. $I_{\infty}\left(n_{i}\right)=Z_{i}$, where $Z_{i}$ is an integer, for $i=1,2, \ldots, G(x)$. Suppose that $Z_{j}=Z_{k}$ where $j \neq k$. If $\left(m_{j}, m_{k}\right)=d$ then, of course, $H_{\infty}\left(M_{j}\right)=H_{\infty}\left(M_{k}\right)$ where $M_{j}=m_{j} / d$ and $M_{k}=m_{k} / d$. Since $M_{j} \neq M_{k}$, we cannot have $M_{j}=1$, so we may suppose that $2 \leqslant M_{j}<M_{k}$. If $M_{j}=p_{1} \ldots p_{s}$ and $M_{k}=q_{1} \ldots q_{t}$, where $p_{1}<\cdots<p_{s}, q_{1}<\cdots<q_{t}$ and $p_{u} \neq q_{v}$, then from (1) it follows that

$$
2^{s} p_{1} \ldots p_{s}\left(1+q_{1}\right) \ldots\left(1+q_{t}\right)=2^{t} q_{1} \ldots q_{t}\left(1+p_{1}\right) \ldots\left(1+p_{s}\right) .
$$

Then $q_{t} \mid\left(1+q_{r}\right)$ for some $r, 1 \leqslant r<t$. This implies that $q_{t}=3$ and $q_{r}=q_{1}=2$, which is a contradiction since we require $1<M M_{j}<M_{k}$. Hence $Z_{j} \neq Z_{k}$, unless $j=k$. Therefore, without loss of generality, $Z_{1}<Z_{2}<\cdots<Z_{G(x)}$ so that $G(x) \leqslant Z_{G(x)}=$ $H_{\infty}\left(N_{P} m_{G(x)}\right)<\tau_{\infty}\left(N_{P} m_{G(x)}\right)$. Since $\tau_{\infty}(n) \leqslant \tau(n)$, and since $N_{P} m_{G(x)} \leqslant x$, and since it follows from Theorem 317 in [4] that $\tau(n) \leqslant 2^{(1+e) \log x / \log \log x}$ if $n \leqslant x$ and $x>n_{0}(\varepsilon)$, we conclude that

$$
\begin{equation*}
G(x)<2^{(1+\varepsilon) \log x / \log \log x \quad \text { for all large } x .} \tag{4}
\end{equation*}
$$

Theorem 4 follows from (3) and (4).
Corollary 4.1. IIf has density zero.
Corollary 4.2. The set of infinitary perfect numbers has density zero.
TABLE 1. The infinitary harmonic numbers in $\left[1,10^{6}\right]$

| $n$ | $H_{\infty}(n)$ | $n$ | $H_{\infty}(n)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | $95550=2 \cdot 3 \cdot 5^{2} 7^{2} 13$ | 14 |
| $0=2 \cdot 3$ | 2 | $136500=2^{2} 3 \cdot 5^{3} 7 \cdot 13$ | 25 |
| $45=3^{2} 5$ | 3 | $163800=2^{3} 3^{2} 5^{2} 7 \cdot 13$ | 24 |
| $60=2^{2} 3 \cdot 5$ | 4 | $172900=2^{2} 5^{2} 7 \cdot 13 \cdot 19$ | 19 |
| $90=2 \cdot 3^{2} 5$ | 4 | $204750=2 \cdot 3^{2} 5^{3} 7 \cdot 13$ | 25 |
| $270=2 \cdot 3^{3} 5$ | 6 | $232470=2 \cdot 3^{4} 5 \cdot 7 \cdot 41$ | 15 |
| $420=2^{2} 3 \cdot 5 \cdot 7$ | 7 | $245700=2^{2} 3^{3} 5^{2} 7 \cdot 13$ | 27 |
| $630=2 \cdot 3^{2} 5 \cdot 7$ | 7 | $257040=2^{4} 3^{3} 5 \cdot 7 \cdot 17$ | 28 |
| $2970=2 \cdot 3^{3} 5 \cdot 11$ | 11 | $409500=2^{2} 3^{2} 5^{3} 7 \cdot 13$ | 30 |
| $5460=2^{2} 3 \cdot 5 \cdot 7 \cdot 13$ | 13 | $464940=2^{2} 3^{4} 5 \cdot 7 \cdot 41$ | 18 |
| $8190=2 \cdot 3^{2} 5 \cdot 7 \cdot 13$ | 13 | $491400=2^{3} 3^{3} 5^{2} 7 \cdot 13$ | 36 |
| $9100=2^{2} 5^{2} 7 \cdot 13$ | 10 | $646425=3^{2} 5^{2} 13^{2} 17$ | 13 |
| $15925=5^{2} 7^{2} 13$ | 7 | $716625=3^{2} 5^{3} 7^{2} 13$ | 21 |
| $27300=2^{2} 3 \cdot 5^{2} 7 \cdot 13$ | 15 | $790398=2 \cdot 3^{4} 7 \cdot 1.7 \cdot 41$ | 17 |
| $36720=2^{4} 3^{3} 5 \cdot 17$ | 16 | $791700=2^{2} 3 \cdot 5^{2} 7 \cdot 13 \cdot 29$ | 29 |
| $40950=2 \cdot 3^{2} 5^{2} 7 \cdot 13$ | 15 | $819000=2^{3} 3^{2} 5^{3} 7 \cdot 13$ | 40 |
| $46494=2 \cdot 3^{4} 7 \cdot 41$ | 9 | $900900=2^{2} 3^{2} 5^{2} 7 \cdot 11 \cdot 13$ | 33 |
| $54600=2^{3} 3 \cdot 5^{2} 7 \cdot 13$ | 20 | $929880=2^{3} 3^{4} 5 \cdot 7 \cdot 41$ | 24 |
| $81900=2^{2} 3^{2} 5^{2} 7 \cdot 13$ | 18 | $955500=2^{2} 3 \cdot 5^{3} 7^{2} 13$ | 28 |

## References

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