

## **Infinite Clusters in Percolation Models**

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The qualitative nature of infinite clusters in percolation models is investigated. The results, which apply to both independent and correlated percolation in any dimension, concern the number and density of infinite clusters, the size of their external surface, the value of their (total) surface-to-volume ratio, and the fluctuations in their density. In particular it is shown that  $N_0$ , the number of distinct infinite clusters, is either 0, 1, or  $\infty$  and the case  $N_0 = \infty$  (which might occur in sufficiently high dimension) is analyzed.

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**KEY WORDS:** Percolation; infinite clusters; cluster density; surface-to-volume ratio.

### **1. INTRODUCTION AND RESULTS**

In this paper, we present results concerning the number and nature of infinite clusters which are applicable to a large class of percolation models. Our analysis is based primarily on ergodic theory and measure theory. In order to give a precise statement of the results and a complete presentation of the proofs, we use a rather formal mathematical style in this paper. For an alternate description (without proofs) from a more physical point of view, see Ref. 1. In this section, we describe the class of models considered, give some basic definitions, and state our results; the proofs are presented in Section 2. We only treat properties of the infinite clusters; rigorous results concerning properties of large finite clusters are discussed in Refs. 2 and 3.

For ease of exposition, we restrict attention in this paper to site percolation in the simple  $d$ -dimensional cubic lattice with nearest-neighbor

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bonds; our results, however, apply (with simple modifications) to rather general lattices and to bond percolation. The models we consider may be defined by a lattice of (site occupation) random variables,  $\{X_k : k \in \mathbb{Z}^d\}$ , where each  $X_k$  either takes the value 1 (corresponding to site  $k$  occupied) or the value 0 (corresponding to site  $k$  not occupied). Such a model may equivalently be defined by the joint distribution  $P$  of  $\{X_k\}$  which is a probability measure on the configuration space,

$$\Omega = \{0, 1\}^{\mathbb{Z}^d} = \{ \omega = (\omega_k : k \in \mathbb{Z}^d) : \text{each } \omega_k = 0 \text{ or } 1 \}$$

(with the standard definition of measurable sets); we assume (without loss of generality) that  $(\Omega, P)$  is the underlying probability space with  $X_k(\omega) = \omega_k$ .

**Example 1.** In classical independent percolation of parameter  $p \in [0, 1]$ , the  $X_k$ 's are independent identically distributed random variables and  $P$  is the infinite product measure:

$$P = \prod_{k \in \mathbb{Z}^d} [p\delta(\omega_k - 1) + (1 - p)\delta(\omega_k)] \tag{1}$$

**Example 2.** In correlated percolation, the  $X_k$ 's are dependent random variables. A class of examples can be obtained by starting with an infinite volume spin-1/2 (i.e.,  $\pm 1$  valued) Ising model  $\{\sigma_k : k \in \mathbb{Z}^d\}$  and defining  $X_k = (1 + \sigma_k)/2$ . For example, with inverse temperature  $\beta$  and Hamiltonian

$$H(\{\sigma_k\}) = - \sum_j h\sigma_j - \sum_j \sum_k J(k - j)\sigma_j\sigma_k$$

$P$  is formally proportional to

$$\exp[-\beta H(\{2\omega_k - 1\})] \prod_k [\delta(\omega_k - 1) + \delta(\omega_k)]$$

The  $X_k$ 's will be dependent if  $J \neq 0$  and  $\beta \in (0, \infty)$ .

For any  $j \in \mathbb{Z}^d$  we consider the shift operator  $T_j$  which acts either on configurations  $\omega \in \Omega$ , events (i.e., measurable sets)  $W \subset \Omega$ , measures  $P$  on  $\Omega$ , or random variables  $X$  on  $\Omega$  according to

$$(T_j\omega)_k = \omega_{k-j}, \quad T_jW = \{T_j\omega : \omega \in W\}$$

$$(T_jP)(W) = P(T_{-j}W), \quad (T_jX)(\omega) = X(T_{-j}\omega)$$

For each  $k \in \mathbb{Z}^d$  and  $\eta = 0$  or  $1$  we consider the measures  $P_k^\eta$  on  $\Omega_k = \{0, 1\}^{\mathbb{Z}^d \setminus \{k\}}$ , defined so that for  $U \subset \Omega_k$ ,

$$P_k^\eta(U) = P(U \times \{\omega_k = \eta\}) \tag{2}$$

$P_k$  is the conditional distribution of  $\{X_j : j \neq k\}$  conditioned on  $X_k = \eta$ .

Throughout this paper, we assume the following three hypotheses on  $P$  (or equivalently on  $\{X_k\}$ ):

**Hypothesis A.**  $P$  is translation invariant; i.e., for any  $j \in \mathbb{Z}^d$ ,  $T_j P = P$ .

**Hypothesis B.**  $P$  is translation ergodic; i.e., if  $j \neq 0$  and  $W$  is an event such that  $T_j W = W$ , then  $P(W) = 0$  or  $1$ .

**Hypothesis C.** For any  $k$ ,  $P_k^0$  and  $P_k^1$  are equivalent measures; i.e., if  $U \subset \Omega_k$ ,  $\eta = 0$  or  $1$ , and  $P_k^\eta(U) \neq 0$ , then  $P_k^{1-\eta}(U) \neq 0$ .

In the independent percolation of Example 1, Hypothesis A is immediate, Hypothesis B is known to be valid (see, e.g., Ref. 4, Theorem 1.2), and Hypothesis C follows (for  $p \neq 0, 1$ ) from the simple fact that

$$p^{-1}P_k^1 = (1 - p)^{-1}P_k^0 = \prod_{j \neq k} [p\delta(\omega_j - 1) + (1 - p)\delta(\omega_j)] \tag{3}$$

In the Ising model percolation of Example 2, Hypotheses A and B will be valid if  $\{\sigma_k\}$  represents a translation-invariant pure phase. Hypothesis C (for  $\beta \neq 0, \infty$ ) is equivalent to assuming that the energy shift due to a single spin flip is finite since

$$dP_k^1/dP_k^0 = \exp[-\beta(H\{\sigma_k = 1; \sigma_j, j \neq k\} - H\{\sigma_k = 0; \sigma_j, j \neq k\})] \tag{4}$$

where the left-hand side of (4) denotes the Radon–Nikodym derivative, so that we have  $0 < dP_k^1/dP_k^0 < \infty$  (which yields Hypothesis C) if

$$\beta|2h + 2 \sum_{j \neq k} [J(k - j) + J(j - k)]\sigma_j| < \infty$$

which is the case if  $\sum_j |J(j)| < \infty$ . Hypothesis C is equivalent to the following reformulation, which can be more convenient in certain situations:

**Hypothesis C'.** For any  $k$ ,

$$0 < E(X_k | \{X_j : j \neq k\}) < 1 \quad (\text{w.p.o.})$$

where w.p.o. means “with probability one.”

Given a particular configuration  $\omega \in \Omega$ , we say that  $i$  is connected to  $j$  if  $i$  and  $j$  are both occupied and there is a nearest-neighbor path of occupied sites leading from  $i$  to  $j$ ; i.e., if there is some finite sequence  $i_0, i_1, \dots, i_n \in \mathbb{Z}^d$  with  $i_0 = i$ ,  $i_n = j$ ,  $\|i_j - i_{j-1}\| = 1$  for  $j = 1, \dots, n$  (where  $\|\cdot\|$  denotes Euclidean distance) and  $X_{i_j} = 1$  for  $j = 0, 1, \dots, n$ . We define  $C(j)$ , the cluster belonging to  $j$ , as  $C(j) = \{i : i \text{ is connected to } j\}$ ; note that

$C(j) = \emptyset$  if  $j$  is not occupied. A set  $C \subset \mathbb{Z}^d$  is called a cluster if  $C = C(j)$  for some  $j$  and is called an infinite cluster if in addition  $|C| = \infty$  where  $|\cdot|$  denotes cardinality. The percolation probability is  $\rho = P(|C(k)| = \infty)$  (which is independent of  $k$ ). We define for any  $F \subset \mathbb{Z}^d$  its lower density,  $\underline{D}(F)$  and upper density  $\overline{D}(F)$  as

$$\underline{D}(F) = \liminf_{n \rightarrow \infty} \frac{|F \cap R_n|}{n^d}, \quad \overline{D}(F) = \limsup_{n \rightarrow \infty} \frac{|F \cap R_n|}{n^d}$$

where

$$R_n = \{j = (j_1, \dots, j_d) \in \mathbb{Z}^d : -n/2 \leq j_l < n/2, \quad l = 1, \dots, d\} \quad (5)$$

$F$  is said to have a density  $D(F)$  if  $\underline{D}(F) = \overline{D}(F) = D(F)$ . We denote by  $H$  the set of all clusters  $C$  and define

$$\begin{aligned} H_0 &= \{C \in H : |C| = \infty\}, & N_0 &= |H_0| \\ F_0 &= \{j \in \mathbb{Z}^d : j \in C \quad \text{for some } C \in H_0\} \end{aligned} \quad (6)$$

$H_0$  is the set of infinite clusters,  $N_0$  the number of infinite clusters, and  $F_0$  the union of infinite clusters.

$N_0$  is a random variable, and for each  $k = 0, 1, 2, \dots$  or  $\infty$  there are of course many configurations  $\omega$  with  $N_0(\omega) = k$ . The following theorem shows, independently of the particular model, that except for  $k = 0, 1, \infty$ , these configurations have probability zero of occurring; in particular one has either (depending on  $d$  and the parameters of the model) no infinite clusters (w.p.o.), or a single infinite cluster (w.p.o.), or infinitely many distinct infinite clusters (w.p.o.). (Note that the theorem refers to clusters of occupied sites only; a single infinite cluster of occupied sites may coexist with a single infinite cluster of unoccupied sites.) The impossibility of  $N_0 > 1$  for  $d = 2$  was shown in Refs. 5 and 6 and there have appeared heuristic arguments (see, for example, Ref. 7) which attempt to extend this result to general  $d$ . It seems, however, that the possibility that  $N_0 = \infty$  or that zero-density infinite clusters might exist (see Theorem 2 below) has not been previously discussed in the literature. The following theorem also shows that there are no infinite clusters if  $\rho = 0$  (which is believed to be the case when  $p = p_c \equiv \inf\{p : \rho > 0\}$  in independent percolation).

**Theorem 1.** Exactly one of the following three events has probability 1: (i)  $N_0 = 0$ ; (ii)  $N_0 = 1$ ; (iii)  $N_0 = \infty$ . If  $N_0 \neq 0$  (w.p.o.), then  $\rho > 0$  and  $D(F_0) = \rho$  (w.p.o.).

The next theorem further analyzes the  $N_0 = \infty$  case into several disjoint possibilities. It shows that the appearance of zero-density (or at

least zero-lower-density) clusters is a necessity. We define the following:

$$\begin{aligned}
 H_1 &= \{ C \in H_0 : D(C) \text{ exists and } D(C) > 0 \} \\
 H_2 &= \{ C \in H_0 : D(C) = 0 \}, \\
 H_3 &= \{ C \in H_0 : \underline{D}(C) = 0 \text{ and } \bar{D}(C) > 0 \}
 \end{aligned}
 \tag{7}$$

Clusters in the class  $H_1$  will be called dense; in the class  $H_2$ , filamentary; in the class  $H_3$ , rough. For  $l = 1, 2, 3$ , we define, in analogy with (6),

$$N_l = |H_l|, \quad F_l = \{ j \in \mathbb{Z}^d : j \in C \text{ for some } C \in H_l \}$$

**Theorem 2.** Assume  $N_0 = \infty$  (w.p.o). It follows that  $H_0 = H_1 \cup H_2 \cup H_3$  (w.p.o.) and  $N_0 = N_1 + N_2 + N_3$  (w.p.o.); in particular exactly one of the following four events has probability 1: (i)  $N_2 = \infty$  and  $N_1 = N_3 = 0$ ; (ii)  $N_2 = \infty$ ,  $N_1 = 1$ , and  $N_3 = 0$ ; (iii)  $N_3 = \infty$  and  $N_1 = N_2 = 0$ ; (iv)  $N_2 = N_3 = \infty$  and  $N_1 = 0$ . For each  $l = 1, 2, 3$  there is a constant  $\rho_l > 0$  such that  $D(F_l) = \rho_l$  (w.p.o.);  $\rho_1 + \rho_2 + \rho_3 = \rho$  and if  $N_l \neq 0$  (w.p.o.), then  $\rho_l > 0$ .

*Remark.* We suspect that cases (iii) and (iv) of Theorem 2 do not occur (at least not in physically interesting models) but that cases (i) and (ii) do occur in sufficiently high dimension. If this happened in independent percolation, then the usual critical point,  $p = p_c \equiv \inf\{ p : \rho > 0 \}$ , should correspond to the transition from  $N_0 = 0$  to case (i) of Theorem 2, while other critical points should exist corresponding first to the transition from case (i) to case (ii) and then from case (ii) to  $N_0 = 1$ . Our next theorem suggests some of the phenomena associated with zero-density infinite clusters. It would be of considerable interest to obtain either a rigorous proof or numerical evidence for the existence of filamentary clusters.

The distance from  $j \in \mathbb{Z}^d$  to  $F \subset \mathbb{Z}^d$  is defined in the usual way as

$$\gamma(j, F) = \inf\{ \|k - j\| : k \in F \}$$

The set of “close-encounter sites” between (distinct) infinite clusters of types  $l$  and  $m$  ( $l, m = 0, 1, 2, 3$ ) is defined as

$$\begin{aligned}
 A_{lm} &= \{ j \in \mathbb{Z}^d : \exists C_1 \in H_l, C_2 \in H_m \text{ with } C_1 \neq C_2 \\
 &\text{such that } \gamma(j, C_1) = \gamma(j, C_2) = 1 \}
 \end{aligned}
 \tag{8}$$

These are vacant sites, which, if occupied, would connect two (or more) previously distinct infinite clusters (of specified type). To formally define the dual notion of “cutting sites” which belong to an infinite cluster and, if vacated, would disconnect that cluster into two (or more) distinct infinite

clusters of specified type, we first define for  $j \in \mathbb{Z}^d$ ,  $\omega \in \Omega$ , the “ $j$ -vacated” configuration,  $\omega[j]$ , as

$$(\omega[j])_k = \begin{cases} \omega_k, & k \neq j \\ 0, & k = j \end{cases}$$

We then define, for  $l = 0, 1, 2, 3$ ,  $H_l[j]$  as

$$(H_l[j])(\omega) = H_l(\omega[j]) \tag{9}$$

and the set of cutting sites, for  $l, m = 0, 1, 2, 3$ ,

$$B_{lm} = \{j \in \mathbb{Z}^d: \exists C_1 \in H_l[j], C_2 \in H_m[j] \text{ with } C_1 \neq C_2, \text{ such that } (C_1 \cup C_2) \subset C(j)\} \tag{10}$$

The next theorem shows that whenever distinct infinite clusters of type  $l$  and  $m$  occur, then there is a positive density of both close-encounter and cutting sites. In the case of independent percolation, these two densities are related to each other in a simple way (see Theorem 5 below).

**Theorem 3.** Choose  $l, m \in \{0, 1, 2, 3\}$ .  $A_{lm} = \emptyset = B_{lm}$  unless

$$N_l \neq 0, \quad N_m \neq 0, \quad \text{and} \quad N_l + N_m = \infty \quad (\text{w.p.o.}) \tag{11}$$

If (11) is valid, then there are constant  $\alpha_{lm} > 0$ ,  $\beta_{lm} > 0$  such that  $D(A_{lm}) = \alpha_{lm}$  (w.p.o.) and  $D(B_{lm}) = \beta_{lm}$  (w.p.o.).

We define the external surface  $\partial^e F$  of a set  $F \subset \mathbb{Z}^d$  as  $\partial^e F = \{j \in F^c: \gamma(j, F)\} = 1$  and there exists an infinite sequence of distinct points  $j_0, j_1, \dots \in F^c$  with  $j_0 = j$  and  $\|j_{m+1} - j_m\| = 1$  for all  $m$ ; i.e., the external surface of  $F$  consists of nearest neighbors of  $F$  which are “connected to  $\infty$ ” through points outside of  $F$ . The following theorem implies that clusters of positive density have either no external surface or else a large one in the sense that the asymptotic surface-to-volume ratio is bounded away from zero. There are other possible definitions of external surface, but the same result would apply to most of them. For a review of numerical results concerning the external surface of large finite clusters, see Ref. 8, Sec. 4.1.2.

**Theorem 4.** For  $l = 0, 1, 2, 3$ , if  $P(\partial^e F_l \neq \emptyset) > 0$ , then there is a constant  $\nu_l > 0$  such that  $D(\partial^e F_l) = \nu_l$  (w.p.o.).

*Remark.* Given  $\omega \in \Omega$ , let  $\omega^*$  be defined by  $(\omega^*)_k = 1 - \omega_k$ . It is easily verified that if  $F_l(\omega) \neq \emptyset$  (w.p.o.) and  $F_0(\omega^*) \neq \emptyset$  (w.p.o.) then  $\partial^e(F_l(\omega)) \neq \emptyset$  (w.p.o.). It follows that in independent percolation, if for a given  $d$ ,  $p_c \equiv \inf\{p: \rho > 0\} < 1/2$ , then there is a nonempty external surface to  $F_0$  (w.p.o.) for  $p_c < p < 1 - p_c$ . It can also be shown in independent percolation that if there exists an infinite cluster of positive density (w.p.o.)

for a certain  $p$  and dimension  $d$ , then there exists an infinite positive-density cluster for the same  $p$  and dimension  $d + 1$ ; since for  $d = 2$  and  $p > p_c(2)$  there is a single infinite cluster (Refs. 5 and 6), it follows that if  $d$  is sufficiently large so that  $p_c(d) < 1 - p_c(2)$ , then there will be an interval  $p_c(2) < p < 1 - p_c(d)$  in which there is an infinite positive-density cluster with a nonempty external surface.

One may also define the (total) surface  $\partial F$  of  $F \subset \mathbb{Z}^d$  as

$$\partial F = \{j \in F^c : \gamma(j, F) = 1\}$$

It follows from Hypothesis C that  $\partial F_l$  is nonempty (w.p.o.) if  $F_l$  is nonempty (w.p.o.) and an analog of Theorem 4 applies to give a well-defined density to  $\partial F_l$ . What is more surprising, perhaps, is that in the case of independent percolation (see Example 1 above), the asymptotic surface-to-volume ratio  $D(\partial F_l)/D(F_l)$  has (at least for  $l = 0, 1$ ) the explicit value  $(1 - p)/p$ . This value was first obtained by nonrigorous arguments in Ref. 9 and independently by numerical methods in Ref. 10. A rigorous proof was recently obtained (for each  $C \in H_0$ ) by Klein and Shamir;<sup>(11)</sup> their result motivated us to search for a simpler proof (based on a kind of duality), which led to the next theorem. In it, we generalize the result to an analogous result about the ratio of the density of close encounter sites to that of cutting sites [see equations (8) and (10) for the appropriate definitions]. We also find that in the case of dependent percolation models satisfying the inequalities of Fortuin, Kastelyn and Ginibre (FKG)<sup>(12)</sup> (see below), the equation  $D(\partial F_l)/D(F_l) = P(X_0 = 0)/P(X_0 = 1)$  is replaced by an inequality, which, in the case of the Ising models of Example 2, bounds the surface-to-volume ratio in terms of the magnetization. A brief discussion of numerical results concerning the surface-to-volume ratio in Ising models is given in Ref. 10. Numerical results concerning the critical point in independent percolation are given in Refs. 13 and 14.

A finite family of random variables  $Y_1, \dots, Y_m$  is said to satisfy the FKG inequalities, if for any real functions  $f_1, f_2$  on  $\mathbb{R}^m$  which are coordinate-wise increasing and such that  $\tilde{f}_j \equiv f_j(Y_1, \dots, Y_m)$  has  $E(\tilde{f}_j^2) < \infty$  ( $j = 1, 2$ ), it follows that

$$\text{Cov}(\tilde{f}_1, \tilde{f}_2) \equiv E(\tilde{f}_1 \tilde{f}_2) - E(\tilde{f}_1)E(\tilde{f}_2) \geq 0$$

An infinite family is said to satisfy the FKG inequalities if every finite subfamily does. The percolation model of Example 1 satisfies the FKG inequalities<sup>(5)</sup> as does that of Example 2 providing  $J(k) \geq 0 \forall k$ .<sup>(12)</sup>

**Theorem 5.** In the case of independent percolation, let  $(F, F^*) = (F_0, \partial F_0)$  or  $(F_1, \partial F_1)$  or  $(B_{lm}, A_{lm})$  (for  $l, m = 0, 1, 2, 3$ ), and assume  $F \neq \emptyset$  (w.p.o.); then  $D(F) > 0, D(F^*) > 0$ , and

$$D(F^*)/D(F) = (1 - p)/p = P(X_0 = 0)/P(X_0 = 1) \tag{12}$$

If  $\{X_k : k \in \mathbb{Z}^d\}$  satisfies the FKG inequalities, then for  $(F, F^*) = (F_0, \partial F_0)$  or  $(F_1, \partial F_1)$  we have

$$D(F^*)/D(F) \leq P(X_0 = 0)/P(X_0 = 1) \tag{13}$$

For each  $i \in \mathbb{Z}^d$ , define the cube  $R_n^i$  of side length  $n$  centered near  $ni$  as

$$R_n^i = \{j \in \mathbb{Z}^d : j = ni + k \text{ for some } k \in R_n\} = R_n + ni$$

where  $R_n$  is defined in (5), and define

$$Z_{l,n}^i = |F_l \cap R_n^i| - E(|F_l \cap R_n^i|) = |F_l \cap R_n^i| - \rho_l n^d \tag{14}$$

The fact that  $D(F_l) = \rho_l$  is equivalent to

$$\forall i, \quad Z_{l,n}^i/n^d \rightarrow 0 \quad (\text{w.p.o.}) \tag{15}$$

The next theorem generalizes this by considering infinite cluster density fluctuations and states that under certain conditions these are of strength  $n^{-d/2}$  and normally distributed as are the density fluctuations in a gas or liquid (away from the critical point). Note that the result *only applies for*  $l = 0$  or  $1$  and implies that for  $l = 0, 1$ , the random variables  $\{\mu_l^i : i \in \mathbb{Z}^d\}$  defined by

$$\mu_l^i = \begin{cases} 1 & \text{if } i \in F_l \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

behave on a macroscopic scale as though they were independent with mean  $\rho_l$  and some positive variance  $V_l$ .

**Theorem 6.** Assume that  $\{X_k : k \in \mathbb{Z}^d\}$  satisfies the FKG inequalities. For  $l = 0$  or  $1$ , suppose  $H_l \neq \emptyset$  (w.p.o.), and define  $K_l(i)$  on  $\mathbb{Z}^d$  by

$$K_l(i - j) = \text{Cov}(\mu_l^i, \mu_l^j) = P(i, j \text{ both belong to } H_l) - \rho_l^2 \tag{17}$$

$K_l(i) \geq 0$  and  $K_l(0) = \rho_l - \rho_l^2 > 0$ . If

$$V_l \equiv \sum_{i \in \mathbb{Z}^d} K_l(i) < \infty \tag{18}$$

then  $\{\tilde{Z}_{l,n}^i \equiv Z_{l,n}^i / V_l^{1/2} n^{d/2} : i \in \mathbb{Z}^d\}$  tend to independent standard normal random variables as  $n \rightarrow \infty$  in the sense that for any  $N$  and distinct  $i_1, \dots, i_N \in \mathbb{Z}^d$  and any  $a_1 \leq b_1, \dots, a_N \leq b_N$ ,

$$P(a_1 \leq \tilde{Z}_{l,n}^{i_1} \leq b_1, \dots, a_N \leq \tilde{Z}_{l,n}^{i_N} \leq b_N) \rightarrow \prod_{j=1}^N \left[ (2\pi)^{-1/2} \int_{a_j}^{b_j} \exp(-z^2/2) dz \right] \tag{19}$$

**Remark.** An analog of Theorem 6 applies to the density fluctuations of  $F_l \cup \partial F_l$  ( $l = 0$  or  $1$ ); this result when combined with Theorem 6 itself gives some information on the fluctuations in surface-to-volume ratio of  $F_0$



and  $F_1$ . For a discussion of such fluctuations in large finite clusters, see Ref. 8, Sec. 4, and for results on infinite clusters, see Ref. 11.

Our final result concerns how the various cases of Theorems 1 and 2 are related to the asymptotic behavior as  $\|i\| \rightarrow \infty$  of the quantity  $G(i)$  defined as

$$G(i - j) = P(i \text{ is connected to } j) \tag{20}$$

Note that  $i$  is not connected to  $j$  unless  $i$  and  $j$  are both occupied. We also define

$$Q_n = n^{-d} \sum_{j \in R_n} G(j) \tag{21}$$

**Theorem 7.** If

$$\liminf_{n \rightarrow \infty} \sum_{j \in \partial R_n} G(j) = 0 \tag{22}$$

then  $H_0 = \emptyset$  (w.p.o.); if

$$\sum_{j \in \mathbb{Z}^d} \|j\|^{-(d-1)} G(j) < \infty \tag{23}$$

then  $H_0 = H_2$  (w.p.o.); if

$$\liminf_{n \rightarrow \infty} Q_n = 0 \tag{24}$$

then  $H_1 = \emptyset$  (w.p.o.). If  $H_0 = H_2$  (w.p.o.), then

$$\lim_{n \rightarrow \infty} Q_n = 0 \tag{25}$$

If  $H_1 \neq \emptyset$  (w.p.o.), then

$$\lim_{n \rightarrow \infty} Q_n = (\rho_1)^2 > 0 \tag{26}$$

*Remark.* One can use  $G(j)$  to define a “mean effective dimension,”

$$\delta = \sup \left\{ \delta' : \lim_{n \rightarrow \infty} n^{-\delta'} \sum_{j \in R_n} G(j) = \infty \right\} \tag{27}$$

If one defines for a particular infinite cluster  $C$  its effective dimension

$$\underline{\delta}(C) = \sup \left\{ \delta' : \lim_{n \rightarrow \infty} n^{-\delta'} |C \cap R_n| = \infty \right\} \tag{28}$$

then it can be shown that

$$\forall C \in H_0, \underline{\delta}(C) \leq \delta \quad (\text{w.p.o.}) \tag{29}$$

It would be interesting to obtain some nontrivial estimates on  $\delta$  and  $\underline{\delta}(C)$ . For more discussion of effective and fractal dimension in percolation and elsewhere, see Refs. 14, 15, and 16.

## 2. PROOFS

The proofs of Theorems 1–5 are based on the following two propositions together with some geometric arguments. The first proposition, which follows from Hypotheses A and B, is a standard fact of ergodic theory.<sup>(17,18)</sup>

**Proposition 8.** Suppose  $X \in L^1(\Omega, P)$ ; then

$$\lim_{n \rightarrow \infty} n^{-d} \sum_{j \in R_n} T_j X = E(X) \quad (\text{w.p.o.}) \tag{30}$$

The next proposition follows from Hypothesis C and is the key to most of our results; it says that by altering configurations at a fixed finite number of sites, one cannot reduce the probability of an event from a positive value to zero. Its validity in the independent percolation of Example 1 (for  $p \neq 0, 1$ ) follows from the facts [see Eq. (3)] that each alteration at a single site introduces a factor of  $[(1-p)/p]^{\pm 1}$ , and  $[(1-p)/p]^{\pm n}$  is nonzero for finite  $n$ . Similarly, in the percolation of Example 2, each alteration at a single site introduces the (random) factor given in Eq. (4) (or its inverse) which is related to the Ising model energy; the validity of the next proposition for Example 2 is thus based on the fact that only a finite energy change is induced by spin flips in a finite volume.

For a region  $V \subset \mathbb{Z}^d$  and a measurable transformation  $\Phi: \Omega \rightarrow \Omega$ , we say that  $\Phi$  is  $V$ -local if  $(\Phi(\omega))_k = \omega_k$  for all  $\omega \in \Omega$  and all  $k \in V^c$ ; we say  $\Phi$  is local if  $\Phi$  is  $V$ -local for some finite  $V$ . Note that the value of  $(\Phi(\omega))_k$  for  $k \in V$  may depend on  $\omega_j$ 's for all  $j \in \mathbb{Z}^d$  (including  $j \in V^c$ ).

**Proposition 9.** If  $\Phi$  is local and  $W$  is any event, then

$$P(W) > 0 \quad \text{implies} \quad P(\Phi(W)) > 0 \tag{31}$$

*Proof.* Let  $V$  be a finite set such that  $\Phi$  is  $V$ -local and let  $S = \{0, 1\}^V$ . For each  $s \in S$ , define  $\Psi_s: \Omega \rightarrow \Omega$  by

$$(\Psi_s \omega)_k = \begin{cases} \omega_k, & k \in V^c \\ s_k, & k \in V \end{cases} \tag{32}$$

Then defining for each  $t, s \in S$ ,  $W_{ts} = W \cap \Psi_t(\Omega) \cap \Phi^{-1}(\Psi_s(\Omega))$  so that

$$W = \bigcup_{t,s \in S} W_{ts}, \quad \Phi(W) = \bigcup_{t,s \in S} \Phi(W_{ts}) = \bigcup_{t,s \in S} \Psi_s(W_{ts})$$

we note that for some  $t, s, P(W_{ts}) > 0$  and it then suffices to show that  $P(\Psi_s(W_{ts})) > 0$ . For  $j \in \mathbb{Z}^d$  and  $\epsilon = 0$  or  $1$ , define  $\Psi_{j,\epsilon}$  by

$$(\Psi_{j,\epsilon} \omega)_k = \begin{cases} \omega_k, & k \neq j \\ \epsilon, & k = j \end{cases} \tag{33}$$

then

$$\Psi_s(W_{ts}) = \left( \prod_{j \in V} \Psi_{j,s_j} \right) (W_{ts}) = \prod_{j \in V : s_j \neq t_j} \Psi_{j,s_j}(W_{ts})$$

and (by induction) it suffices to show that for each  $j$  and any  $U \subset \Omega_j \equiv \{0, 1\}^{\mathbb{Z}^d \setminus \{j\}}$ ,

$$P(U \times \{\omega_j = t_j\}) > 0 \text{ implies } P(U \times \{\omega_j = s_j\}) > 0 \quad (34)$$

but this is the content of Hypothesis C [see Eq. (2)]. ■

*Proof of Theorem 1.* The random variable  $N_0$  and the events  $\{N_0 = k\}$  are invariant under any shift  $T_j$ ; therefore by Hypothesis B there is some  $k_0 = 0, 1, 2, \dots$ , or  $\infty$  such that  $P(N_0 = k_0) = 1$ . Suppose  $k_0 \neq 0, 1$ , or  $\infty$ ; we wish to obtain a contradiction. Let  $W_n$  be the event that  $N_0 = k_0$  and each one of the  $k_0$  infinite clusters has nonzero intersection with the cube  $R_n$ ; then  $P(W_n) \rightarrow P(N_0 = k_0) = 1$  and so for some  $m$ ,  $P(W_m) > 0$ . Let  $V = R_m$  and define  $\Psi_{\bar{s}}$  as in (32) with  $\bar{s}_j = 1 \forall j \in R_m$ ;  $\Psi_{\bar{s}}$  is the transformation which makes all sites in  $R_m$  occupied and leaves all other sites unchanged. By Proposition 9,  $P(\Psi_{\bar{s}}(W_m)) > 0$ , but clearly  $\Psi_{\bar{s}}(W_m) \subset \{N_0 = 1\}$ , thus  $P(N_0 = 1) > 0$ , which contradicts the supposition that  $k_0 \neq 0, 1$  or  $\infty$ . To prove the second half of the theorem, we first note that if  $\rho \equiv P(|C(k)| = \infty) \equiv P(k \in F_0) = 0$ ; then

$$P(\mathbb{Z}^d \cap F_0 \neq \emptyset) = P\left( \bigcup_{j \in \mathbb{Z}^d} \{j \in F_0\} \right) \leq \sum_{j \in \mathbb{Z}^d} P(j \in F_0) = 0$$

and so  $P(F_0 = \emptyset) = 1 = P(N_0 = 0)$ . Finally defining  $\mu_0^i$  as in (16) we have by Proposition 8 that

$$D(F_0) \equiv \lim_{n \rightarrow \infty} n^{-d} \sum_{j \in R_n} T_j \mu_0^0 = E(\mu_0^0) = P(0 \in F_0) = \rho \quad (\text{w.p.o.}) \quad (35)$$

which completes the proof. ■

*Proof of Theorem 2.* We assume throughout that  $N_0 = \infty$  (w.p.o.). Let  $\bar{H}_1 = \{C \in H_0 : \underline{D}(C) > 0\}$  and  $\bar{N}_1 = |\bar{H}_1|$  and suppose that  $P(\bar{N}_1 = 0) \neq 1$ ; then by ergodicity, as in the proof of Theorem 1, there is some  $\bar{k} = 1, 2, \dots$ , or  $\infty$  such that  $P(\bar{N}_1 = \bar{k}) = 1$ . We wish to rule out the possibility that  $\bar{k} > 1$ . Let  $\underline{Y} = \sup\{\underline{D}(C) : C \in H_0\}$ ; then again by ergodicity  $\exists D_1 > 0$  so that  $P(\underline{Y} = D_1) = 1$  and if  $\bar{k} > 1$ , then the event

$$U = \{\exists C_1, C_2 \in H_0 \text{ with } C_1 \neq C_2 \text{ so that } \underline{D}(C_1) + \underline{D}(C_2) > D_1\}$$

has probability 1. Let  $U_n$  be the event

$$U_n = \{\exists C_1, C_2 \in H_0 \text{ with } C_1 \neq C_2 \text{ and } C_1 \cap R_n \neq \emptyset, C_2 \cap R_n \neq \emptyset, \text{ so that } \underline{D}(C_1) + \underline{D}(C_2) > D_1\}$$

then  $P(U_n) \rightarrow P(U)$  and so for some  $m$ ,  $P(U_m) > 0$ . As in the proof of Theorem 1, we consider  $\Psi_{\bar{s}}$  defined by (32) with  $V = R_m$  and  $\bar{s}_j = 1 \forall j \in R_m$ , and conclude by Proposition 9 that  $P(\Psi_{\bar{s}}(U_m)) > 0$ , but each  $\omega \in \Psi_{\bar{s}}(U_m)$  has an infinite cluster containing  $C_1 \cup C_2 \cup R_m$  which has lower density greater than  $D_1$ ; thus  $P(\underline{Y} > D_1) > 0$ , which contradicts  $P(\underline{Y} = D_1) = 1$  and proves that  $\bar{k} = 1$ . Letting  $\bar{F}_1 = \{j \in \mathbb{Z}^d : j \in C \text{ for some } C \in \bar{H}_1\}$ , we have as in the proof of the second part of Theorem 1 that  $\exists \bar{\rho}_1 > 0$  such that  $D(\bar{F}_1) = \bar{\rho}_1$  (w.p.o.), but since  $\bar{N}_1 = 1$  (w.p.o.) this means that the single  $C \in \bar{H}_1$  itself has  $D(C) = \bar{\rho}_1$  and thus  $\bar{H}_1 = H_1$ ,  $\bar{\rho}_1 = \rho_1$ , etc. We are now in the situation where a single cluster has positive density,  $\rho_1$ , and all other clusters have zero lower density. To rule out the possibility that any other cluster has positive upper density, we note that otherwise, for some  $m$ , the event

$$\begin{aligned} \tilde{U}_m = \{ & \exists C_1 \neq C_2 \text{ with } D(C_1) = \rho_1, \bar{D}(C_2) > 0, \\ & C_1 \cap R_m \neq \emptyset, C_2 \cap R_m \neq \emptyset \} \end{aligned}$$

would have positive probability. But then, as before,  $P(\Psi_{\bar{s}}(\tilde{U}_m)) > 0$ , and  $\Psi_{\bar{s}}(\tilde{U}_m) \subset \{\exists C \in H_0 : \underline{D}(C) > 0, \bar{D}(C) > \rho_1\} \subset \{\bar{D}(\bar{F}_1) > \rho_1\}$ , which contradicts the fact that  $D(\bar{F}_1) = \rho_1$ . We have thus proven that if (with positive probability) there is any infinite cluster with positive lower density, we must actually be in case (ii) of Theorem 2 (w.p.o.).

In order to prove that cases (i), (iii), (iv) exhaust the alternatives, we assume that  $\bar{H}_1 = \emptyset$  (w.p.o.) and show that for  $l = 2, 3$  either  $N_l = 0$  (w.p.o.) or  $N_l = \infty$  (w.p.o.). This is not difficult since the proof of Theorem 1 already shows that we need only rule out the case  $N_l = 1$  (w.p.o.); but if  $N_l = 1$  (w.p.o.) then as in the proof above concerning  $\bar{N}_1$  and  $\bar{F}_1$ ; it would follow that  $D(F_l) = \rho_l > 0$  and thus the single infinite cluster  $C$  in  $H_l$  would actually have density  $D(C) = \rho_l > 0$ , which contradicts the definition of  $H_l$  for  $l = 2, 3$ . This completes the proof of the first part of the theorem; the last part is then proved exactly as was the last part of Theorem 1. ■

*Proof of Theorem 3.* We define  $\Psi_{j,\epsilon}$  as in (33) and note that  $\Psi_{0,1}\{0 \in A_{lm}\} = \{0 \in B_{lm}\}$  and  $\Psi_{0,0}\{0 \in B_{lm}\} = \{0 \in A_{lm}\}$ . Since  $P(A_{lm} \neq \emptyset) > 0$  if and only if  $\alpha_{lm} \equiv P\{0 \in A_{lm}\} > 0$  and similarly for  $B_{lm}$  and  $\beta_{lm}$ , and since clearly  $P(A_{lm} \neq \emptyset) = 0$  if (11) is not valid, we see first by Proposition 9 that also  $P(B_{lm} \neq \emptyset) = 0$  if (11) is not valid, and second by Proposition 8 that it suffices to assume (11) and conclude that  $P(A_{lm} \neq \emptyset) > 0$  in order to complete the proof of the entire theorem; this we proceed to do.

Assuming (11) we have, for some  $n$ , that the event

$$\begin{aligned} W_n \equiv \{ & \exists C_1 \in H_l, C_2 \in H_m \text{ with } C_1 \neq C_2 \text{ and} \\ & C_1 \cap R_n \neq \emptyset, C_2 \cap R_n \neq \emptyset \} \end{aligned} \tag{36}$$

has positive probability. We will define an  $R_n$ -local transformation  $\Phi$  such that  $\Phi(W_n) \subset \{\exists j \in A_{lm} \cap R_n\} \subset \{A_{lm} \neq \emptyset\}$ , which will imply by Proposition 9 that  $P\{A_{lm} \neq \emptyset\} > 0$  as desired. We first choose for each  $i, j \in \bar{\partial}R_n \equiv \{j \in R_n : \gamma(j, R_n^c) = 1\}$  a subset  $R_n^{ij} \subset R_n$  such that

- (a)  $R_n^{ij} \setminus \{i, j\} \subset R_n \setminus \bar{\partial}R_n$
- (b)  $\exists$  a nearest-neighbor path in  $R_n^{ij}$  which contains and connects  $i$  and  $j$
- (c)  $\exists k \in R_n^{ij}$  so that (b) is false if  $R_n^{ij}$  is replaced by  $R_n^{ij} \setminus \{k\}$ .

Note that if  $i = j$  we may take  $R_n^{ij} = \{i\}$  and if  $\|i - j\| = 1$  we may take  $R_n^{ij} = \{i, j\}$ . We next choose an ordering  $\alpha : \bar{\partial}R_n \rightarrow \{1, \dots, |\bar{\partial}R_n|\}$ . We then define  $\Phi_1 = \Psi_s$  as in (32) with  $V = R_n$  and  $s_j = 0 \forall j \in R_n$  and we define

$$\Phi(\omega) = \begin{cases} \omega & \text{if } \omega \notin W_n \\ \Phi_2(\Phi_1(\omega)) & \text{if } \omega \in W_n \end{cases}$$

where  $\Phi_2$  is defined for  $\omega \in \Phi_1(W_n)$  as follows: Let  $(i, j) = (i(\omega), j(\omega))$  be the  $i, j \in \bar{\partial}R_n$  with the smallest value (lexicographically) of the ordering  $(\alpha_i, \alpha_j)$  such that  $\exists C_1 \in H, C_2 \in H_m$  with  $C_1 \neq C_2$  such that  $\gamma(i, C_1) = 1, \gamma(j, C_2) = 1$ ; such an  $(i, j)$  exists by the definitions of  $W_n$  and  $\Phi_1$ . We then define  $\Phi_2 = \Psi_s$  analogously to (32) but with  $V = R_n$  and

$$s_l = s_l(\omega) = \begin{cases} 1 & \text{if } l \in R_n^{i(\omega), j(\omega)} \\ 0 & \text{otherwise} \end{cases}$$

For  $\omega \in \Phi_1(W_n)$ ,  $\Phi_2(\omega)$  has a cutting point which is just the  $k$  of part (c) of the definition of  $R_n^{ij}$  given above. Thus  $\Phi_2(\Phi_1(W_n)) \subset \{A_{lm} \cap R_n \neq \emptyset\}$  as desired and since  $\Phi$  is  $R_n$ -local, the proof is complete. ■

**Proof of Theorem 4.** Theorem 4 follows from Proposition 8 as in the proof of the last part of Theorem 1. ■

**Proof of Theorem 5.** Using Proposition 8 as in the proof of the last part of Theorem 1, we see that it suffices in the first part of the theorem to show

$$P(X_0 = 0) / P(X_0 = 1) = P(0 \in F^*) / P(0 \in F)$$

or equivalently

$$P(0 \in F) = P(0 \in (F \cup F^*)) \cdot P(X_0 = 1) \tag{37}$$

and in the second part of the theorem to show that

$$P(0 \in F) \geq P(0 \in (F \cup F^*)) \cdot P(X_0 = 1) \tag{38}$$

Now, by Theorem 2, we may replace  $(F_1, \partial F_1)$  in (37) and (38) by  $(\bar{F}_1, \bar{\partial} \bar{F}_1)$  without affecting probabilities, where  $\bar{F}_1 = \{i : i \in C \text{ for some } C \in H_0 \text{ with}$

$\underline{D}(C) > 0\}$ . If we define for  $\omega \in \Omega$ ,  $\omega[j]$  and  $\omega\langle j \rangle$  by

$$(\omega[j])_k = \begin{cases} \omega_k, & k \neq j \\ 0, & k = j \end{cases} \quad (\omega\langle j \rangle)_k = \begin{cases} \omega_k, & k \neq j \\ 1, & k = j \end{cases}$$

then the following duality is satisfied:

$$\omega \in \{0 \in F\} \Rightarrow \omega[0] \in \{0 \in F^*\}, \quad \omega \in \{0 \in F^*\} \Rightarrow \omega\langle 0 \rangle \in \{0 \in F\}$$

It then follows that

$$\begin{aligned} \{0 \in F\} &= \{0 \in F \cup F^*\} \cap \{X_0 = 1\}, \\ \{0 \in F^*\} &= \{0 \in F \cup F^*\} \cap \{X_0 = 0\} \end{aligned} \tag{39}$$

and also that for some

$$U \subset \Omega_0 = \{0, 1\}^{\mathbb{Z}^d \setminus \{0\}}, \quad \{0 \in F \cup F^*\} = U \times \{\omega_0 = 0 \text{ or } 1\} \tag{40}$$

From (40) and the definition of independent percolation [see Eq. (1)], we see that in the first part of the theorem  $\{0 \in F \cup F^*\}$  and  $\{X_0 = 1\}$  are independent events and thus (37) follows as desired. In the second part of the theorem; we note that for  $(F, F^*) = (F_0, \partial F_0)$  or  $(\bar{F}_1, \partial \bar{F}_1)$  the random variable

$$Y = \begin{cases} 1 & \text{if } 0 \in F \cup F^* \\ 0 & \text{if } 0 \notin F \cup F^* \end{cases}$$

is an increasing function of the basic occupation variables,  $\{X_j\}$ , and thus if the FKG inequalities are valid,

$$\text{Cov}(Y, X_0) \equiv E(YX_0) - E(Y)E(X_0) \geq 0 \tag{41}$$

But  $E(X_0) = P(X_0 = 1)$ ,  $E(Y) = P(0 \in F \cup F^*)$ , and by (39),  $E(YX_0) = P(0 \in F)$ , which yields (38) and completes the proof. ■

**Proof of Theorem 6.** Define  $\bar{\mu}_0^i = \mu_0^i$  [as in (16)] and

$$\bar{\mu}_1^i = \begin{cases} 1 & \text{if } i \in C \text{ for some } C \in H_0 \text{ with } \underline{D}(C) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

Now by Theorem 2,  $\bar{\mu}_l^i = \mu_l^i$  (w.p.o.) and so we have from (14)

$$Z_{l,n}^i = \sum_{j \in R_n^i} [\bar{\mu}_l^j - E(\bar{\mu}_l^j)] \quad (\text{w.p.o.}) \tag{43}$$

For  $l = 0, 1$  the  $\bar{\mu}_l^i$ 's (but not the  $\mu_l^i$ 's) are increasing functions of the basic occupation variables,  $\{X_k\}$ , and thus they also satisfy the FKG inequalities [which immediately implies  $k_l(i) \geq 0$ ]. Moreover  $\{\bar{\mu}_l^i : i \in \mathbb{Z}^d\}$  is translation

invariant, and by (18) it is assumed that

$$V_l \equiv \sum_{i \in \mathbb{Z}^d} \text{Cov}(\bar{\mu}_l^0, \bar{\mu}_l^i) < \infty \tag{44}$$

The theorem now follows from the general results of Ref. 19. ■

The following standard proposition will be used in the proof of Theorem 7. We write  $W_n$  i.o. to denote the event that  $W_n$  occurs for infinitely many  $n$  and we write  $W_n$  a.f.o. to denote that  $W_n$  occurs for all but finitely many  $n$ .

**Proposition 10.** Let  $W_n$  be a sequence of events and  $Y_n$  a sequence of random variables; then

$$\begin{aligned} \sum_{n=1}^{\infty} P(W_n) < \infty & \quad \text{implies} \quad P(W_n \text{ i.o.}) = 0 \\ \sum_{n=1}^{\infty} E(|Y_n|) < \infty & \quad \text{implies} \quad Y_n \rightarrow 0 \quad (\text{w.p.o.}) \end{aligned} \tag{45}$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} P(W_n) = 0 & \quad \text{implies} \quad P(W_n \text{ a.f.o.}) = 0 \\ \liminf_{n \rightarrow \infty} E(|Y_n|) = 0 & \quad \text{implies} \quad \liminf_{n \rightarrow \infty} |Y_n| = 0 \quad (\text{w.p.o.}) \end{aligned} \tag{46}$$

*Proof.* Equation (45) for  $W_n$  is just the Borel–Cantelli lemma. Now by Tchebyshev’s inequality, for any  $\epsilon > 0$ ,  $\sum P(|Y_n| \geq \epsilon) \leq \epsilon^{-1} \sum E(|Y_n|)$ , and thus letting  $W_n = \{|Y_n| \leq \epsilon\}$ , we have  $P(|Y_n| \geq \epsilon \text{ i.o.}) = 0$ . Thus for each  $m$ ,  $P(\limsup |Y_n| \leq 1/m) = 1$ , and letting  $m \rightarrow \infty$  we have  $P(\lim |Y_n| = 0) = 1$ . To obtain (46) from (45) simply choose an appropriate subsequence. ■

*Proof of Theorem 7.* Since

$$P(0 \text{ is connected to } \partial R_n) \leq \sum_{j \in \partial R_n} G(j) \tag{47}$$

we have from (45) that if (22) is valid then  $P(0 \text{ connected to } \partial R_n \text{ a.f.o.}) = 0$  and thus  $\rho_0 \equiv P(|C(0)| = \infty) = P(0 \text{ connected to } \partial R_n \ \forall n) = 0$ , so by Theorem 1  $H_0 = \emptyset$  (w.p.o.). Now let

$$Y_n = |C(0) \cap R_n| / n^d = n^{-d} \sum_{j \in R_n} \lambda_j \tag{48}$$

where  $\lambda_j$  is the indicator function of the event,  $\{0 \text{ is connected to } j\}$ , so that

$E(\lambda_j) = G(j)$  and  $E(Y_n) = E(|Y_n|) = Q_n$ . If (23) is valid, then (for  $d > 1$ ),

$$\begin{aligned} \sum_n E(|Y_n|) &= \sum_n Q_n = \sum_n n^{-d} \sum_{j \in R_n} G(j) = \sum_{k \in \mathbb{Z}^d} \left( G(k) \sum_{n: R_n \ni k} n^{-d} \right) \\ &\leq \sum_{k \in \mathbb{Z}^d} O(\|k\|^{-(d-1)}) G(k) < \infty \end{aligned} \quad (49)$$

and so by (45)  $D(C(0)) \equiv \lim Y_n = 0$  (w.p.o.) which implies  $\rho_1 = \rho_3 = 0$  and so  $\rho_0 = \rho_2$  and  $H_0 = H_2$  (w.p.o.). Similarly, if (24) is valid, then  $\lim \inf E(|Y_n|) = 0$  and so by (41)  $\lim \inf |Y_n| = \underline{D}(C(0)) = 0$  (w.p.o.), so  $\rho_1 = 0$  and  $H_1 = \emptyset$  (w.p.o.). On the other hand, if  $H_0 = H_2$  (w.p.o.), then  $\lim Y_n = 0$  (w.p.o.) and since  $0 \leq Y_n \leq 1$ , we have by dominated convergence that  $Q_n = E(Y_n) \rightarrow 0$  as desired. If  $H_1 \neq \emptyset$ , then we are in case (ii) of Theorem 1 or 2 and so  $D(C(0)) \equiv \lim Y_n = 0 \cdot P(0 \in H_2) + \rho_1 \cdot P(0 \in H_1) = (\rho_1)^2$ , so by dominated convergence  $Q_n = E(Y_n) \rightarrow (\rho_1)^2$  as desired. This completes the proof. ■

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