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## Infinite Abelian Groups

Joaquin Pascual

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INFINITE ABELIAN GROUPS

by

Joaquin Pascual

A report submitted in partial fulfillment  
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Plan B

Approved:

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Joaquin Pascual

## NOTATION

$\mathbb{Z}$  : Set of integers

$\mathbb{Q}$  : Set of rationals

$\mathbb{Z}_p$  : Group of integer modulo  $p$

$\{a_1, \dots, a_n\}$  : Set whose elements are  $a_1, \dots, a_n$

$[a_1, \dots, a_n]$  : Subgroup generated by  $a_1, \dots, a_n$

$\sigma(m)$  : Cyclic group of order  $m$

$\sigma(p^\infty)$  :  $p$ -primary component of rationals modulo one

$tG$  : Torsion subgroup of  $G$

$dG$  : Maximal divisible subgroup of  $G$

$G[p]$  :  $\{x \in G : px = 0\}$

$nG$  :  $\{nx : x \in G\}$

$\Sigma A_k$  : Direct sum of the groups  $A_k$  (almost all coordinates are 0)

$k \in K$

$\Pi A_k$  : Direct product of the groups  $A_k$

$k \in K$

To Amelia

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## INTRODUCTION

When the theory of groups was first introduced, the attention was on finite groups. Now, the infinite abelian groups have come into their own. The results obtained in infinite abelian groups are very interesting and penetrating in other branches of Mathematics. For example, every theorem that is stated in this paper may be generalized for modules over principal ideal domains and applied to the study of linear transformations.

This paper presents the most important results in infinite abelian groups following the exposition given by J. Rotman in his book, Theory of Groups: An Introduction. Also, some of the exercises given by J. Rotman are presented in this paper. In order to facilitate our study, two classifications of infinite abelian groups are used. The first reduces the study of abelian groups to the study of torsion groups, torsion-free groups and an extension problem. The second classification reduces to the study of divisible and reduced groups. Following this is a study of free abelian groups that are, in a certain sense, dual to the divisible groups; the basis and fundamental theorems of finitely generated abelian groups are proved. Finally, torsion groups and torsion-free groups of rank 1 are studied.

It is assumed that the reader is familiar with elementary group theory and finite abelian groups. Zorn's lemma is applied several times as well as some results of vector spaces.

## PRELIMINARY RESULTS

The following results will be used in the support of this paper, but are not directly a part of it.

1. If  $K$  and  $S$  are groups, an extension of  $K$  by  $S$  is a group  $G$  such that
  - a.  $G$  contains  $K$  as a normal subgroup.
  - b.  $G/K \approx S$ .
2. Every finite abelian group  $G$  is a direct sum of  $p$ -primary group.
3. Every finite abelian group  $G$  is a direct sum of primary cyclic groups.
4. If  $G = \sum_{i=1}^n H_i$ , then

$$mG = \sum_{i=1}^n mH_i$$

where  $m$  is a positive integer.

5. If  $G = \sum_{i=1}^n H_i$ , then

$$G[p] = \sum_{i=1}^n (H_i[p])$$

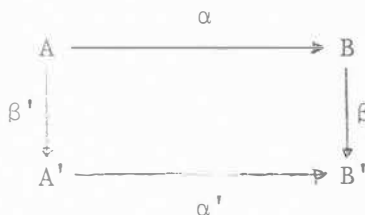
6. Every vector space has a basis.
7. Two bases for a vector space  $V$  have the same number of elements.

## INFINITE ABELIAN GROUPS

All groups under consideration are abelian and are written additively. The trivial group is the one having one element and is denoted by 0.

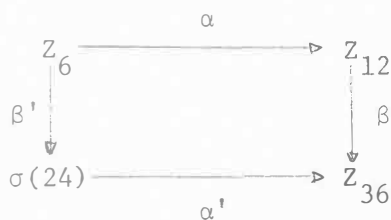
Definition

In the following diagram, capital letters denote groups and the arrows denote homomorphisms.



We say that the diagram commutes if  $\beta\alpha = \alpha'\beta'$ .

The following is one example of a commuting diagram



where  $\mathbb{Z}_6$ ,  $\mathbb{Z}_{12}$ , and  $\mathbb{Z}_{36}$  are the groups modulo 6, 12 and 36 respectively and  $\sigma(24)$  is a cyclic group of order 24.

$$\sigma : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_{12}$$

$$n \longrightarrow 2n$$

$$\beta : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{36}$$

$$m \longrightarrow 3_m$$

$$\beta' : \mathbb{Z}_6 \longrightarrow \sigma(24)$$

$$n \longrightarrow a^{3n}$$

where  $a$  is the generator of  $\sigma(24)$ .

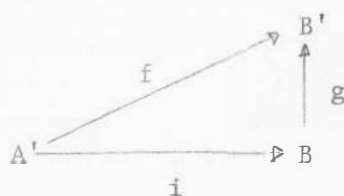
$$\alpha' : \sigma(24) \longrightarrow \mathbb{Z}_{36}$$

$$a^n \longrightarrow 2n$$

Consider now  $\beta\alpha(n) = \beta(\alpha n) = \beta(2n) = 6n$ .  $\alpha'\beta'(n) = \alpha'(\beta'n) = \alpha'(a^{3n}) = 6n$ . Then the above diagram commutes.

### Definition

A triangular diagram of the following type is a special type of commuting diagram



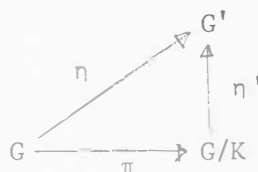
where  $i$  is an identity homomorphism and commutes if  $gi = f$ ; we also say that  $g$  extends  $f$ .

Example

Let  $G'$  equal the image of the group  $G$  by the homomorphism  $\eta$ . Because of the fundamental theorem of the homomorphism for groups, it is possible to find a factorization of  $\eta$ .

$\eta = \pi\eta'$  where  $\pi$  is the natural homomorphism from  $G$  to  $G/K$  ( $K$  is the kernel of  $\eta$ ) and  $\eta'$  is a homomorphism from  $G/K$  to  $G'$  such that

$$(aK)\eta' = a\eta$$



Consequently, this triangular diagram commutes.

If we have a large diagram composed of squares and triangles, we say that the diagram commutes if each component diagram commutes.

Definition

Let

$$\cdots A_{k+2} \xrightarrow{f_{k+2}} A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \cdots$$

be a sequence of groups and homomorphisms. This sequence is exact in case the image of each map is equal to the kernel of the next map.

Suppose  $A$  and  $B$  are isomorphic groups by  $f$ . Then,

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is an exact sequence.

### Theorem 1

If  $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$  is an exact sequence, then  $B$  is an extension of  $A$  by  $C$ .

### Proof

The image of  $0 \longrightarrow A$  is the kernel of  $g$ , but this image is  $0$ ; thus, the kernel of  $g$  is  $0$  and consequently,  $g$  is one-to-one. In the homomorphism  $C \longrightarrow 0$  all elements of  $C$  are mapped onto  $0$ . Therefore, the kernel is all of the set  $C$  and by definition,  $C$  is the image of  $f$ . Therefore,  $f$  is onto.

Now,  $A$  is isomorphic to  $A'$ , where  $A'$  is a normal subgroup of  $B$  and the image of  $A$  by  $g$ .

$f$  is onto and its kernel is  $A'$ , thus, by the fundamental theorem of the homomorphism

$$B/A' \approx C.$$

This proves that  $B$  is the extension of  $A$  by  $C$ .

### Theorem 2

In the exact sequence

$$\cdots \longrightarrow A_{k+2} \xrightarrow{f_{k+2}} A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \longrightarrow \cdots$$

$f_{k+2}$  is onto if and only if  $f_k$  is one-to-one.

Proof

Suppose  $f_{k+2}$  is onto, then the image of  $f_{k+2}$  is  $A_{k+1}$ . Then the kernel of  $f_{k+1}$  is  $A_{k+1}$ . Consequently, the image of  $f_{k+1}$  is 0 and is also the kernel of  $f_{k+2}$ . Therefore,  $f_k$  is one-to-one.

Suppose now that  $f_k$  is one-to-one, thus the kernel of  $f_k$  is 0 and this kernel is the image of  $f_{k+1}$ , so the kernel of  $f_{k+1}$  is  $A_{k+1}$  and  $A_{k+1}$  is the image of  $f_{k+2}$ . Therefore,  $f_{k+2}$  is onto.

Definition

The torsion subgroup of an abelian group  $G$  denoted  $tG$  is the set of all elements in  $G$  of finite order.

Since  $G$  is abelian, the set of all elements of finite order is a subgroup of  $G$ .

A group  $G$  is torsion in the case  $tG = G$ ;  $G$  is a torsion-free group in the case  $tG = 0$ .

Theorem 3

Every abelian group  $G$  is an extension of a torsion group by a torsion-free group.

Proof

We need to prove that there exists a normal subgroup of  $G$  that is a torsion group and the quotient group of  $G$  by this torsion group is a torsion-free group.

By definition,  $tG$  is a torsion group and  $tG$  is normal in  $G$ . We shall now prove that  $G/tG$  is a torsion-free group.

Suppose  $n\bar{x} = \bar{0}$  for some  $\bar{x} \in G/tG$  and some integer  $n \neq 0$ .

$\bar{x} = x + tG$  with  $x \in G$ ,  $\bar{0} = n\bar{x} = nx + tG$ , then  $nx \in tG$ ; hence there is an integer  $m \neq 0$  such that  $m(nx) = (mn)x = 0$ . Thus  $x$  has finite order,  $x$  is in  $tG$  and  $\bar{x} = \bar{0}$ . This proves the theorem.

Let  $K$  be a non-empty set and for each  $k \in K$ , let there be given a group  $A_k$ . The set  $K$  is called an index set.

#### Definition

The direct product of the  $A_k$ , denoted  $\prod_{k \in K} A_k$  is the group consisting

of all elements  $\langle a_k \rangle$  in the cartesian product of the  $A_k$  under the binary operation,

$$\langle a_k \rangle + \langle a'_k \rangle = \langle a_k + a'_k \rangle$$

i.e., componentwise addition. We do not require that  $A_k \neq A_r$  if  $k \neq r$  for  $k, r \in K$ ; thus the same group can be counted many times.

The subgroup of  $\prod_{k \in K} A_k$  consisting of all elements  $\langle a_k \rangle$  such that

only finitely many  $a_k$  are nonzero is denoted  $\sum_{k \in K} A_k$  and called

direct sum of the  $A_k$ .

If the index set  $K$  is finite, then

$$\prod_{k \in K} A_k = \sum_{k \in K} A_k.$$

#### Theorem 4

Let  $\{A_k\}$  be a family of abelian groups, then

$$t(\prod_{k \in K} A_k) \subset \prod_{k \in K} tA_k$$

$$t(\Sigma A_k) = \Sigma tA_k.$$

Proof

Let  $y$  be any element of  $t(\Pi A_k)$ , so  $my = 0$  for some integer  $m \neq 0$ ; that is,

$$y = \langle a_k \rangle$$

$$my = \langle ma_k \rangle = \langle 0 \rangle = 0$$

then,  $ma_k = 0$  for all  $k \in K$  and any  $a_k$  has finite order. Hence,  $y = \langle a_k \rangle$  is one element of  $\Pi tA_k$ . Therefore,  $t(\Pi A_k) \subset \Pi tA_k$ .

In order to prove that  $t(\Sigma A_k) = \Sigma tA_k$ , it suffices to show that  $\Sigma tA_k \subset t(\Sigma A_k)$  because  $t(\Sigma A_k) \subset \Sigma tA_k$  is a finite case of the first part of the theorem.

Consider any element  $\langle a_k \rangle$  in  $\Sigma tA_k$ . Then any  $a_k$  has a finite order. Let  $m$  be the least common multiple of the order of the  $a_k$ 's. Since  $ma_k = 0$ , for all  $a_k$ , then also  $\langle ma_k \rangle = 0$ , so  $\langle a_k \rangle$  belongs to  $t(\Sigma A_k)$ . This completes the proof.

Now we give an example that shows the inclusion  $t(\Pi A_k) \subset \Pi tA_k$  is proper.

Suppose that  $t(\Pi A_k) = \Pi tA_k$ . Let  $x = \langle b_k \rangle \in \Pi tA_k$  such that  $b_k$  is the generator of the  $\sigma(p^k)$  for each  $k \in K$ . The element  $x$  has infinite order since, for each  $m$  there exists  $p^k$  such that  $m < p^k$ . But  $x$  also is in  $t(\Pi A_k)$ . Then  $x$  cannot have infinite order because this would be a contradiction. Therefore, the inclusion

$$t(\Pi A_k) \subset \Pi tA_k$$

is proper.

Definition

The function  $\pi_i$  defined by  $\pi_i : \prod_{k \in K} A_k \rightarrow A_i$ ,  $\pi_i(\langle a_k \rangle) = a_i$

is called the  $i$ th projection.

It is obvious that the  $i$ th projection is a homomorphism from  $\prod_{k \in K} A_k$  onto  $A_i$ .

Theorem 5

Let  $\{A_k\}$  be a family of subgroups of  $G$ , then  $G \approx \Sigma A_k$  if and only if every nonzero element  $g$  has a unique expression of the form  $g = a_{k_1} + \dots + a_{k_n}$  where  $a_{k_i} \in A_{k_i}$ , the  $k_i$  are distinct and each  $a_{k_i} \neq 0$ .

Proof

Assume that  $G \approx \Sigma A_k$ . Let  $\langle a_k \rangle$  be any element in  $\Sigma A_k$ . Almost all coordinates of  $\langle a_k \rangle$  are zero. Let  $a_{k_1}, a_{k_2}, \dots, a_{k_n}$  be the coordinates of  $\langle a_k \rangle$  different from zero with  $a_{k_i} \in A_{k_i}$ . Consider the elements in  $\Sigma A_k$   $\langle a_{k_1} \rangle, \langle a_{k_2} \rangle, \dots, \langle a_{k_n} \rangle$ , where  $\langle a_{k_1} \rangle$  has all the coordinates zero except in the  $k_1$ th place that has  $a_{k_1}$ . Hence,

$$\langle a_k \rangle = \langle a_{k_1} \rangle + \langle a_{k_2} \rangle + \dots + \langle a_{k_n} \rangle.$$

Let  $f$  be the isomorphism from  $G$  onto  $\Sigma A_k$  and let  $x \in G$  such that  $f(x) = \langle a_k \rangle$ . Therefore, the inverse of  $f$ ,  $f^{-1}$ , maps  $x = f^{-1}(\langle a_k \rangle) = f^{-1}(\langle a_{k_1} \rangle + \langle a_{k_2} \rangle + \dots + \langle a_{k_n} \rangle) = f^{-1}(\langle a_{k_1} \rangle) + f^{-1}(\langle a_{k_2} \rangle) + \dots + f^{-1}(\langle a_{k_n} \rangle)$ .

In order to finish the proof, we must show that  $f^{-1}\langle a_{k_i} \rangle \in A_{k_i}$ .

Consider the subgroup  $A'_{k_i} \subset \Sigma A_k$ , where  $A'_{k_i} = \{\langle a_{k_i} \rangle\}$  and  $\langle a_{k_i} \rangle$

has all the coordinates 0 except the  $k_i$ th that is  $a_{k_i} \in A_{k_i}$ . It

is obvious that  $A_{k_i}$  is isomorphic to  $A'_{k_i}$  under the map  $a_{k_i} \mapsto \langle a_{k_i} \rangle$ .

Since  $f^{-1}$  is an isomorphism, then  $f^{-1}(A'_{k_i}) \cong A'_{k_i}$  and consequently,

$f^{-1}(A'_{k_i}) \cong A_{k_i}$ . Therefore,  $f^{-1}(\langle a_{k_i} \rangle)$  is in  $A_{k_i}$  and any element

$g \in G$  has a unique expression of the form  $g = a_{k_1} + \dots + a_{k_n}$ .

Conversely, suppose that the last statement is true. We define the function  $\eta$  from  $G$  into  $\Sigma A_k$  by  $\eta(g) = \langle a_k \rangle$  if  $g = a_{k_1} + \dots + a_{k_n}$ ,

where  $a_{k_i} \in A_{k_i}$ . The function  $\eta$  is well defined because if  $g$  has

two images then by the uniqueness of the representation of  $g$ , both

images are equals.  $\eta$  is also one-to-one. Suppose that the elements

$g_1 = a_{k_1} + \dots + a_{k_n}$ ,  $g_2 = b_{k_1} + \dots + b_{k_m}$  are different and have

the same image  $\langle C_k \rangle$ , then  $\langle C_k \rangle = \langle b_k \rangle = \langle a_k \rangle$  but by hypothesis

$a_{k_i} \neq b_{k_i}$  for some  $k_i$ ; hence,  $\eta$  is one-to-one. Let  $\langle a_k \rangle$  be any

element in  $\Sigma A_k$ , hence  $a_{k_1} + \dots + a_{k_m}$  represent some unique  $g$

in  $G$ , then  $\langle a_k \rangle$  is the image of  $g$  by  $\eta$  and so  $\eta$  is onto.

If  $g_1$  and  $g_2$  have the representation given above, then

$$\eta(g_1) = \langle a_k \rangle \quad \eta(g_2) = \langle b_k \rangle$$

$$\eta(g_1 + g_2) = \eta[a_{k_1} + \dots + a_{k_n} + (b_{k_1} + \dots + b_{k_m})] =$$

$$\langle a_k + b_k \rangle = \langle a_k \rangle + \langle b_k \rangle =$$

$$\eta(g_1) + \eta(g_2).$$

Therefore,  $\eta$  is an isomorphism of  $G$  into  $\Sigma A_k$ . This completes the proof.

### Theorem 6

Let  $\{A_k\}$  be a family of subgroup of  $G$ . Then  $G \approx \Sigma A_k$  if and only if  $G = [\bigcup_k A_k]$ , and for every  $i$ ,  $A_i \cap [\bigcup_{k \neq i} A_k] = 0$ .

### Proof

Suppose  $G \approx \Sigma A_k$ , then by theorem 5 any element  $g$  in  $G$  has a unique expression of the form  $g = a_{k_1} + \dots + a_{k_n}$ , where  $a_{k_i} \in A_{k_i}$  and  $a_{k_i} \neq 0$ . But  $a_{k_1} + \dots + a_{k_n} \in [\bigcup_k A_k]$ , so  $G \subset [\bigcup_k A_k]$  and obviously  $G \subset [\bigcup_k A_k]$ . Then  $G = [\bigcup_k A_k]$ . Let  $g$  be one element of  $A_i \cap [\bigcup_{k \neq i} A_k]$ , then  $a_i = a_{k_1} + \dots + a_{k_n}$ , but  $g$  has a unique representation, so  $a_i = a_{k_1} = \dots = a_{k_n} = 0$ . This proves the first part of the theorem.

Suppose now that  $G = [\bigcup_k A_k]$  and  $A_i \cap [\bigcup_{k \neq i} A_k] = 0$  for every  $i$ .

Let  $g$  be any element of  $G$ , i. e.,  $g \in [\bigcup_k A_k]$ , then we claim that

$g = a_{k_1} + \dots + a_{k_n}$  is a unique representation of  $g$  where  $a_{k_i} \in A_{k_i}$

and  $a_{k_i} \neq 0$ .

Suppose that  $g$  has other representation,  $g = b_{k_1} + \dots + b_{k_m}$ ,  
 $g - g = (a_{k_1} - b_{k_1}) + \dots + (a_{k_n} - b_{k_n}) = 0$  but  $A_i \cap [\bigcup_{k \neq i} A_k] = 0$ .  
 Then,  $a_{k_1} - b_{k_1} = \dots = a_{k_n} - b_{k_n} = 0$ . So,  $a_{k_1} = b_{k_1}, \dots, a_{k_n} =$   
 $b_{k_n}$ . Therefore, any element in  $G$  has a unique representation and by  
 theorem 5,  $G \cong \Sigma A_k$ .

#### Remark 1

It is easy to see that if  $G = A \oplus B$ , then the direct summand  
 $B$  is isomorphic to the factor group  $G/A$ .

#### Theorem 7

A subgroup  $A$  of  $G$  is a direct summand of  $G$  if and only if there  
 is a homomorphism  $p : G \rightarrow A$  such that  $p(a) = a$  for every  $a \in A$ .

#### Proof

Suppose that  $A$  is a direct summand of  $G$ . Then, the projection  
 $p$  defined as before is a homomorphism of  $G$  onto  $A$  and  $p(a) = a$  for  
 every  $a \in A$ .

Conversely, if there is a homomorphism  $p : G \rightarrow A$ , with the property  
 $p(a) = a$  for every  $a \in A$ , then we claim that the kernel  $K$  of  $p$  is such  
 that  $G = K \oplus A$ . First of all, we will show that  $K \cap A = 0$ . Let  $b$   
 be an element in  $K \cap A$ , then  $p(b) = 0$  because  $b \in K$  and  $p(b) = b$   
 because  $b \in A$ . Then  $b = 0$ .

Also, we must show that  $[A \cup K] = G$ . It is evident that  $[A \cup K] \subset G$ .  
 Let  $b$  be any element in  $G$ , then  $p(b) = b$  if  $b \in A$ . Suppose,  $f(b) = c$   
 and  $b \notin G - A$ , so  $f(b) - c = 0$ ,  $f(b - c) = 0$ , or  $b - c \in K$ . Therefore,

$b - c = a$ ;  $b = a + c$  where  $a \in K$  and  $c \in A$  so any element  $b \in G$  is also in  $[A \cup K]$ . Therefore,  $G = [K \cup A]$  and by theorem 6,  $G = A \oplus K$ .

### Definition

Let  $x \in G$  and let  $n$  be an integer.  $x$  is divisible by  $n$  if there is an element of  $y \in G$  with  $ny = x$ .

### Lemma 1

Let  $x \in G$  have order  $n$ . If  $(m, n) = 1$ , then  $x$  is divisible by  $m$ .

### Proof

If  $x \in G$  has order  $n$  and  $(m, n) = 1$ , then there exists integers  $p$  and  $q$  such that  $mp + nq = 1$ . Hence,  $x(mp + nq) = x$ ,  $(xm)p + (xn)q = x$ .

Since  $xn = 0$  and  $(xm)p = (xp)m$ , if we let  $y = xp$ , then  $my = x$  and thus,  $x$  is divisible by  $m$ .

### Theorem 8

There exists an abelian group  $G$  whose torsion subgroup is not a direct summand.

### Proof

Let  $P$  be the set of all primes and let  $G = \prod_{p \in P} \sigma(p)$ . We claim that  $tG$  is not a direct summand.

Assume  $tG$  is a direct summand; then, by Remark 1,  $G \cong (G/tG \oplus tG)$ . Now, we shall prove that  $tG = \sum \sigma(p)$ . Evidently,  $\sum \sigma(p) \subset tG$ . Suppose  $x = \langle x_p \rangle \in G$  and  $mx = 0$ , for some integer  $m \neq 0$ , then  $mx_p = 0$  for each  $p$ . Since  $x_p \in \sigma(p)$  and by the fact that the order of the element divides the order of the group, then  $m \equiv 0 \pmod{p}$  for

every  $p$  and  $x_p \neq 0$ . There are only finitely many coordinates  $x_p$  different from zero, otherwise,  $m$  is divisible by infinitely many distinct primes and this is impossible. Hence,  $tG \subset \Sigma \sigma(p)$  and so  $tG = \Sigma \sigma(p)$ .

Our next step is to prove that  $G/tG$  has an element different from zero and divisible by every prime  $p$ . Consider the element  $\langle a_p \rangle + tG$  in  $G/tG$  where  $a_p$  is the generator of  $\sigma(p)$  for every prime  $p$ . If  $q$  is a prime, then by lemma 1, for each prime  $p \neq q$  there exists  $x_p \in \sigma(p)$  with  $qx_p = a_p$ . Let  $\langle x_q \rangle \in G$  be such that any component has the above property except  $x_q = 0$ . Thus

$$q\langle x_p \rangle = \langle a_p \rangle - \langle y \rangle$$

where  $\langle y \rangle$  has 0 in each coordinate save the  $q$ th where it has  $a_q$ .

Therefore,  $\langle y \rangle \in tG$  and

$$q(\langle x_p \rangle + tG) = q\langle x_p \rangle + tG =$$

$$\langle a_p \rangle - \langle y \rangle + tG =$$

$$\langle a_p \rangle + tG.$$

Since  $G/tG$  is a direct summand of  $G$ , then  $G/tG$  is isomorphic to some subgroup of  $G$ . Therefore,  $G$  needs to have some element divisible by every prime. Suppose that this is the case. Assume that the nonzero element  $\langle x_q \rangle \in G$  is divisible by every prime  $p$ , then  $p\langle y_q \rangle = \langle x_q \rangle$  for some  $\langle y_q \rangle \in G$ . Hence,  $\langle py_q \rangle = \langle x_q \rangle$  i. e.,  $py_q = x_q$  for every prime  $q$ . In particular, if  $q = p$ , then  $py_p = x_p = 0$ . Therefore, if  $\langle x_q \rangle$  is divisible by every prime, then each component of  $\langle x_q \rangle$  is 0

and  $\langle x \rangle = 0$ . This is a contradiction. Therefore, our assumption that  $tG$  is a direct summand of  $G$  is false.

### Definition

Let  $p$  be a prime. A group  $G$  is  $p$ -primary (or is  $p$ -group) in case every element in  $G$  has order of a power of  $P$ .

### Theorem 9

Every torsion group  $G$  is the direct sum of  $p$ -primary groups.

### Proof

Let  $G_p$  be the  $p$ -primary subgroup of  $G$ , i. e.,  $G_p$  is the set of elements of  $G$  that have order of a power of  $p$ . We want to prove that

$G = \sum G_p$ . Let  $x \in G$ ,  $x \neq 0$ , and let the order of  $x$  be  $n$ . By the

fundamental theorem of the arithmetic,  $n = p_1^{e_1} \cdot \dots \cdot p_h^{e_h}$  where

the  $p_{k_i}$  are distinct primes and the exponents  $e_i \geq 1$ . Let

$n_{k_i} = \frac{n}{p_{k_i}^{e_i}}$  and consider the greatest common divisor of the  $n_{k_i}$ 's.

It is easy to see that

$$(n_{k_1}, \dots, n_{k_h}) = 1.$$

Therefore, there exists integers  $m_i$  such that  $\sum_{i=1}^h m_i n_{k_i} = 1$  and

hence,  $\sum_{i=1}^h m_i n_{k_i} x = x$ .

Now  $p_{k_i}^{e_i} (m_i n_{k_i} x) = m_i (p_{k_i}^{e_i} n_{k_i} x) = m_i (nx) = 0$ . Therefore, the

element  $m_i n_{k_i} x \in Gp_{k_i}$ . Also  $m_i n_{k_i} x \neq 0$  for otherwise  $m_i n_{k_i} = sn$

or  $m_i = sp_{k_i}$  and this contradicts the fact that

$$\sum_{i=1}^h m_i n_{k_i} = 1.$$

We claim that any  $x$  in  $G$  can be written as unique form  $x = xp_{k_1} +$

$\dots + xp_{k_h}$ , where  $xp_{k_i} \in Gp_{k_i}$ , the  $p_{k_i}$  are distinct and each  $xp_{k_i} \neq 0$ .

We proved above that  $xp_{k_i} = m_i n_{k_i} x$ ; that is,  $x = \sum_{i=1}^h m_i n_{k_i} x$ . Suppose

that  $x$  has another representation,  $x = y_{k_1} + \dots + y_{k_n}$ . Thus,

$$\sum_{i=1}^h m_i n_{k_i} x = y_{k_1} + \dots + y_{k_n}.$$

$$n \sum_{i=1}^h m_i n_{k_i} x = n(y_{k_1} + \dots + y_{k_n})$$

$$\sum_{i=1}^h nm_i n_{k_i} x = ny_{k_1} + \dots + ny_{k_n} = 0.$$

Then,  $ny_{k_1} = ny_{k_2} = \dots = ny_{k_n} = 0$  so that the order of the  $y_{k_i}$ 's

divides  $n$  and the divisors of  $n$  are the  $p_{k_i}$ 's. Therefore,  $y_{k_i} \in Gp_{k_i}$

for  $i = 1, \dots, h$  and  $y_{k_i} = xp_{k_i}$ . By theorem 5,  $G = \sum Gp_{k_i}$ .

Theorem 10

Let  $G$  and  $H$  be a torsion group.  $G \approx H$  if and only if  $G_p \approx H_p$  for every prime  $p$ .

Proof

Let  $f$  be an isomorphism of  $G$  onto  $H$  and  $f^{-1}$  be the inverse of  $f$ . Let  $x \in G_p$ , and  $p^\alpha$  be the order of  $x$ . Then  $p^\alpha f(x) = f(p^\alpha x) = f(0) = 0$ . Therefore,  $f(G_p) \subset H_p$  and by symmetry,  $f^{-1}(H_p) \supset G_p$ . This means  $f(G_p) = H_p$ ; thus,  $f|_{G_p}$  is an isomorphism from  $G_p$  onto  $H_p$  so we have  $G_p \approx H_p$ .

Conversely, if  $f_p$  is an isomorphism of  $G_p$  onto  $H_p$ , for every  $p$ , then the function  $f : G \rightarrow H$  defined by  $f\langle x_p \rangle = \langle f_p(x_p) \rangle$  is an isomorphism. In fact, let  $x, y \in G$ , then

$$\begin{aligned} f(x + y) &= f\langle x_p + y_p \rangle = \langle f_p(x_p + y_p) \rangle = \\ &\langle f_p(x_p) + f_p(y_p) \rangle = \langle f_p(x_p) \rangle + \langle f_p(y_p) \rangle = \\ &f(x) + f(y). \end{aligned}$$

$f$  is one-to-one. Let  $x, y \in G$ ,

$$\begin{aligned} y &= y_{p_1} + \dots + y_{p_n} \\ x &= x_{p_1} + \dots + x_{p_n}. \end{aligned}$$

Suppose  $x \neq y$  and  $f(x) = f(y)$ , then

$$f_{p_1}(y_{p_1}) + \dots + f_{p_n}(y_{p_n}) = f_{p_1}(x_{p_1}) + \dots + f_{p_m}(x_{p_m})$$

but the  $f_{p_i}$ 's are isomorphisms, then

$$f_{p_i}(x_{p_i}) = f_{p_i}(y_{p_i})$$

for all  $i$ , so that  $x = y$  contradicting our hypothesis that  $x \neq y$ .

Therefore  $f$  is one-to-one. Let  $y \in H$ ,  $y = y_{p_i} + \dots + y_{p_k} =$

$$f_{p_i}(x_i) + \dots + f_{p_k}(x_k) = f(x_i + \dots + x_k) = f(x). \text{ Then, } f \text{ is}$$

onto. Therefore,  $f$  is an isomorphism of  $G$  onto  $H$ .

Up here we have studied arbitrary abelian groups and are making some important reductions. Theorem 3 reduces the study of arbitrary abelian groups to the study of torsion groups and torsion-free groups. Theorems 8 and 9 reduce the study of torsion groups to the study of  $p$ -primary groups. We will now study a generalization of the groups of rationals and the group of reals, called the divisible groups.

#### Definition

A group  $G$  is divisible if each  $x \in G$  is divisible by every integer  $n > 0$ .

#### Example

The addition group of the rational numbers, denoted by  $Q$ , is divisible. Given any rational  $a$  and any integer  $n > 0$ , there exists  $a' = \frac{a}{n} \in Q$  such that  $na' = a$ . Also the following groups are divisible: the additive group of reals, the additive group of complex, and the multiplicative group of the reals.

Theorem 11

A quotient of a divisible group is divisible.

Proof

Let  $G$  be a divisible group and  $H$  a subgroup of  $G$ . For any integer  $n > 0$  and a given  $a + H \in G/H$  we assert that there exists  $b + H \in G/H$  such that  $n(b + H) = a + H$ . In fact, it is always possible to find  $n$  and  $b$  such that  $nb = a$  and therefore the element  $b + H \in G/H$  has the property that  $n(b + H) = n(b + H) = a + H$ . Since this is always possible,  $G/H$  is divisible.

The converse of this theorem is not true and the following is an example.

In theorem 8, we constructed the group  $G = \prod_{p \in P} \sigma(p)$  and also we proved that  $G$  is not divisible. However, we will prove that  $G/tG$  is divisible.

Let  $\langle x_p \rangle + tG \in G/tG$  and  $n$  any non-prime integer greater than 0. Since  $x_p \in \sigma(p)$ , it is divisible by any  $n < p$ , by lemma 1, and also if  $n > p$ , because  $n \equiv r \pmod{p}$  where  $r < p$ . Thus  $\langle x_p \rangle + tG$  is divisible by any non-prime integer. We will now prove that  $\langle x_p \rangle + tG$  is divisible by every prime. Let  $q$  be any prime, then the above result holds except for  $x_q \in \sigma(q)$ . We know that for any  $y_q \in \sigma(q)$ ,  $qy_q = 0$ . Let  $\langle y_q \rangle \in tG$ , where all the coordinates of  $\langle y_q \rangle$  are zero except the  $q$ th that is  $x_q \in \sigma(q)$ . Of course,  $\langle y_q \rangle \in tG$ . Let  $\langle x_p \rangle + tG \in G/tG$ , then there exists  $\langle z_p \rangle + tG$  such that

$$q(\langle z_p \rangle + tG) = (\langle x_p \rangle - \langle y_q \rangle) + tG =$$

$$\langle x_p \rangle + tG.$$

Therefore,  $G/tG$  is divisible.

### Remark 2

It is clear that a direct sum (direct product) of groups is divisible if, and only if, each summand (factor) is divisible.

### Lemma 2

A torsion-free divisible group is a vector space over  $Q$ .

### Proof

Let  $G$  be a torsion-free divisible group. We define the scalar multiplication as follows: for any  $\frac{a}{b} \in Q$  and  $x \in G$ ,  $\frac{a}{b}x = ay$ , where  $y \in G$  and  $by = x$ . This scalar multiplication is well defined because of the uniqueness of the number  $y$ , i. e., for a given integer  $n$  and  $x \in G$ ,  $ny = x$ ,  $y$  is unique. Suppose there exists  $y_1$  such that  $ny_1 = x$ . Then  $n(y - y_1) = 0$ . This means  $y - y_1$  is either 0 or an element of finite order; since  $G$  is torsion-free,  $y - y_1 = 0$ . Therefore  $y = y_1$ . Now, we shall show that this scalar multiplication satisfies the axioms of a vector space.

1. For any  $\frac{a}{b}, \frac{c}{d} \in Q$ ,  $x \in G$ ,  $\frac{a}{b}x = ay_1$  with  $by_1 = x$ ,  $\frac{c}{d}x = cy_2$  with  $dy_2 = x$ ,  $(\frac{a}{b} + \frac{c}{d})x = \frac{ad + cb}{bd}x = (ad + cb)y_3$  where  $bdy_3 = x$ .

Now,  $\frac{a}{b}x + \frac{c}{d}x = ay_1 + cy_2$ , but  $x = by_1 = dy_2 = dby_3$ ; thus,  $ay_1 + cy_2 = ady_3 + cby_3 = (\frac{a}{b} + \frac{c}{d})x$ . Therefore,  $(\frac{a}{b} + \frac{c}{d})x = \frac{a}{b}x + \frac{c}{d}x$ .

2.  $\frac{a}{b}(x + y) = ay_1$  with  $by_1 = x + y$  and  $\frac{a}{b}x = ay_2$  with  $by_2 = x$ ,  $\frac{a}{b}y = ay_3$  with  $by_3 = y$ . Then,  $x + y = b(y_2 + y_3)$  and  $\frac{a}{b}x + \frac{a}{b}y = a(y_2 + y_3) = ay_1 = \frac{a}{b}(x + y)$  so the scalar multiplication is distributive over addition.

$$3. (a) \left(\frac{a}{b} \cdot \frac{c}{d}\right)x = acy_1, \text{ with } bdy_1 = x.$$

$$(b) \frac{a}{b}\left(\frac{c}{d}x\right) = \frac{a}{b}(cy_2) = ay_3, \text{ where } dy_2 = x, by_3 = cy_2.$$

Since  $by_1 = y_2$  and  $y_3 = cy_1$ , then (a) and (b) are equal.

4.  $1.x = x$  for any  $x \in G$ . Therefore, the group  $G$  over  $Q$  is a vector space.

### Corollary 1

Let  $V$  be a vector space over  $F$ . Considering  $V$  as an abelian group,  $V$  is the direct sum of copies of  $F$ .

### Proof

Let  $B = \{x_k : k \in K\}$  be a basis of  $V$  and let  $F_k$  denote the one-dimensional vector space generated by  $x_k$ .

Let  $f$  be the function from  $F_k$  onto  $F$  such that for any  $ax_k \in F_k$   $f(ax_k) = a$ . It is clear that  $f$  is one-to-one, onto and also  $f(bx_k + ax_k) = f(a + b)x_k = a + b$ . Therefore,  $F_k$  is isomorphic to the additive group of  $F$ .

We claim that the additive group  $V$  is isomorphic to  $\sum_{k \in K} F_k$ . Any vector  $x$  in  $V$  has a unique expression  $x = \sum_{k_i} r_{k_i} x_{k_i}$ , where the  $r_{k_i} \neq 0$

and all the  $x_{k_i}$  are distinct; furthermore, each  $r_{k_i} x_{k_i} \in F_{k_i}$ . By

theorem 5,  $V \approx \sum F_k$ .

### Lemma 3

An abelian group with  $pG = 0$  is a vector space over  $\mathbb{Z}_p$ .

### Proof

Let  $\bar{k}$  denote the residue class of the integer  $k$  in  $\mathbb{Z}_p$ .

Define a scalar multiplication on  $G$  by  $\bar{k}x = kx$ , where  $x \in G$ .

This operation is well defined for if  $\bar{k} = \bar{k}'$ , then  $k - k' = mp$  for some integer  $m$  so that  $(k - k')x = mp_x = 0$ ; hence,  $\bar{k}x = \bar{k}'x$ . It is easy to verify the axioms of a vector space in this case.

### Corollary 2

1. Every torsion-free divisible group  $G$  is a direct sum of copies of  $Q$ .
2. An abelian group  $G$  in which any nonzero element has prime order  $p$  is a direct sum of copies of  $\sigma(p)$ .

### Proof

1. By lemma 2,  $G$  is vector space over  $Q$ . Therefore, by corollary 1,  $G$  is the direct sum of copies of  $Q$ .
2. By lemma 3,  $G$  is a vector space over  $Z_p$  and by corollary 1,  $G$  is the direct sum of copies of  $Z_p$  but  $Z_p \approx \sigma(p)$ , then by theorem 10,  $\Sigma Z_p \approx \Sigma \sigma(p)$ . Therefore,  $G \approx \Sigma \sigma(p)$ .

### Theorem 12

Let  $V$  and  $W$  be vector spaces over  $F$ , then  $V$  and  $W$  are isomorphic if and only if  $V$  and  $W$  have the same dimension.

### Proof

Let  $B_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$  be a basis of  $V$  and  $B_2 = \{\beta_1, \beta_2, \dots, \beta_n, \dots\}$  be the image of  $B_1$  by the isomorphism  $f$ , i. e.,

$$f(\alpha_1) = \beta_1$$

$$f(\alpha_2) = \beta_2$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$f(\alpha_n) = \beta_n$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Let  $x$  be any element of  $V$ , then  $x$  has a unique expression of the form  $x = a_{k_1} \alpha_{k_1} + \dots + a_{k_n} \alpha_{k_n}$  with  $a_{k_i} \in F$  and  $\alpha_{k_i} \in B_1$ .

$$y = f(x) = a_{k_1} \beta_{k_1} + \dots + a_{k_n} \beta_{k_n}.$$

Since any  $y$  in  $W$  is a linear combination of the  $\beta_i$ 's, then  $B_2$  spans  $W$ . Suppose that  $B_2$  is a linear dependent; that is, there is a subset  $\{\beta_{k_1}, \dots, \beta_{k_n}\}$  of  $B_2$  such that  $a_{k_1} \beta_{k_1} + \dots + a_{k_n} \beta_{k_n} = 0$  where the  $a_{k_i}$ 's are not all 0. Suppose  $a_{k_i} \neq 0$ , then  $a_{k_1} f(\alpha_{k_1}) + \dots + a_{k_n} f(\alpha_{k_n}) = 0$ .  $f(a_{k_1} \alpha_{k_1} + \dots + a_{k_n} \alpha_{k_n}) = 0$  but  $f$  is an isomorphism so  $a_{k_1} \alpha_{k_1} + \dots + a_{k_n} \alpha_{k_n} = 0$ . With  $a_{k_i} \neq 0$  and the set  $\{\alpha_{k_1}, \dots, \alpha_{k_n}\}$  is linearly dependent. This contradicts the fact that  $B_1$  is a basis. Therefore,  $B_2$  is linearly independent and a basis of  $W$  with the same number of elements as that of  $B_1$ .

Conversely, suppose now that  $V$  and  $W$  have the same dimension, then if  $B_1 = \{\alpha_1, \dots, \alpha_n, \dots\}$  is a basis of  $V$  and  $B_2 = \{\beta_1, \beta_2, \dots, \beta_n, \dots\}$  is a basis of  $W$  and the mapping

$$f(\alpha_1) = \beta_1$$

$$f(\alpha_2) = \beta_2$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$f(\alpha_n) = \beta_n$$

is one-to-one. Now, we extend the mapping  $f$  as follows: if  $x \in V$  and  $x = a_{k_1} \alpha_{k_1} + \dots + a_{k_n} \alpha_{k_n}$ , then  $f(x) = a_{k_1} \beta_{k_1} + \dots + a_{k_n} \beta_{k_n}$ .

It is clear that  $f$  is well defined and one-to-one.

Let  $y \in W$ , then  $y = a_{k_1} \beta_{k_1} + \dots + a_{k_m} \beta_{k_m}$  and  $y = f(a_{k_1} \alpha_{k_1} + \dots + a_{k_m} \alpha_{k_m}) = f(y)$ . So,  $f$  is onto. It is clear,  $f(x + y) = f(x) + f(y)$ , for every  $x, y \in V$ . Therefore,  $V \cong W$  as a vector space.

### Corollary 3

Let  $V$  and  $W$  be vector spaces over  $F$ . As abelian groups,  $V \cong W$ , if and only if,  $V$  and  $W$  have the same dimension.

### Proof

By theorem 12,  $V \cong W$  as a vector space. Let  $f$  be an isomorphism from  $V$  to  $W$ . Then  $f$  maps  $V$  as abelian group onto the abelian group  $W$  and one-to-one. Let  $x, y \in V$ , then  $f(x + y) = f(x) + f(y)$ . Therefore,  $V \cong W$  as abelian groups.

### Lemma 4

The group  $Q/Z$  is a torsion and divisible group.

### Proof

Let  $\frac{a}{b} + Z \in Q/Z$ , the order of  $\frac{a}{b} + Z$  is  $b$ . Given any integer  $n$  and  $\frac{a}{b} + Z$ , then the element  $y = \frac{a}{nb} + Z \in Q/Z$  is such that  $ny = \frac{a}{b} + Z$ . Therefore,  $Q/Z$  is a torsion and divisible group.

The  $p$ -primary component of  $Q/Z$  is a subgroup and consequently it is also a divisible group.

Definition

If  $p$  is a prime,  $\sigma(p^\infty)$  denotes the  $p$ -primary component of  $Q/Z$ . Let  $A^{(p)}$  denote the set of all rationals between 0 and 1 of the form  $m/p^n$ , where  $m, n \geq 0$ . We define on  $A^{(p)}$  the binary operation "addition modulo 1" as usual. For example, if  $p = 3$ , then  $\frac{1}{3} + \frac{2}{3} = 0$ ,  $\frac{1}{3} + \frac{8}{9} = \frac{2}{9}$ , etc.

Theorem 13

$A^{(p)}$  is a  $p$ -primary group and  $Q/Z \cong \Sigma A^{(p)}$ .

Proof

First of all, the operation "addition modulo 1" is well defined and it is associative and commutative; 0 is the identity and  $-\frac{m}{p^n}$  is the inverse of  $\frac{m}{p^n}$ . The order of  $\frac{m}{p^n}$  is  $p^n$ , therefore,  $A^{(p)}$  is a  $p$ -primary group.

Let  $x \in \sigma(p^\infty)$ , thus  $x$  has order of a power of  $p$ , say  $p^n$ ,  $x = \frac{a}{b} + Z$  with  $(a, b) = 1$ . So,  $p^n x = \frac{ap^n}{b} + Z = \bar{0}$ . Then  $\frac{ap^n}{b} = h \in Z$  or  $hb = ap^n$ ; since  $(b, a) = 1$ , then  $b = rp^n$  for some integer  $r$  that means  $x = \frac{a}{rp^n} + Z$ , but this element does not have order  $p^n$

so  $r = 1$  and the element  $x$  of order  $p^n$  has the form  $\frac{a}{p^n} + Z$ .

Consider now the mapping  $f$  from  $\sigma(p^\infty)$  into  $A^{(p)}$  defined by

$f(\frac{m}{p^r} + Z) = \frac{m}{p^r}$ . Let  $(\frac{m_1}{p^{r_1}} + Z) \neq (\frac{m_2}{p^{r_2}} + Z)$  be elements of  $\sigma(p^\infty)$

and suppose  $f(\frac{m_1}{p^{r_1}} + Z) = f(\frac{m_2}{p^{r_2}} + Z)$ , thus  $\frac{m_1}{p^{r_1}} = \frac{m_2}{p^{r_2}} \pmod{1}$ .

So,  $\frac{m_1}{p r_1} - \frac{m_2}{p r_2} = k$  for some integer  $k$ . This means  $\frac{m_1}{p r_1} - \frac{m_2}{p r_2} + Z = \bar{0}$ .

$\frac{m_1}{p r_1} + Z = \frac{m_2}{p r_2} + Z$ . Therefore,  $f$  is one-to-one and onto because for

any  $\frac{m}{p r} \in A^{(p)}$ ,  $f(\frac{m}{p r} + Z) = \frac{m}{p r}$ . Also,  $f(\frac{m_1}{p r_1} + Z + \frac{m_2}{p r_2} + Z) =$

$f(\frac{m_1}{p r_1} + \frac{m_2}{p r_2} + Z) = \frac{m_1}{p r_1} + \frac{m_2}{p r_2} = f(\frac{m_1}{p r_1} + Z) + f(\frac{m_2}{p r_2} + Z)$ . So,  $f$  is

an isomorphism and  $A^{(p)} \cong \sigma(p^\infty)$ .

We know by theorem 9,  $Q/Z \cong \Sigma\sigma(p^\infty)$  and by theorem 10,  $\Sigma\sigma(p^\infty) \cong \Sigma A^{(p)}$ ; therefore,  $Q/Z \cong \Sigma A^{(p)}$ .

#### Theorem 14

Let  $a_1, a_2, \dots, a_n, \dots$ , be nonzero elements of  $\sigma(p^\infty)$  such that  $pa_1 = 0, pa_2 = a_1, \dots, pa_{n+1} = a_n, \dots$ . If  $[a_n]$  is the cyclic subgroup of  $\sigma(p^\infty)$  generated by  $a_n$ , then  $[a_n] \cong \sigma(p^n)$ ,  $[a_n] \subset [a_{n+1}]$  for all  $n$ , and  $\sigma(p^\infty) = \bigcup_{n=1}^{\infty} [a_n]$ .

#### Proof

Consider  $p^n a_n$ . We know  $a_1 = pa_2, a_2 = pa_3, \dots, a_{n-1} = pa_n$  then  $p^n a_n = p^{n-1}(pa_n) = p^{n-2}pa_{n-1} = \dots = pa_1 = 0$ . Therefore, the order of  $[a_n]$  is  $p^n$  and by the well known theorem that two cyclic groups of the same order are isomorphic,  $[a_n] \cong \sigma(p^n)$ .

Let  $b$  be any element of  $[a_n]$ . Thus,  $b = ra_n$ , where  $r$  is some integer less than  $p^n$ . But  $a_n = pa_{n+1}$ , then  $b = rpa_{n+1}$ ; therefore,  $b \in [a_{n+1}]$  and  $[a_n] \subset [a_{n+1}]$ .

It is obvious that  $\bigcup_{n=1}^{\infty} [a_n] \subset \sigma(p^{\infty})$ . Consider now  $x \in \sigma(p^{\infty})$  of order  $p^r$ , so  $x = \frac{a}{p^r} + Z$ . We claim that  $x \in [a_n]$ . Let  $a_n = \frac{b}{p^r} +$

$Z$ . We consider two cases.

1. Suppose  $b$  divides  $a$ , so  $a = qb$ , therefore  $qa_n = \frac{bq}{p^r} + Z =$

$$\frac{a}{p^r} + Z = x.$$

2. Neither  $b$  divides  $a$  nor  $a$  divides  $b$ . Since  $b < p^r$ ,  $a < p^r$ , then there exists some integer  $h$  such that  $hb = a \pmod{p^n}$ . Therefore,  $ha_n = \frac{hb}{p^r} + Z = \frac{a}{p^r} + Z = x$  that imply  $x \in \bigcup_{n=1}^{\infty} [a_n]$  so that  $\sigma(p^{\infty}) =$

$$\bigcup_{n=1}^{\infty} [a_n].$$

#### Corollary 4

Every proper subgroup of  $\sigma(p^{\infty})$  is finite and the set of subgroups is well ordered by inclusion.

#### Proof

Suppose there exists an infinite group  $G$  properly contained in  $\sigma(p^{\infty})$ . We will show that this is impossible. Let  $x \in G$ , since also  $x \in \sigma(p^{\infty})$ ,  $x$  has finite order, say  $p^n$ , so  $x = \frac{a}{p^n} + Z$  and since the order of  $x$  is the same as the order of  $a_n$ , then  $[x] = [a_n]$ . The order of the elements of  $G$  are either bounded or not. Suppose  $p^r$  is a bound of the order of the elements of  $G$ . Then, by theorem 14,  $G \subset [a_{r+1}]$  that contradicts our hypothesis that  $G$  is infinite. If the order of the elements of  $G$  are unbounded, then there exists  $y_i \in G$  such that  $y_i \in [a_i]$  for every  $i$ . But  $[y_i] = [a_i]$ , and

$\bigcup_{i=1}^{\infty} [y_i] = \sigma(p^{\infty})$ . This contradicts the hypothesis that  $G$  is a proper subgroup of  $\sigma(p^{\infty})$ . Therefore,  $G$  is finite.

Now we will prove that the set  $M$  of subgroups of  $\sigma(p^{\infty})$  is well ordered by inclusion. Since  $\sigma(p^{\infty}) = \bigcup_{n=1}^{\infty} [a_n]$  and all proper subgroups are finite, then, for any two subgroups  $G_1, G_2$ , either  $G_1 \subset G_2$  or  $G_2 \subset G_1$ . Hence, the elements of any subset  $S$  of  $M$  are contained in some  $[a_n]$ . Therefore,  $S$  has a first element.

### Corollary 5

$\sigma(p^{\infty})$  has the descending chain conditions (DCC) but not the ascending chain condition (ACC).

### Proof

By theorem 14, given a subgroup  $G$  of  $\sigma(p^{\infty})$ ,  $G$  is finite and  $[a_n] \subset G \subset [a_{n+1}]$  for some  $n$ . But  $[a_n] \supset [a_{n-1}] \supset \dots \supset [a_1] \supset 0$ . Therefore,  $\sigma(p^{\infty})$  has the DCC.

By theorem 14  $\sigma(p^{\infty}) = \bigcup_{n=1}^{\infty} [a_n]$  with  $[a_n] \subset [a_{n+1}]$ ; therefore, any ascending chain cannot stop after a finite number of steps.

### Theorem 15

Let  $G$  be an ascending union of infinite cyclic groups  $C_n$  such that  $C_n = [c_n]$  and  $(n+1)c_{n+1} = c_n$ , for  $n = 1, 2, \dots$ . Then  $G$  is isomorphic to the additive group of rationals.

### Proof

Let  $Q_n = [\frac{1}{n!}]$ ,  $n = 1, 2, \dots$ . Clearly,  $Q_n \subset Q_{n+1}$  and  $Q = \bigcup_{n=1}^{\infty} Q_n$ .

Define the map  $\theta : G \rightarrow Q$  by  $\theta(mc_n) = \frac{m}{n!}$ , where  $m$  is an integer.

We must prove that  $\theta$  is well defined, i. e., if  $m_1 c_n = m_2 c_r$ , where  $m_1$ ,  $m_2$  and  $n$ ,  $r$  are integers, then  $\theta(m_1 c_n) = \theta(m_2 c_r)$ . Suppose  $n \leq r$ .

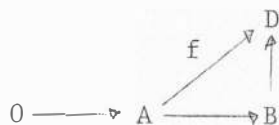
Since  $nc_n = c_{n-1}$ ,  $(n-1)c_{n-1} = c_{n-2}$ ,  $\dots$ ,  $(r+1)c_{r+1} = c_r$ , then,  $c_n = \frac{r!}{n!} c_r$ . Hence,  $m_1 c_n = m_1 \left(\frac{r!}{n!}\right) c_r = m_2 c_r$ . Since  $C_r$  is an infinite cyclic group,  $m_1 \left(\frac{r!}{n!}\right) = m_2 c_r$  implies that  $m_1 \left(\frac{r!}{n!}\right) = m_2$  and so that  $\frac{m_1}{r!} = \frac{m_2}{n!}$  which means  $\theta(m_1 c_n) = \theta(m_2 c_r)$ . Consequently,  $\theta$  is well defined.

Since  $\theta(c_n) = \frac{1}{n!}$ ,  $\theta(C_n) = Q_n$ , it follows that  $\theta$  is onto. Let  $a, b \in G$ . We may suppose  $a, b \in C_n$  for some  $n$ . Hence,  $a = m_1 c_n$ ,  $b = m_2 c_n$  and  $a + b = (m_1 + m_2) c_n$ ;  $\theta(a + b) = (m_1 + m_2) \frac{1}{n!} = \frac{m_1}{n!} + \frac{m_2}{n!} = \theta(a) + \theta(b)$ . Thus  $\theta$  is a homomorphism.

Consider now the kernel of  $\theta$ . Suppose that  $\theta(a) = 0$  for some  $a \in G$ . We have  $a \in C_n$ ,  $a = mc_n$ . Then,  $\theta(a) = \frac{m}{n!} = 0$  and this is true only if  $m = 0$ . Hence,  $a = 0$ . Therefore the kernel is 0 and  $\theta$  is an isomorphism.

### Definition

Let  $A$  be a subgroup of  $B$ , and let  $f : A \rightarrow D$  be a homomorphism. We say that  $D$  has the injective property in case  $f$  can be extended to a homomorphism  $F : B \rightarrow D$ ; in other words, an  $F$  exists making the adjoined diagram commute.



Theorem 16

A group  $D$  is divisible if, and only if,  $D$  has the injective property.

Proof

Suppose  $D$  is divisible and there exists a homomorphism  $f$  from  $A$  into  $D$  where  $A$  is a subgroup of  $B$ . We will prove that there is an  $F : B \rightarrow D$  that extends  $f$ .

Consider the set  $S^*$  of all pairs  $(S, H)$ , where  $S$  is a subgroup of  $B$  containing  $A$  and  $h$  is a homomorphism from  $S$  to  $D$  that extends  $f$ .  $S^*$  is not empty, for  $(A, f) \in S^*$ . We partially order  $S^*$  by  $(A_1, h_1) \leq (S_2, h_2)$  in case  $S_1 \subset S_2$  and  $h_2$  extends  $h_1$ . Let  $\{(S_\alpha, h_\alpha)\}$  be a simply ordered subset of  $S^*$  and define  $(S_o, h_o)$  as follows:  $S_o = \bigcup_{\alpha} S_{\alpha}$ ; if  $s \in S_o$ , then  $s \in S_{\alpha}$  for some  $\alpha$ , thus defining  $h_o(s) = h_{\alpha}(s)$ . We claim that  $(S_o, h_o) \in S^*$  and it is an upper bound of  $\{(S_{\alpha}, h_{\alpha})\}$ .  $S_o = \bigcup_{\alpha} S_{\alpha}$ . Then  $S_o$  contains  $A$  and  $h_o$  extends  $f$  because the  $h_{\alpha}$ 's are extensions of  $f$ ; so,  $(S_o, h_o) \in S^*$ . Suppose now that  $(S_o, h_o)$  is not an upper bound of  $\{(S_{\alpha}, h_{\alpha})\}$ , then there is  $(S_1, h_1)$  such that  $S_o \subset S_1$  and  $h_1$  extends  $h_o$ . But this is impossible because  $S_o = \bigcup_{\alpha} S_{\alpha}$  and  $S_1 \subset \bigcup_{\alpha} S_{\alpha}$ . By Zorn's lemma, there exists a maximal pair,  $(M, h)$ . We shall prove that  $M = B$ .

Suppose there is an element  $b \in B$  that is not in  $M$ . Let  $M_1 = M + [b]$ . It is clear that  $M$  is a proper subgroup of  $M_1$ , so it suffices to extend  $h$  to  $M_1$  to reach a contradiction.

Case 1.  $M \cap [b] = 0$ . Then  $M_1 = M \oplus [b]$ . Define  $g : [b] \rightarrow D$

to be the zero map. There is a map  $F : M_1 \rightarrow D$  extending  $h$  and  $g$ .

In fact, any element  $a$  in  $M_1$  has a unique expression  $a = a_1 + b_1$ , where  $a_1 \in M$  and  $b_1 \in [b]$ . Define  $F(a) = h(a) + g(b_1) = h(a)$ .

Clearly,  $F$  is a homomorphism and  $F$  is an extension of  $h$ .

Case 2.  $M \cap [b] \neq 0$ . Let  $k$  be the smallest positive integer for which  $kb \in M$ ; then, every element  $y$  in  $M_1$  has the unique expression  $y = m + tb$ , where  $t > k$ . Let  $c = kb$ . Since  $c \in M$ ,  $h(c)$  is well defined and, by the divisibility of  $D$ , there is an element  $x \in D$  with  $kx = h(c)$ . Define  $F : M_1 \rightarrow D$  by  $F(m + tb) = h(m) + tx$ . It is clear that  $F$  is well defined and for any  $y_1 = m_1 + t_1b$ ,  $y_2 = m_2 + t_2b$  in  $M_1$

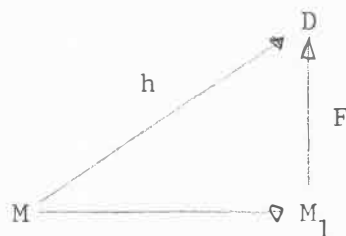
$$F(y_1 + y_2) = F[(m_1 + m_2) + (t_1 + t_2)b] =$$

$$h(m_1 + m_2) + (t_1 + t_2)x =$$

$$h(m_1) + t_1x + h(m_2) + t_2x =$$

$$F(y_1) + F(y_2).$$

Hence,  $F$  is a homomorphism and the following diagram commutes



This contradicts the fact that  $M$  is maximal. Therefore,  $M = B$ .

Conversely, assume now that the group  $G$  has the injective property.

Let  $x \in G$  and define  $f_x : n\mathbb{Z} \rightarrow G$  by  $f_x(np) = px$ . It is clear that  $f$

is a homomorphism since  $f_x(nq + np) = f_x n(q + p) = (q + p)x = qx + px = f_x(nq) + f_x(np)$ . Since  $G$  has the injective property and  $n\mathbb{Z}$  is a subgroup of  $Q$ , the following diagram commutes

$$\begin{array}{ccccc}
 & & & G & \\
 & & & \uparrow & \\
 & f_x & & \uparrow q_x & \\
 0 & \longrightarrow & n\mathbb{Z} & \longrightarrow & Q
 \end{array}$$

Therefore, for any  $x \in G$  there exists a homomorphism  $f_x$  and its extension  $g_x$ . Consider the set  $B$  of all homomorphisms  $g_x$ . Now, let  $y$  be any element in  $G$  and let  $m$  be any integer. Then there exists some homomorphism  $g_x \in B$  such that  $g_x(r) = y$ , where  $r$  is some rational different from zero. Since  $r$  is divisible by  $m$ , then  $mr' = r$  which implies  $mg_x(r') = y$ . Set  $g_x(r') = y' \in G$ , thus,  $my' = y$ . Therefore, any nonzero element in  $G$  is divisible by every integer. Consequently,  $G$  is divisible.

#### Corollary 6

Let  $D$  be a subgroup of  $G$  where  $D$  is divisible. Then  $D$  is a direct summand of  $G$ .

#### Proof

Consider the diagram

$$\begin{array}{ccc}
 & & D \\
 & \nearrow I & \uparrow \\
 0 & \longrightarrow & D \longrightarrow G
 \end{array}$$

where  $I$  is the identity map. By theorem 16, there is a homomorphism  $p : G \rightarrow D$  such that  $p(d) = d$  for every  $d \in D$ . By theorem 7,  $D$  is a direct summand of  $G$ .

### Theorem 17

A group  $G$  is divisible if and only if  $pG = G$  for every prime  $p$ .

### Proof

If  $G$  is divisible for any integer  $n$ ,  $nG = G$ ; in particular, for every prime  $p$ ,  $pG = G$ .

Conversely, suppose now that for every prime  $p$ ,  $pG = G$ . We have to prove that any  $x \in G$  is divisible for every integer  $n$ . By hypothesis, any element  $x \in G$  is divisible by every prime  $p$ .

Our first step will be to show that every  $x \in G$  is divisible by every power of any prime, i. e.,  $x$  is divisible by  $p^r$ .

By hypothesis,  $x$  is divisible by  $p$ . Thus, there is some  $y \in G$  such that  $py = x$ , but  $y$  is also divisible by  $p$ . Then there is some  $y_1 \in G$ ,  $py_1 = y$ . But  $y_1$  also is divisible by  $p$ . Then  $py_2 = y_1$  for some  $y_2 \in G$ . Repeating this process  $r$  times we will find  $py_{r-1} = y_{r-2}$  and, putting all this together,  $p^r y_{r-1} = x$ . Therefore, any  $x$  in  $G$  is divisible by any power of any integer.

Let  $n$  be any integer and  $x$  be some element of  $G$ . By the fundamental theorem of the arithmetic,  $n = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ , where the  $p_i$  are different primes and  $r_i$ , integers. There exists some  $z_1 \in G$  such that  $p_1^{r_1} z_1 = x$  and by step one, the following equations hold.

$$p_2^{r_2} z_2 = z_1$$

$$p_3^{r_3} z_3 = z_2$$

$$p_n^{r_n} z_n = z_{n-1}$$

Thus,  $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} z_n = x$ , or  $nz_n = x$ . Therefore,  $x$  is divisible by  $n$ .

### Theorem 18

A  $p$ -primary group  $G$  is divisible if and only if  $G = pG$ .

### Proof

If  $G$  is divisible, any element  $x \in G$  is divisible by every integer; in particular, by any prime  $p$ . Then  $pG = G$  for every prime  $p$ .

Conversely, suppose  $pG = G$ . We claim that  $G$  is divisible by every prime; then, by theorem 17, the theorem follows.

First of all, we prove that  $G$  is divisible by any power of  $p$ . Let  $x \in G$ , since  $pG = G$ , then there is  $y_1 \in G$  such that  $py_1 = x$ ; also,  $y_1$  is divisible by  $p$ , then for some  $y_2 \in G$ ,  $py_2 = y_1$ . If we repeat this process  $n$  times and put together all the equalities, we get  $p^n y_n = x$ . Therefore,  $x$  is divisible by any power of  $p$ . Let  $q$  be any prime and  $x \in G$  of order  $p^m$ . Since  $(p^m, q) = 1$ , there exists integer  $h$  and  $r$  such that  $hp^m + rq = 1$ . Hence,

$$x(hp^m + rq) = x$$

$$xp^mh + qrx = x$$

$$q(rx) = x$$

$$qy = x.$$

Therefore,  $x$  is divisible by any prime  $q$ .

### Definition

If  $G$  is an abelian group,  $dG$  is the subgroup of  $G$  generated by all divisible subgroups of  $G$ .

### Lemma 5

$dG$  is a divisible subgroup of  $G$ .

### Proof

Let  $n > 0$  and let  $x \in dG$ ; then,  $x = x_1 + x_2 + \dots + x_n$  where  $x_i$  is in a divisible subgroup  $D_i$  of  $G$ . Since  $D_i$  is divisible, there is an element  $y_i \in D_i$  with  $my_i = x_i$  for a given integer  $m$  and every  $i$ . Hence,  $y_1 + y_2 + \dots + y_n \in dG$  and  $x = my_1 + \dots + my_n = m(y_1 + \dots + y_n)$ .

### Definition

An abelian group  $G$  is reduced if  $dG = 0$ .

### Theorem 19

Every abelian group  $G = dG \oplus R$  where  $R$  is reduced.

### Proof

By corollary 6,  $dG$  is a direct summand of  $G$ . So,  $G = dG \oplus R$  for some subgroup  $R$ . If  $R$  contains a divisible group  $M$ , then  $dG \cap R$  is not empty, but  $dG \cap R = 0$  by hypothesis. Then,  $M = 0$  and  $R$  is reduced.

Theorem 20

The abelian groups  $G, H$  are isomorphic if and only if  $dG \approx dH$  and  $G/dH \approx H/dH$ .

Proof

Suppose  $G \approx H$ , then by theorem 19,  $G = dG \oplus R$  and  $H = dH \oplus R_2$ . Let  $f : G \rightarrow H$  be an isomorphism and consider the restriction of  $f$  to  $dG$ . Let  $x \in dG$  and  $f(x) = y$ . Now,  $ny_1 = x$  and  $f(ny_1) = nf(y_1) = y$ , so that  $y \in dH$ ,  $f(dG) \subset dH$ . Let  $y \in dH$  and  $n$  any integer, then  $y = y_1 n$ . Since  $f$  is one-to-one there exists  $x, x_1 \in G$  such that  $f(x) = y$  and  $f(x_1) = y_1$ . Then,  $f(nx_1) = nf(x_1) = ny_1 = y = f(x)$  and since  $f$  is one-to-one, so  $nx_1 = x$ . Therefore,  $f|_{dG}$  is onto and thus an isomorphism of  $dG$  onto  $dH$ . Since  $dG \approx dH$ ,  $G \approx H$ , then  $G/dG \approx H/dH$ .

Conversely, suppose  $dH \approx dG$  and  $G/dG \approx H/dH$ . By theorem 19,  $G = dH \oplus R_1$  and  $H = dH \oplus R_2$ , where  $R_1, R_2$  are reduced. But we know  $G/dG \approx R_1$ ,  $H/dH \approx R_2$ . Therefore  $G \approx H$ .

Lemma 6

Let  $G$  and  $H$  be divisible  $p$ -primary groups. Then,  $G \approx H$  if and only if  $G[p] \approx H[p]$ .

Proof

Let  $f$  be an isomorphism from  $G$  onto  $H$ . The image of  $G[p]$  by  $f$  is a subgroup of  $H$ . Let  $x \in G[p]$ , then  $px = 0$  and  $f(px) = 0$ ;  $pf(x) = 0$ , so  $f(x) \in H[p]$ . If  $y \in H[p]$ , then  $py = 0$  and  $y = f(x)$  for some  $x \in G$ ; so that  $pf(x) = 0$ ,  $f(px) = 0$  and since  $f$  is an isomorphism,  $px = 0$ . Therefore,  $G[p] \approx H[p]$ .

Now we will prove the sufficient conditions. Let  $f : G[p] \rightarrow H[p]$  an isomorphism. We may consider  $f$  as a mapping  $G[p] \rightarrow H$ . Then, by theorem 16,  $H$  and  $G$  have the injective property; that is, there exists a homomorphism  $F : G \rightarrow H$  extending  $f$ . We claim that  $F$  is an isomorphism.

Let  $x \in G$  with order  $p^n$ . We know  $p^{n-1}x \in G[p]$  and  $f(p^{n-1}x) = y \in H[p]$  but  $H$  is divisible; then there is  $y_1 \in H$  such that  $p^{n-1}y_1 = y$ . We define  $F(x) = y_1$ . Because of the uniqueness of  $y_1$ ,  $F$  is well defined. Let  $x_1, x_2$  be in  $G$  with order  $p^r, p^n$  respectively,  $x_1 \neq x_2$  and suppose  $F(x_1) = F(x_2)$ . This implies  $y_1 = y_2$ , where  $f(p^{r-1}x_1) = p^{r-1}y_1$  and  $f(p^{n-1}x_2) = p^{n-1}y_2$ . Suppose  $n > r$ ,  $p^n y_1 = 0$ ,  $p^r y_1 = 0$ , then  $p^{n-1}y_1 = p^{n-1}y_2 = 0$  but  $p^{n-1}x_2 \neq 0$ , so  $f(p^{n-1}x_2) \neq 0$ . Therefore  $F$  is one-to-one.

Let  $y \in H$  with order  $p^n$ .  $p^{n-1}y \in H[p]$  and for some  $x_1 \in G[p]$ ,  $f(x_1) = p^{n-1}y$ . Hence,  $G$  is divisible. There is  $x \in G$  with  $p^{n-1}x = x_1$ .  $f(x_1) = f(p^{n-1}x) = p^{n-1}y$ , then  $F(x) = y$ . Consequently,  $F$  is onto and an isomorphism.

### Theorem 21

Every divisible group  $D$  is the direct sum of copies of  $Q$  and of copies of  $\sigma(p^\infty)$  for various prime  $p$ .

### Proof

$D$  is divisible. Then any subgroup of  $D$  is also divisible; in particular  $tD$  so that,  $D \simeq tD \oplus D/tD$ . It was shown earlier in theorem 11 that  $D/tD$  is a torsion-free divisible group. Thus it is a direct sum of copies of  $Q$  by corollary 2.

$tD$  is the direct sum of  $p$ -primary groups by theorem 9. Let  $H$  be the  $p$ -primary component of  $tD$ ;  $H$  is divisible and  $H[p]$  is a vector space over  $\mathbb{Z}_p$  by lemma 3. Let  $r$  be the dimension of this vector space and  $G$  be the direct sum of  $r$  copies of  $\sigma(p^\infty)$ . Since the direct sum of  $p$ -primary divisible groups is  $p$ -primary divisible group, so  $G$  is  $p$ -primary divisible group. The dimension of  $\sigma(p^\infty)[p]$  is 1 and  $G[p] = \sum_{i=1}^r \sigma(p^\infty)[p]$ . Hence,  $G[p]$  has dimension  $r$ . Therefore,  $H[p] \cong G[p]$  because both are vector spaces over  $\mathbb{Z}_p$  and have the same dimension and by lemma 6,  $G \cong H$ . This proves the theorem.

#### Notation

Let  $D$  be a divisible group. Then  $D^\infty = D/tD$  and  $D_p = (tD)[p]$ .

#### Theorem 22

If  $D$  and  $D'$  are divisible groups, then  $D \cong D'$  if and only if

- (1)  $D^\infty \cong D'^\infty$ ; (2) for each  $p$ ,  $D_p \cong D'_p$ .

#### Proof

We know that  $D = tD \oplus D^\infty$  and  $D' = tD' \oplus D'^\infty$ . Suppose  $f : D \rightarrow D'$  is an isomorphism. Consider now the image of  $tD$  by  $f$ . If  $x \in tD$  and  $x$  has order  $n$ , then  $nx = 0$ ,  $f(nx) = nf(x) = 0$ , so  $f(x) \in tD'$ . Let  $y \in tD'$  with order  $m$ . There is  $x \in D$  such that  $f(x) = y$  and  $mf(x) = 0$ ,  $f(nx) = 0$ , then  $nx = 0$  and  $x \in tD$ . Since  $f$  is one-to-one and the restriction of  $f$  to  $tD$  is onto  $tD'$ , it implies  $tD \cong tD'$  and  $D^\infty \cong D'^\infty$ . By theorem 9  $tD = \sum tD_p$ ,  $tD' = \sum tD'_p$ . Since  $tD \cong tD'$  by theorem 10,  $tD_p \cong tD'_p$  and by lemma 6 this implies  $tD_p[p] \cong tD'_p[p]$  for each prime  $p$ . Therefore,  $D_p \cong D'_p$ .

Suppose now (1)  $D^\infty \approx D'^\infty$  (2)  $D_p \approx D'_p$  for each  $p$ . By lemma 6,  $D_p \approx D'_p$  implies that  $tD \approx tD'$ . Since  $D \approx tD \oplus D^\infty$ ,  $D' \approx tD' \oplus D'^\infty$  then  $D \approx D'$ .

The above theorem can be stated as follows: If  $D$  and  $D'$  are divisible groups, then  $D \approx D'$  if, and only if, (1)  $D^\infty$  and  $D'^\infty$  have the same dimension; (2)  $D_p$  and  $D'_p$  have the same dimension for each  $p$ . Note that  $D^\infty$  is a vector space over  $Q$  and  $D_p$  is a vector space over  $Z_p$ .

### Theorem 23

If  $G$  and  $H$  are torsion-free divisible groups, each of which is isomorphic to a subgroup of the other, then  $G \approx H$ .

#### Proof

By lemma 2,  $G$  and  $H$  are vector spaces over  $Q$ . Since  $G$  is isomorphic to a subgroup  $H_1$  of  $H$ , then the dimension of  $G$  is the same as the dimension of  $H_1$ . Also,  $H$  is isomorphic to a subgroup  $G_1$  of  $G$  so that the dimension of  $H$  is the same as the dimension of  $G_1$ . By Cantor-Schroder-Bernstein's theorem, the dimension of  $H$  is the same as the dimension of  $G$ , and by theorem 12,  $H \approx G$ .

### Theorem 26

Let  $G, H$  be torsion-free divisible groups and  $G \oplus G \approx H \oplus H$ , then  $G \approx H$ .

#### Proof

By theorem 12,  $G \oplus G$  and  $H \oplus H$  have the same dimension as a vector space over  $Q$ . We will consider two cases. (1) When the

dimension of  $G \oplus G$  is finite; (2) when the dimension of  $G \oplus G$  is infinite.

(1) If the dimension of  $G$  is  $n$ , then the dimension of  $G \oplus G$  is  $2n$  and also  $H \oplus H$  has dimension  $2n$ , so that,  $H$  has dimension  $n$ . Therefore,  $H \approx G$ .

(2) If  $G \oplus G$  has infinite dimension, then the dimension of  $G$  is the same as the dimension of  $G \oplus G$  since the cross product of two infinite sets of the same cardinal has the same cardinal as each set.

Therefore, the dimension of  $G \oplus G$  is equal to the dimension of  $G$  and to the dimension of  $H$ . By theorem 12,  $H \approx G$ .

#### Definition

$F$  is a free abelian group on  $\{x_k\}$  in case  $F$  is a direct sum of infinite cyclic groups  $Z_k$  where  $Z_k = [x_k]$ .

#### Theorem 27

If  $F$  is free on  $\{x_k\}$ , every nonzero element  $x \in F$  has the unique expression

$$x = m_{k_1} x_{k_1} + \dots + m_{k_n} x_{k_n}$$

where the  $m_{k_i}$  are nonzero integers and the  $k_i$  are distinct.

#### Proof

By theorem 5 any element  $x \in \sum Z_k$  has a unique expression

$$x = m_{k_1} x_{k_1} + \dots + m_{k_n} x_{k_n}$$

where the  $m_{k_i}$  are nonzero integers and the  $k_i$  are distinct. This proves the theorem.

### Theorem 28

Let  $F = \sum_{i \in I} Z_i$  and  $G = \sum_{j \in J} Z_j$  be free abelian groups. Then,

$F \approx G$  if, and only if,  $J$  and  $I$  have the same number of elements.

### Proof

Suppose  $F \approx G$  and  $F$  is free on  $\{x_i\}$ ,  $G$  is free on  $\{y_i\}$ . Let  $p$  be prime. Then  $F/pF$  and  $G/pG$  are vector spaces over  $Z_p$  by lemma

3. We claim that the  $\{x_i + pF\}$  is a basis for  $F/pF$ . Let

$\{x_{k_1} + pF, \dots, x_{k_n} + pF\}$  be any subset of  $\{x_i + pF\}$  and suppose

$$\bar{m}_1(x_{k_1} + pF) + \dots + \bar{m}_n(x_{k_n} + pF) = \bar{0}$$

where  $\bar{m}_i \in Z_p$ . Hence, we have

$$\bar{m}_1 x_{k_1} + \bar{m}_2 x_{k_2} + \dots + \bar{m}_n x_{k_n} + pF = \bar{0}$$

or

$$m_1 x_{k_1} + m_2 x_{k_2} + \dots + m_n x_{k_n} = 0.$$

But, by theorem 24, it implies  $m_1 = m_2 = \dots = m_n = 0$ , so that  $\{x_i + pF\}$  is a linearly independent set and also is maximal since there is no  $y + pF$  such that  $B = \{x_i + pF\} \cup \{y + pF\}$  is linearly independent because  $y = m_1 x_{k_1} + \dots + m_h x_{k_h}$  and  $y + pZ$  cannot

be linearly independent with  $\{x_i + pF\}$ . Therefore  $\{x_i + pF\}$  is a basis for  $F/pF$ . Proceeding as above, we get that  $\{y_i + pG\}$  is also a basis for  $G/pG$ . Since  $F \approx G$ ,  $pF$  is isomorphic to  $pG$ . Therefore,  $F/pF \approx G/pG$ . By the well known theorem, the cardinal of  $\{x_i + pF\}$  is the same as the cardinal of  $\{y_i + pG\}$  that implies  $I$  and  $J$  have the same number of elements.

Conversely, if  $I$  and  $J$  have the same number of elements,

$F = \sum_{i \in I} Z_i$  and  $G = \sum_{j \in J} Z_j$  have the same number of direct summands

with  $Z_j \approx Z_i$ , then,  $F \approx G$ .

### Definition

Let  $F$  be free on  $\{x_i : i \in I\}$ . The rank of  $F$  is the cardinal of  $I$ . If  $I$  is finite, we say that  $F$  has finite rank.

Theorem 25 states that the necessary and sufficient condition in order that two groups be isomorphic is that they have the same rank. As in vector spaces, if  $I$  is finite and has  $n$  elements, we say that  $F$  has rank  $n$ . Also, the above theorem gives the duality between the rank of a free abelian group and the dimension of a vector space. In order to stress this analogy, we make the following definition.

Definition. A basis of a free abelian group  $F$  is a free set of generators of  $F$ .

### Theorem 29

Let  $F$  be free with basis  $\{x_k\}$ ,  $G$  an arbitrary abelian group and  $f : \{x_k\} \rightarrow G$  any function. There is a unique homomorphism  $g : F \rightarrow G$  such that

$$g(x_k) = f(x_k)$$

for all  $k$ .

### Proof

Let  $Z_k = [x_k]$ . We define  $g : F \rightarrow G$  by  $g(x) = g(m_1 x_{k_1} + \dots + m_n x_{k_n}) = m_1 f(x_{k_1}) + \dots + m_n f(x_{k_n})$ .

The mapping  $g$  is well defined since any element  $x \in Z_k$  has a unique expression as a linear combination of the  $x_i$  and the function of  $f$  is single-valued. Let  $x = m_1 x_{k_1} + \dots + m_n x_{k_n}$  and  $y = n_1 x_{k_1} + \dots + n_1 x_{h_1}$ , then  $g(x + y) = g(m_1 x_{k_1} + \dots + m_n x_{k_n} + n_1 x_{k_1} + \dots + n_1 x_{h_1}) = m_1 f(x_{k_1}) + \dots + m_n f(x_{k_n}) + n_1 f(x_{k_1}) + \dots + n_1 f(x_{h_1}) = g(x) + g(y)$ . Then  $g$  is a homomorphism. Suppose

now that there is another homomorphism  $g'$  such that  $g'(x_i) = f(x_i)$ .

If  $x = n_1 x_{k_1} + \dots + n_r x_{k_r}$ ,  $g(x) = n_1 f(x_{k_1}) + \dots + n_r f(x_{k_r})$

and  $g'(x) = n_1 g'(x_{k_1}) + \dots + n_r g'(x_{k_r})$  but  $g'(x_{k_i}) = f(x_{k_i})$ .

Hence,  $g'$  coincides with the mapping that we have defined.

### Corollary 7

Every abelian group  $G$  is a quotient of a free abelian group.

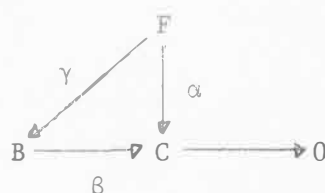
### Proof

We first state that if  $X$  is any set, then there exists a free abelian group  $F$  have  $X$  a basis. If  $X$  contains just one element,

$x$ , then an infinite cyclic group  $Zx$  can be constructed that has  $x$  as a generator. Let  $Zx = \{nx : n \in \mathbb{Z}\}$  and define the addition of two elements by  $nx + mx = (m + n)x$ . It is clear that this operation is well defined and associative. The element  $0x$  is the identity and  $-nx$  is the inverse of  $nx$ . Therefore,  $Zx$  is an infinite cyclic group. For the general case, set  $F = \sum_{x \in X} Zx$ . In order to prove the corollary, set  $F = \sum_{x \in G} Zx$ . By theorem 26, the identity mapping  $I : G \rightarrow G$ ,  $I(x) = x$  can be extended to a homomorphism  $g : F \rightarrow G$ . Since  $I$  is the identity,  $g$  is onto and by the fundamental theorem of homomorphism,  $F/K \cong G$ , where  $K$  is the kernel of  $g$ . Therefore,  $G$  is the quotient group of a free abelian group.

### Definition

Let  $\beta : B \rightarrow C$  be a homomorphism of  $B$  onto  $C$ . We say that  $F$  has the projective property in case that if  $\alpha : F \rightarrow C$  is a homomorphism, then there is a homomorphism  $\gamma : F \rightarrow B$  with  $\beta\gamma = \alpha$ , i. e., there is an  $\alpha$  making the following diagram commute.



### Theorem 30

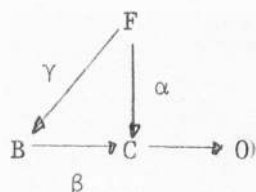
An abelian group  $F$  is free if, and only if, it has the projective property.

### Proof

Suppose  $F$  is free and on the above diagram is given  $\beta$  and  $\alpha$ .

Let  $\{x_k\}$  be a basis for  $F$ . For each  $k$  there is an element  $b_k \in B$  such that  $\beta(b_k) = \alpha(x_k)$  because  $\beta$  is onto. Define the function  $f(x_k) = b_k$  from  $\{x_k\}$  into  $B$ . By theorem 26, there is a unique homomorphism  $\gamma$  such that  $\gamma(x_k) = b_k$  for all  $x_k \in \{x_k\}$ . In order to finish the proof of the theorem, we have to show that  $\gamma\beta = \alpha$  and for this purpose it suffices to evaluate each on the set of generators of  $F$ . But  $\beta\gamma(x_k) = \beta(b_k) = \alpha(x_k)$  as required.

Conversely, suppose  $F$  has the projective property, i. e., the following diagram commutes.



Since every abelian group is the quotient of a free abelian group, (corollary 7), let  $B$  be a free abelian group such that  $B/B' \cong F$ .

Set  $C = F$  and  $\beta$  the natural homomorphism from  $B$  onto  $F$ ,  $\alpha = I$ .

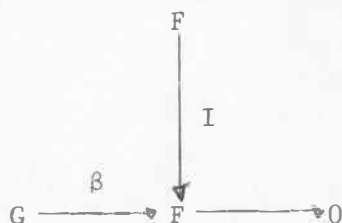
By hypothesis, the diagram commutes,  $\gamma\beta = I$ ; since  $\beta$  is onto and  $\gamma\beta$  is the identity mapping, then  $\beta$  is also one-to-one and consequently an isomorphism of  $B$  onto  $F$ . Therefore,  $F$  is free.

### Corollary 8

Let  $G$  be an abelian group and let  $\beta : G \rightarrow F$  be a homomorphism onto, where  $F$  is free. Then,  $G = B \oplus S$ , where  $S \cong F$  and  $B$  is the kernel of  $\beta$ .

### Proof

Consider the diagram



where  $I$  is the identity map. By hypothesis,  $F$  is free. Then  $F$  has the projective property. That is, there exists a homomorphism  $\gamma : F \rightarrow G$  with  $\beta\gamma = I$ . But  $\gamma$  is one-to-one because, if not,  $\beta\gamma$  cannot be one-to-one and  $\beta\gamma$  is the identity mapping. Then, the image  $S$  of  $F$  by  $\gamma$  is isomorphic to  $F$ . We claim that  $G = B \oplus S$ .

Let  $x \in B \cap S$ . Hence,  $\beta(x) = 0$  because  $x \in B$  and  $x = \gamma(y)$  where  $y \in F$ , so that  $\beta\gamma(y) = 0$  which implies  $y = 0$ . Therefore,  $x = 0$  and  $B \cap S = 0$ . Consider now  $[B \cup S]$ . It is obvious that  $[B \cup S] \subset G$ . Let  $x \in G$  and  $x \neq 0$ . Then either  $x \in S$  or not. If  $x \in S$ , then  $x \in [B \cup S]$ . If  $x \notin S$ , then  $\beta(x) = y$ . If  $y = 0$ , then  $x \in B$  and so  $x \in [B \cup S]$ . Suppose  $\beta(x) = y \neq 0$ . Since  $\beta\gamma = I$ , then  $\beta\gamma(y) = y$ . Let  $\gamma(y) = x'$ , so  $\beta(x') = y = \beta(x)$  or  $\beta(x - x') = 0$  which implies  $x - x' \in B$ . That is,  $x - x' = b$  where  $b \in B$ . Therefore,  $x = b + x'$ , but  $b + x' \in [B \cup S]$ . So,  $[B \cup S] = G$ . By theorem 6, we then have  $G = \text{kernel } \beta \oplus S$ .

### Theorem 31

Every subgroup  $H$  of a free abelian group  $F$  is free. Moreover,  $\text{rank } H \leq \text{rank } F$ .

### Proof

Let  $\{x_k : k \in K\}$  be a basis of  $F$ . Define  $F(I) = \sum_{k \in I} [x_k]$

where  $I$  is a subset of the index set  $K$ . Consider now the set  $S^*$  of all pairs  $(B, I)$  where  $I \subset K$  and  $H \cap F(I)$  is free with a basis  $B$  such that the cardinality of  $B$  is less or equal to the cardinal of  $I$ . Such pairs do exist, i. e.,  $(\phi, \phi)$ .

The relation defined on  $S^*$  by  $(B, I) \leq (B', I')$  where  $B \subset B'$  and  $I \subset I'$  is a partial order relation. Let  $(M, J)$  be such that  $M = \bigcup_i B_i$  and  $J = \bigcup_i I_i$ . It is trivial that  $(M, J)$  contains the  $(B_i, I_i)$ , but we must verify that  $(M, J) \in S^*$  in order that it be an upper bound. Since  $M$  is the union of ascending independent sets, then  $M$  is also independent. Also,  $J \subset K$  and since  $F(J) = \bigcup_i F(I_i)$ , then the cardinal of  $M$  is less or equal to the cardinal of  $J$ . Also,  $M$  is a basis for  $H \cap [\bigcup_i F(I_i)]$  since  $H \cap [\bigcup_i F(I_i)] = \bigcup_i [H \cap F(I_i)]$  and  $M$  contains the  $B_i$  that are the basis for the  $H \cap F(I_i)$ . Hence,  $\bigcup_i B_i = M$  is a basis for  $\bigcup_i [H \cap F(I_i)]$ . Therefore,  $(M, J) \in S^*$ . Then, Zorn's theorem can be applied so there exists a maximal pair  $(B_0, I_0)$ . We claim that  $I_0 = K$  which will complete the proof. Since  $F(K) = F$  and  $F \cap H = H$ ,  $B_0$  will be a basis for  $H$ .

Suppose  $I_0 \neq K$ , i. e., there is an index  $k \notin I_0$ ; set  $I_0^* = \{I_0, k\}$ . Then,  $F(I_0) \subset F(I_0^*)$  and

$$\begin{aligned} \frac{F(I_0^*) \cap H}{F(I_0) \cap H} &= \frac{F(I_0^*) \cap H}{F(I_0^*) \cap H \cap F(I_0)} \\ &\approx \frac{(F(I_0^*) \cap H) + F(I_0)}{F(I_0)} \subset \frac{F(I_0^*)}{F(I_0)} \end{aligned}$$

by the second isomorphic theorem. Since  $I_0^*$  and  $I_0$  are different by one element, then  $F(I_0^*)/F(I_0) \cong Z$  so that the original quotient is 0 or  $Z$  (every non-trivial subgroup of  $Z$  is cyclic and isomorphic to  $Z$ ). If the quotient is 0, then  $F(I_0^*) \cap H = F(I_0) \cap H$ . Therefore,  $(B_0, I_0^*) \in S^*$  and is larger than the maximal pair  $((B_0, I_0)$  which is a contradiction. Suppose now that the quotient is isomorphic to  $Z$ . Then, by corollary 8,  $F(I_0^*) \cap H = F(I_0) \cap H \oplus L$ , where  $L \cong Z$ . The pair  $(B_0^*, I_0^*) \in S^*$  and is larger than  $(B_0, I_0)$ , a contradiction. Therefore,  $I_0 = K$  as we claimed.

### Theorem 32

An abelian group  $G$  is finitely generated if, and only if, it is a quotient of a free abelian group of finite rank.

#### Proof

Let  $G = [a_1, a_2, \dots, a_n]$  and  $F = \sum_{a_i \in G} Z_{a_i}$  where  $Z_{a_i}$

denotes the infinite cyclic group generated by  $a_i$ . Consider the function  $f : F \rightarrow G$  defined by  $f(a_i) = a_i$ . By theorem 27 there exists a unique homomorphism  $g : F \rightarrow G$  that extends  $f$ . Since  $F$  has rank  $n$ ,  $G$  is the quotient group of a free abelian group of finite rank.

Suppose now  $G$  is the quotient group of a free abelian group  $F$  of finite rank. Then  $F/K \cong G$ , where  $K$  is a subgroup of  $F$ . By theorem 28,  $K$  is also free and  $\text{rank } K \leq \text{rank } F$ . Let  $y + K \in F/K$ . Since  $y \in F$ , then  $y = m_1 x_{k_1} + \dots + m_n x_{k_n}$ , where the  $x_{k_i}$  are in the basis  $\{x_i\}$  of  $F$ . Hence

$$y + K = m_1 x_{k_1} + \dots + m_n x_{k_n} + K =$$

$$(m_1 x_{k_1} + K) + (m_2 x_{k_2} + K) + \dots + (m_n x_{k_n} + K).$$

This means that  $y + K \in [x_1 + K, \dots, x_n + K]$ . Then,  $F/K \subset [x_1 + K, \dots, x_n + K]$ .

Consider any element of  $y \in [x_1 + K, \dots, x_n + K]$ . Then  $y = (m_{k_1} x_{k_1} + K) + (m_{k_2} x_{k_2} + K) + \dots + (m_{k_r} x_{k_r} + K) = m_{k_1} x_{k_1} + m_{k_2} x_{k_2} + \dots + m_{k_r} x_{k_r} + K$ . So that  $y \in F/K$ . Therefore,  $F/K$

is finitely generated.

#### Corollary 9

A direct summand  $B$  of a finitely generated abelian group  $G$  is also finitely generated.

#### Proof

Let  $B$  be a direct summand of  $G$ . Since  $G$  is finitely generated, there exists a free abelian group  $F$  of finite rank such that  $F/H \simeq G$ .  $B$  is a direct summand of  $G$ . Then there exists a direct summand of  $F/H$  isomorphic to  $B$ . By the correspondent theorem, there exists a subgroup  $M$  containing  $H$  such that  $M/H \simeq B$ . By theorem 28,  $M$  is free abelian. Then,  $B$  is finitely generated.

#### Corollary 10

Every subgroup  $H$  of a finitely generated abelian group  $G$  is itself finitely generated.

Proof

Let  $F/A \cong G$  where  $F$  is free abelian of finite rank and  $A$  is a subgroup of  $F$ . Since  $F/A \cong G$  and  $H$  is a subgroup of  $G$ , then there exists a subgroup  $H' \subset F/A$  such that  $H' \cong H$ . By the correspondent theorem there exists a subgroup  $F'$  of  $F$  containing  $A$  such that  $F'/A \cong H'$ . Therefore,  $H$  is isomorphic to  $F'/A$  which is the quotient of a free abelian group of finite rank and by theorem 29,  $H$  is finitely generated.

Theorem 33

Every abelian group  $G$  can be imbedded in a divisible group.

Proof

By corollary 7, there is a free abelian group  $F$  with  $G \cong F/R$  for some subgroup  $R$  of  $F$ . Let  $F = \sum_{i \in I} Z_i$  since the infinite cyclic group  $Z_i$  is isomorphic to the additive group of integers, it follows that  $F \subset \sum Q$  since each  $Z_i$  can be imbedded in a copy of  $Q$ . Therefore,  $G \cong F/R \subset (\sum Q)/R_1$  and this group is divisible being a quotient of a divisible group.

Corollary 11

An abelian group  $G$  is divisible if, and only if, it is a direct summand of every group containing it.

Proof

We know by corollary 6 that if  $G$  is a divisible subgroup of  $D$  then  $G$  is a direct summand. This proves the sufficiency.

In order to prove the necessity, we imbedded  $G$  in a divisible group  $D$  that is always possible by Theorem 30.  $G$  is a direct summand of  $D$  which is divisible; therefore,  $G$  is also divisible.

### Theorem 34

Every finitely generated subgroup  $A$  of  $Q$  is cyclic.

#### Proof

Suppose  $A$  has generators  $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$ . Let  $b = \prod_{i=1}^n b_i$

and consider the function  $f : A \rightarrow \mathbb{Z}$  defined by  $f(x) = bx$  for every  $x \in A$ . First of all, we must show that  $f$  is well defined. For any  $x \in A$ , the expression

$$x = m_1 \frac{a_1}{b_1} + \dots + m_n \frac{a_n}{b_n} = \frac{\sum_{i=1}^n m_i \prod_{\substack{j=1 \\ j \neq i}}^n b_j a_i}{\prod_{i=1}^n b_i}$$

is unique. Then,  $f(x) = \sum_{i=1}^n m_i \prod_{\substack{j=1 \\ j \neq i}}^n b_j a_i$  which is always well defined.

Now, if  $x \neq y$ , then  $f(x) = bx$  and  $f(y) = by$ , which are not equal unless  $x = y$ ;  $f$  is one-to-one. We will prove now that  $f$  is a homomorphism with kernel 0 which will complete the proof.

For any  $x, y \in A$ ,

$$f(x + y) = b(x + y) = bx + by = f(x) + f(y)$$

$f$  is a homomorphism.

If  $f(x) = 0$ , then  $bx = 0$  so that  $x = 0$ . Therefore, the kernel of  $f$  is  $0$ .

### Definition

Let  $G$  be a torsion-free group and  $x \in G$  define

$$\langle x \rangle = \{y \in G : my \in [x] \text{ for some } m \in \mathbb{Z}, m \neq 0\}.$$

### Lemma 7

The group  $\langle x \rangle$  is isomorphic to a subgroup of  $Q$ .

### Proof

If  $y \in \langle x \rangle$ , then  $my = nx$  where  $m$  and  $n$  are integers and  $m \neq 0$ .

Define the function  $f : \langle x \rangle \rightarrow Q$  by  $f(y) = \frac{n}{m}$ , where  $m, n$  are such that

$my = nx$ . Since the numbers  $m, n$  are uniquely determined, then the

function  $f$  is well defined. Let  $y, z \in \langle x \rangle$ , then there exists  $m_1,$

$m_2$ , and  $n_1, n_2$  such that  $m_1 y = n_1 x$  and  $m_2 z = n_2 x$ . Suppose  $z \neq y$ , thus,

if  $n_1 = n_2$ , it implies  $m_1 y = m_2 z$  and  $m_1 \neq m_2$ . Otherwise  $y = z$  which

contradicts our hypothesis. This proves that if  $x \neq y$ , then  $\frac{n_1}{m_1} \neq \frac{n_2}{m_2}$

which means that  $f(x) \neq f(y)$ . Therefore,  $f$  is one-to-one. Now, if

we prove that  $f$  is a homomorphism, the theorem will be proved. Let

$y_1, y_2 \in \langle x \rangle$  and  $m_1 y_1 = n_1 x, m_2 y_2 = n_2 x$ . Hence,  $m_1 m_2 y_1 = m_2 n_1 x,$

$m_1 m_2 y_2 = m_1 n_2 x$  and  $m_1 m_2 (y_1 + y_2) = (m_2 n_1 + m_1 n_2)x$ , so that  $f(y_1 + y_2) =$

$$\frac{m_2 n_1 + m_1 n_2}{m_1 m_2} = f(y_1) + f(y_2). \text{ Therefore, } f \text{ is a homomorphism as}$$

we claimed.

### Lemma 8

If  $G$  is torsion-free and  $x \in G$ , then  $G/\langle x \rangle$  is also torsion-free.

Proof

Suppose  $\bar{y} \in G/\langle x \rangle$  has finite order  $n$ , then  $n(y + \langle x \rangle) = \bar{0}$ . That means  $ny \in \langle x \rangle$ , but this cannot be the case because  $\bar{y} \neq 0$ . Therefore,  $G/\langle x \rangle$  is torsion-free.

Theorem 35 (Basis Theorem)

Every finitely generated abelian group  $G$  is the direct sum of cyclic groups.

Proof

Let  $G = [x_1, x_2, \dots, x_n]$ . We prove the theorem by induction on  $n$ . If  $n = 1$ , then  $G$  is cyclic and we are done. Suppose  $n > 1$ . We will consider two cases where, in the first case,  $G$  is torsion-free and the second case is general.

Case 1.  $G$  is torsion-free. By lemma 8,  $G/\langle x_n \rangle$  is torsion-free and it is generated by  $n - 1$  elements. We know by induction hypothesis that  $G/\langle x_n \rangle$  is free abelian and that there exists a homomorphism of  $G$  onto  $G/\langle x_n \rangle$ . Then, by corollary 8,  $G = \langle x_n \rangle \oplus F$ , where  $F$  is free abelian. Since  $\langle x_n \rangle$  is the direct summand of a finitely generated group, by corollary 9,  $\langle x_n \rangle$  is also finitely generated. By lemma 7,  $\langle x_n \rangle$  is isomorphic to a subgroup of  $Q$  and we proved in theorem 31 that a finitely generated subgroup of  $Q$  is cyclic. Therefore,  $\langle x_n \rangle$  is cyclic and  $G$  is free abelian which means a direct sum of cyclic groups.

Case 2. This is a general case. We already know that  $G/tG$  is torsion-free and since  $G$  is finitely generated, by theorem 29,  $G/tG$  is finitely generated. Therefore, by case 1,  $G/tG$  is free

abelian. Since  $G/tG$  is free abelian and  $G/tG$  is the homomorphic image of  $G$  by the natural homomorphism, then, by corollary 8,  $G = tG \oplus F$ , where  $F$  is free abelian. By corollary 9,  $tG$  is finitely generated and torsion; therefore, it is a finite group. By the basis theorem for finite groups,  $tG$  is a direct sum of cyclic groups and the proof is complete.

Now, we give the fundamental theorem of finitely generated abelian groups.

### Theorem 36

Every finitely generated abelian group  $G$  is the direct sum of primary and infinite cyclic groups and the number of summands of each kind depend only on  $G$ .

### Proof

We proved earlier that  $G \approx tG \oplus (G/tG)$ . The fundamental theorem for finite groups given us has the uniqueness of the decomposition of  $tG$  into direct sum of cyclic groups. By theorem 25,  $G/tG$  has a unique number of cyclic summands.

### Definition

A subgroup  $S$  of  $G$  is pure in  $G$  in case  $nG \cap S = nS$  for every integer  $n$ .

An alternative definition of a pure subgroup is the following. A subgroup  $S$  of  $G$  is pure if for any element  $h \in S$ ,  $h = ny$  for any integer  $n$  and  $y \in G$  which implies  $h = nh_1$  with  $h_1 \in S$ .

It is clear that both definitions are equivalent. Both

say that if an element of  $S$  is divisible by  $n$  in  $G$ , it is also divisible by  $n$  in  $S$ .

One of the simplest examples of a non-pure subgroup is the following. Let  $G$  be the additive group of integer module 4.  $G = \{0, 1, 2, 3\}$  and  $S = \{0, 2\}$ . Then, 2 is a multiple of 2 in  $G$  but not in  $S$ , so  $S$  is not pure.

The following are some of the most simple properties of the pure groups. We omit the proof of most of them.

1. Any direct summand of  $G$  is pure in  $G$ .

Proof. Let  $G = H \oplus S$ . We know  $nG = nH \oplus nS$ . Therefore,  $nG \cap S = nS$ .

2. If  $G/S$  is torsion-free, then  $S$  is pure in  $G$ .

Proof. Let  $y \in S$  and  $y = nx$ , where  $x \in G$ . Consider  $x + S$ . Then  $n(x + S) = nx + S = y + S = \bar{0}$ . If  $nx \in S$ ,  $x \in S$ , since  $G/S$  is a torsion group, so that  $nG = nS$ .

3. Since  $G/tG$  is torsion-free, then  $tG$  is pure. Furthermore, we gave an example in theorem 8 that  $tG$  is not a direct summand of  $G$ . Therefore, a pure subgroup need not be a direct summand.

4. If  $G$  is torsion-free, a subgroup  $S$  of  $G$  is pure if and only if  $G/S$  is torsion-free.

Proof. Sufficiency is property 3. Suppose now that  $S$  is pure in  $G$  and consider  $y + S \neq \bar{0}$ . Then, if  $n(y + S) = \bar{0}$ ,  $ny = ny_1$  where  $y_1 \in S$  so that  $n(y - y_1) = 0$ . But, by hypothesis,  $G$  is torsion-free. Therefore,  $n(y - y_1) = 0$  implies  $y - y_1 = 0$ .

5. Purity is transitive, i. e., if  $K$  is pure in  $H$  and  $H$  is pure in  $G$ , then  $K$  is pure in  $G$ .

6. Any intersection of pure subgroups of a torsion-free group  $G$  is pure.

7. A pure subgroup of a divisible group is divisible.

8. The ascending union of pure subgroups is pure.

9. Let  $S$  be pure in  $G$  and let  $\bar{y} \in G/S$ . Then  $y$  can be lifted to  $x \in G$ , where  $x$  and  $\bar{y}$  have the same order.

Proof. Let  $y = (y_1 + S) \in G/S$  and let  $n$  be the order of  $y$ . Then,  $ny \in S$  and there is some  $z \in S$  such that  $ny_1 = nz$ . Let  $x = z - y_1$ . Then  $nx = nz - ny_1 = 0$  as we desired. If  $y$  has infinite order, then the element  $y_1$  has the required property.

#### Lemma 9

Let  $T$  be pure in  $G$ . If  $T \subset S \subset G$  and  $S/T$  is pure in  $G/T$ , then  $S$  is pure in  $G$ .

#### Proof

Suppose  $ng = s$ , where  $s \in T$  and  $g \in G$ . Then  $n\bar{g} = \bar{s}$ , where  $\bar{s}$  denotes the coset of  $s$  in  $S/T$ . By the purity of  $S/T$ , there exists an element  $s' \in S/T$  with  $n\bar{s}' = \bar{s}$ . Rewriting this equation in  $G$  we get

$$ns' - s = t$$

for some  $t \in T$ . Hence  $ns' - ng = t$  but  $T$  is pure. Then there is  $t' \in T$  such that  $n(s' - g) = nt'$  or  $ns' - ng = nt'$ . Thus,  $s = n(s' - t')$  and  $s' - t' \in S$ . Therefore  $S$  is pure in  $G$ .

#### Lemma 10

A  $p$ -primary group  $G$  which is not divisible contains a pure cyclic subgroup.

Proof

Suppose there is an  $x \in G[p]$  which is divisible by  $p^k$  but not by  $p^{k+1}$ . Let  $p^k y = x$ . We claim that this element exists and  $[y]$  is pure in  $G$ .

By lemma 1, in a  $p$ -primary group every element is divisible by any integer prime to  $p$ . From this lemma we need to check only the divisibility by powers of  $p$  in order to prove that  $[y]$  is pure. Suppose  $y_1 = p^r y$  and  $y_1$  is divisible by  $p^h$  in  $G$ , i. e.,  $p^r y = p^h z$ ,  $z \in G$ . If  $h > r$ , we have  $x = p^{k-r} p^r y = p^{k-r} p^h z$ , or  $x = p^{k+1} (p^{h-r-1} z)$  contradicting the hypothesis that  $x$  is not divisible by  $p^{k+1}$ .

Now, we will prove that our assumption that there is an  $x \in G[p]$  which is divisible by  $p^k$  but not by  $p^{k+1}$  is true. Suppose that each  $x \in G[p]$  is divisible by every power of  $p$ . If this is the case, we will prove that  $pG = G$ , so that  $G$  is divisible by theorem 18 contradicting our hypothesis. Let  $y \in G$  and  $p^k$  equal its order. Then  $p^{k-1} y = x$  with  $x \in G[p]$ . Since  $x$  is divisible by every power of  $p$ , so  $x = p^k z_1$ . Then  $p^{k-1} y = p^k z_1$ ,  $p^{k-1} (y - pz_1) = 0$ . This implies  $p^{k-2} (y - pz_1) \in G[p]$ . Therefore  $p^{k-2} (y - pz_1)$  is divisible by  $p^{k-1}$ , i. e.,  $p^{k-2} (y - pz_1) = p^{k-1} z_2 = 0$  which implies that  $p^{k-3} (y - pz_1 - pz_2) \in G[p]$ . Repeating this process  $k$  times we get

$$y - pz_1 - pz_2 - \dots - pz_k = 0$$

$$y = p(z_1 + \dots + z_k).$$

That is, every  $y$  in  $G$  is in  $pG$ . Therefore,  $G = pG$ .

Definition

A subset  $X$  of nonzero elements of a group  $G$  is independent in case  $\sum_{\alpha} m_{\alpha} x_{\alpha} = 0$  which implies each  $m_{\alpha} x_{\alpha} = 0$ , where  $x_{\alpha} \in X$  and  $m_{\alpha} \in \mathbb{Z}$ .

Lemma 11

A set of nonzero elements of  $G$  is independent if, and only if,

$$[X] = \sum_{x \in X} [x].$$

Proof

Suppose  $X$  is independent. Let  $x_0 \in X$  and let  $y \in [x_0]$

$[X - \{x_0\}]$ . Then  $y = mx_0$  and  $y = \sum_{\alpha} m_{\alpha} x_{\alpha}$ , where each  $x_{\alpha} \neq x_0$ .

Therefore,

$$-mx_0 + \sum_{\alpha} m_{\alpha} x_{\alpha} = 0$$

so that, by the independence of  $X$ , each term is 0. Hence,

$$0 = mx_0 = y. \text{ By theorem 6, } [X] = \sum_{x \in X} [x].$$

Conversely, suppose  $[X] = \sum_{x \in X} [x]$ . By theorem 5, every  $x \in [X]$  has a unique expression

$$x = m_1 x_1 + \dots + m_n x_n.$$

Then, if  $0 = \sum_{\alpha} m_{\alpha} x_{\alpha}$ , each term is 0. Otherwise, we have distinct representation for 0.

Definition

A subset  $X$  of  $G$  is pure independent if  $X$  is independent and  $[X]$  is a pure subgroup of  $G$ .

Lemma 12

Let  $G$  be a  $p$ -primary group. If  $X$  is maximal pure independent, (i. e.,  $X$  is contained in no larger pure independent), then  $G/[X]$  is divisible.

Proof

Suppose  $G/[X]$  is not divisible. Then, by lemma 10, it contains a pure cyclic subgroup  $[\bar{y}]$ . By property 9 of pure subgroups,  $\bar{y}$  may be lifted to an element  $y \in G$ , where  $y$  and  $\bar{y}$  have the same order. We claim that  $X^* = X \cup \{y\}$  is pure independent, which will contradict the maximality of  $X$ . First of all

$$[X] \subset [X^*] \subset G$$

and  $[X^*]/[X] = [\bar{y}]$  which is pure in  $G/[X]$ . Therefore, by lemma 9,  $[X^*]$  is pure in  $G$ . Secondly,  $X^*$  is independent. Suppose,  $my + \sum_{\alpha} m_{\alpha} x_{\alpha} = 0$ ,  $x_{\alpha} \in X$ ,  $m_{\alpha} \in \mathbb{Z}$ . In  $G/[X]$ , this equation becomes  $m\bar{y} = 0$  which means that the order of  $\bar{y}$  is  $m$  and since  $y$  and  $\bar{y}$  have the same order, then  $my = 0$ . Hence  $\sum_{\alpha} m_{\alpha} x_{\alpha} = 0$  and by the independence of  $X$ , each  $m_{\alpha} x_{\alpha} = 0$ . Therefore,  $X^*$  is independent and so it is pure independent.

Definition

Let  $G$  be a torsion group. A subgroup  $B$  of  $G$  is a basic subgroup of  $G$  in the following cases.

1.  $B$  is a direct sum of cyclic groups.
2.  $B$  is pure in  $G$ .
3.  $G/B$  is divisible.

Theorem 37

Every torsion group  $G$  contains a basic subgroup.

Proof

By theorem 9 every torsion group has a decomposition as a direct sum of  $p$ -primary group. Then, if we show that every  $p$ -primary group has a basic subgroup, the theorem follows.

Assume, therefore, that  $G$  is  $p$ -primary.

If  $G$  is divisible, then  $G$  is isomorphic to  $\sigma(p^\infty)$ . Then  $B = 0$  is a basic subgroup. If  $G$  is not divisible, then  $G$  does contain pure independent subsets by lemma 10.

Let  $Y$  be the set of all pure independent subsets of  $G$ . Partially order  $Y$  by ordinary inclusion. Let  $\{Y_\alpha\}$  be a simply ordered subset of  $Y$ , i. e., the  $Y_\alpha$  are pure independent subsets of  $G$  and given any two of them, one contains the other. Let  $Y_1$  be the union of these  $Y_\alpha$ . But by property 8, the ascending union of pure subgroups is pure, so  $Y_1$  is pure. Consider now  $\sum_{\alpha} m_{\alpha} x_{\alpha} = 0$  where  $m_{\alpha} \in \mathbb{Z}$  and  $y_{\alpha} \in Y_{\alpha}$ . Since the  $Y_{\alpha}$  are partially ordered by inclusion, then  $y_{\alpha} \in Y_1$  and there exists some  $\alpha'$  such that  $y_{\alpha} \subset Y_{\alpha'}$ , for every  $\alpha$ . Hence  $\sum_{\alpha} m_{\alpha} x_{\alpha} = 0$  implies  $m_{\alpha} x_{\alpha} = 0$  for each  $\alpha$ . Therefore,  $Y_1$  is independent. The set  $Y$  satisfies the hypothesis of Zorn's lemma. Therefore, there is a maximal pure independent subset  $X$  of  $G$ . The previous two lemmas show that  $B = [X]$  is a basic subgroup.

Corollary 12

Every torsion group is an extension of a direct sum of cyclic groups by a divisible group.

Proof

The theorem follows from the previous theorem and the definitions of extension and basic groups.

Corollary 13 (Prüfer)

Let  $G$  be a subgroup of bounded order, i. e.,  $nG = 0$  for some integer  $n > 0$ . Then  $G$  is a direct sum of cyclic groups.

Proof

Since  $G$  is of bounded order,  $G$  is torsion and by theorem 34,  $G$  contains a basis  $B$ . Let  $y + B \in G/B$ . Since  $G/B$  is divisible,  $y + B$  is divisible by  $n$ . So  $y + B = n(x + B) = \bar{0}$ . Therefore,  $G/B = 0$  and, consequently,  $G = B$ .

Theorem 38

Let  $G$  be an abelian group and  $H$  a pure subgroup such that  $G/H$  is a direct sum of cyclic groups. Then,  $H$  is a direct summand of  $G$ .

Proof

For each cyclic summand of  $G/H$ , pick a generator  $y_i$ , i. e.,  $G/H = \sum \langle y_i \rangle$ . By property 9 of pure subgroup, we can select elements  $x_i \in G$  such that the  $x_i$  and  $y_i$  have the same order. Let  $K = \sum \langle x_i \rangle$ . We claim that  $G = H \oplus K$ .

If we prove that  $[H + K] = G$  and  $H \cap K = 0$ , then by theorem 6, the theorem follows.

1.  $[H + K] = G$ . Let  $t$  be any element in  $G$  and  $\bar{t}$  equal the image of  $t$  by the natural homomorphism  $\eta$  from  $G$  into  $G/H$ .

Hence  $\bar{t} = \sum m_i y_i$ , where  $m_i \in \mathbb{Z}$ . Now,  $\eta(t - \sum m_i x_i) = \eta(t) - \eta(\sum m_i x_i) = \eta(t) - \sum m_i \eta(x_i) = \bar{t} - \sum m_i y_i = \bar{0}$ , then  $t - \sum m_i x_i \in H$ .

Since  $\sum m_i x_i \in K$ , we have  $t \in [H + K]$ .

2.  $H \cap K = 0$ . Let  $w \in H \cap K$ , then  $w \in K$ , so that  $w = \sum m_i x_i$  and  $\eta(w) = \sum m_i y_i = \bar{0}$  because  $w$  is also in  $H$ . If  $y_i$  has infinite order then  $m_i = 0$ . If  $y_i$  has finite order  $n_i$ , then  $m_i$  must be a multiple of  $n_i$ . In either case, since  $x_i$  and  $y_i$  have the same order,  $a_i x_i = 0$  for every  $i$  so that  $w = 0$ .

### Theorem 39

Let  $S$  be a pure subgroup of  $G$  with  $nS = 0$  for some  $n > 0$ .

Then,  $S$  is a direct summand of  $G$ .

### Proof

Let  $f : G \rightarrow G/(S + nG)$  be the natural map. It is obvious that this quotient is of bounded order since  $n(G/(S + nG)) = \bar{0}$ . Also,  $G/(S + nG)$  is the direct sum of cyclic groups by corollary 13. Let  $G/(S + nG) = \sum \sigma(r_\alpha)$  where  $\bar{x}_\alpha$  is a generator of  $\sigma(r_\alpha)$ . For each,  $\bar{x}_\alpha$  is raised to  $x_\alpha \in G$ . Then  $r_\alpha x_\alpha \in S + nG$  since the order of  $x_\alpha$  mod.  $(S + nG)$  is  $r_\alpha$  and so

$$r_\alpha x_\alpha = s_\alpha + nh_\alpha$$

where  $s_\alpha \in S$  and  $h_\alpha \in G$  with  $r_\alpha$  dividing  $n$ . Thus, we have

$$s_\alpha = r_\alpha \left( x_\alpha - \frac{h}{r_\alpha} h_\alpha \right).$$

Since  $S$  is pure, there is  $s'_\alpha \in S$  with  $s_\alpha = r_\alpha s'_\alpha$ . Therefore,

$$r_\alpha y_\alpha = nh_\alpha \text{ and } f(y_\alpha) = \bar{x}_\alpha.$$

Let  $K = [nG \cup \{y_\alpha\}]$ . We claim that  $G = S \oplus K$ . We must prove that  $S \cap K = 0$  and  $S + K = G$ .

1.  $S \cap K = 0$ . Let  $x \in S \cap K$ . Since  $x \in K$ ,  $x = \sum_\alpha y_\alpha + nh$ . Also  $x \in S$ , then  $f(x) = \bar{0}$  so that  $\bar{0} = \sum_\alpha \bar{x}_\alpha$ . Hence,  $r_\alpha$  divides  $m_\alpha$  for each  $\alpha$ . But we know  $r_\alpha y_\alpha \in nG$  so that  $m_\alpha y_\alpha \in nG$ . Therefore,  $x = \sum_\alpha y_\alpha + ny \in nG$ . But  $S \cap nG = 0$  since for any element  $y$  in  $G$  that is also in  $S$ , we have  $ny = 0$ . Consequently,  $x = 0$ .

2.  $S + K = G$ . Let  $x \in G$ . Then  $f(x) = \sum_\alpha \bar{x}_\alpha = f(\sum_\alpha y_\alpha)$  so  $f(x - \sum_\alpha y_\alpha) = \bar{0}$ , or  $x - \sum_\alpha y_\alpha = s + nh \in S + nG$ . Therefore,  $x = s + (nh + \sum_\alpha y_\alpha) \in S + K$ .

#### Corollary 14

If  $tG$  is of bounded order, then  $tG$  is a direct summand of  $G$ . In particular,  $tG$  is a direct summand if  $tG$  is finite.

#### Proof

By property 3,  $tG$  is pure in  $G$ . Then, if  $tG$  is of bounded order, by the above theorem,  $tG$  is a direct summand of  $G$ . And, of course, if  $tG$  is finite,  $tG$  is of bounded order.

#### Definition

A group  $G$  is indecomposable if  $G \neq 0$  and if  $G \approx H \oplus K$ .

Then, either  $H$  or  $K$  is 0.

#### Corollary 15

An indecomposable abelian group  $G$  is either torsion or torsion-free.

Proof

Suppose that  $G$  is an indecomposable group that is neither torsion nor torsion-free. Otherwise, we do not have anything to prove. Hence,  $tG$  is a proper subgroup of  $G$ . If  $tG$  is divisible, then, by corollary 6,  $tG$  is a direct summand of  $G$  which contradicts our hypothesis so that  $tG$  is not divisible. By theorem 9,  $tG$  is the direct sum of  $p$ -primary groups so, by lemma 10,  $tG$  contains a cyclic and pure group  $\sigma(p)$ . It follows from theorem 36, that  $\sigma(p)$  is a direct summand of  $G$ , a contradiction.

Theorem 40

A torsion group  $G$  is indecomposable if and only if  $G$  is primary and cyclic or  $G \simeq \sigma(p^\infty)$  for some prime  $p$ .

Proof

The sufficiency condition is obvious. Suppose  $G$  is torsion and indecomposable. By theorem 9,  $G$  is the direct sum of  $p$ -primary groups so that  $G$  is  $p$ -primary for some prime  $p$ . If  $G$  is of bounded order, then, by corollary 10,  $G$  is the direct sum of cyclic groups and  $G$  is indecomposable so that  $G$  is cyclic.

Suppose now that  $G$  is not of bounded order. If  $G$  is not divisible, it follows from lemma 10 that  $G$  has a pure cyclic subgroup  $\sigma(p)$  and by theorem 36  $\sigma(p)$  is a direct summand of  $G$ , a contradiction. Therefore,  $G$  is divisible and by theorem 21, it is a direct sum of copies of  $Q$  and  $\sigma(p^\infty)$  for distinct  $p$ . Since  $G$  is torsion, we cannot have the case that  $G$  has as a direct summand copies of  $Q$ , and, because  $G$  is indecomposable,  $G \simeq \sigma(p^\infty)$ .

Theorem 41

Let  $G$  be an infinite abelian group with every proper subgroup finite; then  $G \approx \sigma(p^\infty)$  for some  $p$ .

Proof

Since every proper subgroup of  $G$  is finite, then  $G$  is torsion. By theorem 9,  $G$  is the direct sum of  $p$ -primary group. Suppose that  $G$  has infinite summand, then there exists a subgroup of  $G$  that is not finite. Since the finite direct sum of finite summand is finite,  $G$  cannot be decomposed as a direct sum of proper subgroups. Therefore,  $G$  is indecomposable and, by theorem 37, it follows that  $G \approx \sigma(p^\infty)$  because  $G$  cannot be cyclic.

Theorem 42

If an infinite abelian group  $G$  is isomorphic to every proper subgroup, then  $G \approx \mathbb{Z}$ .

Proof

Let  $x \in G$  and  $x \neq 0$ . Consider the cyclic group generated by  $x$ ,  $[x] = D$ . By hypothesis,  $G \approx D$  and since  $D$  is infinite cyclic group, it follows that  $G \approx \mathbb{Z}$ .

Now we will study a restricted class of torsion-free groups--those of rank 1.

Definition

The rank of a torsion-free group  $G$  is the number of elements in a maximal independent subset  $G$ .

Since a free abelian group is torsion-free, then our two notions of rank coincide for these groups. Below, we give some theorems and notation for this type of group.

### Theorem 43

Every torsion-free group  $G$  can be imbedded in a vector space  $V$  over  $Q$ .

### Proof

By theorem 30 the group  $G$  can be imbedded into a division group  $G$ . Consider the natural map  $f : D \rightarrow D/tD$ . Since  $G$  is torsion-free, any element in  $G$  is not in  $tD$ , so if  $x \in G$ ,  $f(x) \neq 0$  and, consequently,  $f(G) \subset D/tD$ . By theorem 11,  $D/tD$  is torsion-free divisible group and by lemma 2,  $D/tD$  is a vector space over  $Q$ . Therefore,  $G$  is imbedded in the vector space  $D/tD$ .

### Theorem 44

A torsion-free group  $G$  has rank at most  $r$  if, and only if,  $G$  can be imbedded in an  $r$ -dimensional vector space over  $Q$ .

### Proof

If the rank of the torsion-free group  $G$  is less or equal to  $r$ , then, by the theorem above,  $D/tD$  is a vector space over  $Q$  containing  $G$ . Suppose now that  $D/tD$  has dimension  $q$  less than  $r$ . Let  $\{x_1, \dots, x_r\}$  be a maximal linearly independent set in  $G$ . Hence,  $\{\bar{x}_1, \dots, \bar{x}_r\}$  is linearly independent set in  $D/tD$ ,

where  $f(x_1) = \bar{x}_1$ . But this is a contradiction of our hypothesis since the dimension of  $D/tD$  is  $q$ . Therefore, dimension of  $D/tD \geq r$ .

Conversely, if  $G$  can be imbedded in a vector space  $V$  over  $Q$  of dimension at most  $r$ , then any subspace of  $V$  has dimension at most  $r$ . Let  $\bar{G}$  be the subspace of  $V$  such that  $G \approx \bar{G}$  and let  $\{\bar{x}_1, \dots, \bar{x}_s\}$  be a basis of  $\bar{G}$ . Then the corresponding set in  $G$ ,  $\{x_1, \dots, x_s\}$ , is maximal linearly independent. Hence, the rank of  $G$  is  $s$  with  $s \leq r$ . Because of this theorem, the rank of a torsion-free group is well defined by the above definition. Thus, any two maximal linearly independent sets of  $G$  will be a basis of the vector space  $\bar{G}$  over  $Q$ .

#### Theorem 45

Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of torsion-free groups. Then,  $\text{rank } A + \text{rank } C = \text{rank } B$ .

#### Proof

We know by theorem 1 that  $B$  is an extension of  $A$  by  $C$ . Let  $\{x_1, \dots, x_r\}$  be a maximal independent set in  $A$  so the rank of  $A$  is  $r$ . Since the groups  $A$  and  $B$  are torsion-free, we can identify the rank of groups with the dimension of the subspace over  $Q$  in which they are imbedded. Hence, we can extend the set  $\{x_1, \dots, x_r\}$  to a maximal independent set in  $B$ ,

$\{x_1, \dots, x_r, x_{r+1}, \dots, x_n\}$ , since  $A \subset B$ . Also,  $C \approx B/A$  and  $\text{rank } C = \text{rank } B/A$ . But  $A$  and  $B$  are torsion-free. Then,  $\text{rank } B/A = n - r$ . Therefore,  $\text{rank } A + \text{rank } C = \text{rank } B$ .

#### Corollary 16

Any torsion-free group of rank 1 is indecomposable.

#### Proof

Suppose  $G = G_1 \oplus G_2$ . But

$$0 \longrightarrow G_1 \xrightarrow{f} G \xrightarrow{g} G_2 \longrightarrow 0$$

is an exact sequence if we define  $f$  and  $g$  by the identity and projective mappings, respectively. By the above theorem,  $\text{rank } G = \text{rank } G_1 + \text{rank } G_2$ , but the rank of  $G_1$  and  $G_2$  is at least 1 for each one. Therefore,  $G$  is indecomposable.

#### Corollary 17

Any torsion-free group  $G$  of rank 1 is an isomorphic subgroup of  $Q$ .

#### Proof

By theorem 43,  $G$  can be imbedded in 1-dimensional vector space  $V$  over  $Q$ . Since  $V \approx Q$  and  $G$  is imbedded in  $V$ ,  $G$  is isomorphic to some subgroup of  $Q$ .

The following subgroups of  $Q$  are non-isomorphic.

$G_1$  : All rationals whose denominator is square-free.

$G_2$  : All dyadic rationals, i. e., all rationals of the form  $\frac{m}{2^k}$ .

$G_3$  : All rationals whose decimal expansion is finite.

Let  $p_1, p_2, p_3, \dots, p_n, \dots$  be the sequence of primes.

### Definition

A characteristic is a sequence

$$(K_1, K_2, \dots, K_n, \dots)$$

where each  $K_n$  is a non-negative integer or the symbol  $\infty$ .

If  $G$  is a subgroup of  $Q$  and  $x \in G$  is nonzero, then  $x$  determines a characteristic in the following way. We put  $K_n = 0$  if  $py = x$  has no solution in  $G$ ,  $K_n = K$  if  $p_n^k y = x$  has solution but  $p_n^{k+1} y = x$  has no solution.  $K_n = \infty$  if all the equations  $p_n^i y = x$  have solutions for every  $i$ .

It is useful to write each nonzero integer  $m$  as a formal infinite product,  $m = \prod p_i^{\alpha_i}$ , where the  $p_i$  range over all the primes and  $\alpha_i \geq 0$ . If the element  $a$  is replaced by  $ma$ , where  $m$  is a nonzero integer, then there is no change in  $K_n$  if it is  $\infty$ , but it is finite and equal to  $K \geq 0$  and  $m = c_n^c m'$  with  $(p_n, m') = 1$ . Then, after the change, it will be  $K_n = K + c$ .

Let  $m = \prod p_i^{\alpha_i}$  and  $n = \prod p_i^{\beta_i}$  be given integers. If  $a \in G$  has the characteristic  $(K_1, K_2, K_3, \dots)$ , then by the definition of characteristic, there is an  $x \in G$  such that  $mx = na$  and only if  $\alpha_i \leq K_i + \beta_i$  for every  $i$  (we use by convention  $\infty + \beta_i = \infty$ ).

The groups  $Z, Q, G_1, G_2, G_3$  (the last three defined as above) are of rank 1 and all contain  $x = 1$ .

The characteristic of  $x = 1$  in each group is

$$Z : (0, 0, 0, \dots)$$

$$Q : (\infty, \infty, \infty, \dots)$$

$$G_1 : (1, 1, 1, \dots)$$

$$G_2 : (\infty, 0, 0, \dots)$$

$$G_3 : (\infty, 0, \infty, \dots).$$

Distinct nonzero elements of the same group may have distinct characteristics. For example, in  $Z$  the characteristic of 6 is

$$(0, 1, 1, 0, 0, \dots),$$

while the characteristic of 1 is

$$(0, 0, 0, 0, \dots).$$

#### Definition

Two characteristics are equivalent if (1) they have  $\infty$  in the same coordinates and (2) they differ in, at most, a finite number of coordinates.

It is obvious that this is an equivalence relation. An equivalence class of characteristics is called type.

#### Lemma 13

Let  $G$  be a subgroup of  $Q$ , and let  $x$  and  $x'$  be nonzero elements of  $G$ . Then, the characteristics of  $x$  and  $x'$  are equivalent.

#### Proof

Suppose first that  $x' = mx$  for some integer  $m$ . Then, the characteristics of  $x$  and  $x'$  are equivalent because the characteristic of  $x$  differs from the characteristics of  $mx$  in a finite

number of coordinates as we remarked above. Now, since  $G$  is a subgroup of  $Q$ , there are integers  $m$  and  $n$  such that

$$mx = nx'.$$

The characteristic of  $x$  is equivalent to that  $mx$  and this one to  $nx'$  which is equivalent to that of  $x'$ .

As a result of this lemma, if  $G$  is a torsion-free group of rank 1 (a subgroup of  $Q$ ), we may define the type of  $G$ ,  $\Gamma(G)$ , as the type of any nonzero element of  $G$ .

#### Theorem 46

Let  $G$  and  $G'$  be a torsion-free group of rank 1. Then,  $G \approx G'$  if and only if  $\Gamma(G) = \Gamma(G')$ .

#### Proof

Suppose  $f : G \rightarrow G'$  is an isomorphism. If  $x \in G$  is nonzero, then if  $p_1^n y = x$ ,  $f(p_1^n y) = p_1^n f(y) = f(x)$ ; that is,  $x$  and  $f(x)$  are divisible by the same powers of  $p_1$  for every  $i$ . Hence,  $x$  and  $f(x)$  have equivalent characteristics. Therefore,  $\Gamma(G) = \Gamma(G')$ .

Assume that  $\Gamma(G) = \Gamma(G')$  and that  $G$  and  $G'$  are subgroups of  $Q$ . If  $a$  and  $a'$  are two elements in  $G$  and  $G'$ , respectively, then their characteristics  $(K_1, K_2, K_3, \dots)$  and  $(K_1', K_2', K_3', \dots)$  differ in only a finite number of places. If we agree that the notation  $\infty - \infty$  means 0, then we may define a ration number  $\lambda$  by

$$\lambda = \prod_i p_i^{K_i - K_i'}.$$

It follows from the definition of equivalence and our convention concerning  $\alpha$  that almost all the  $K_i - K'_i = 0$ .

Define  $f : G \rightarrow Q$  by  $f(x) = ux$ , where  $u = \lambda \frac{a'}{a}$ . Since, by distributivity,  $f(x + y) = u(x + y) = ux + uy = f(x) + f(y)$ ; thus,  $f$  is a homomorphism. Now, a rational number  $x$  is in  $G$  if and only if there are integers  $m = \prod p_i^{\alpha_i}$  and  $n = \prod p_i^{\beta_i}$  with  $mx = na$  and  $\alpha_i \leq \beta_i + K_i$  for all  $i$ . A rational  $y$  is in  $G'$  if, and only if, there are integers  $m$  and  $n$  with  $my = na'$  and  $\alpha_i \leq \beta_i + K'_i$  for all  $i$ . We claim that  $f(G) \subset G'$ . If  $x \in G$ , then  $mx = na$  and  $\alpha_i \leq \beta_i + K_i$ . Hence  $m(ux) = nua = (n\lambda)a'$ . Since  $\alpha_i \leq (\beta_i + K_i - K'_i) + K_i$ , then  $ux = f(x) \in G'$ . Now, in a similar manner, define  $g : G' \rightarrow Q$  by  $g(x') = u^{-1}x'$ . It is obvious that  $g$  is a homomorphism. Let  $y' \in G'$ , then  $my' = na'$  with  $\alpha_i \leq \beta_i + K'_i$ . Hence  $mu^{-1}y' = nu^{-1}a'$  but  $u^{-1} = \lambda^{-1} \frac{a}{a'}$ , so  $m(u^{-1}y') = (n\lambda^{-1})a$ . Since  $\alpha_i \leq (\beta_i + K'_i - K_i) + K_i$ , it follows that  $u^{-1}y' = g(y') \in G$ . Therefore,  $g(G') \subset G$  and  $f$  and  $g$  are inverse so that  $G \approx G'$ .

#### Theorem 47

If  $\Gamma$  is a type, then there exists a group of  $G$  of rank 1 with  $\Gamma(G) = \Gamma$ .

#### Proof

Let  $(K_1, K_2, K_3, \dots)$  be a characteristic of  $\Gamma$ . We define the group  $G$  as the subgroup of  $Q$  generated by all rationals of the form  $\frac{1}{m}$  where for all  $n$ ,  $p_n^t$  divides  $m$  if and only if  $t \leq K_n$ . We must prove that the rank of  $G$  is one. Let

$\frac{1}{m_1}$  and  $\frac{1}{m_2}$  be elements of  $G$ . We will prove that they are not independent. Suppose that there exists integers  $h_1$  and  $h_2$  such that

$$h_1 \frac{1}{m_1} + h_2 \frac{1}{m_2} = 0.$$

If  $(h_1, h_2) \neq 1$ , we can simplify the above equation. Thus, suppose that  $(h_1, h_2) = 1$ . Hence

$$h_1 m_2 + h_2 m_1 = 0$$

$$h_1 m_2 = -h_2 m_1$$

which implies that  $m_1$  and  $m_2$  have equivalent characteristics and the elements  $\frac{1}{m_1}$  and  $\frac{1}{m_2}$  are dependent. Therefore, the rank of  $G$  is 1.

Also, we must prove that the element 1 has the given characteristic which is equivalent to proving that the equation

$$p_n^r x = 1$$

always has a solution in  $G$  for every  $n$  if and only if  $r \leq K_n$ .

Since  $x$  belongs to  $G$ , then  $x = \frac{h}{m}$ , where  $m$  is divisible by  $p_n^t$

for all  $n$  if, and only if,  $t \leq K_n$ . Consequently, the above equation

always has solutions and 1 has the given characteristic.

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