Infinite Dimensional Analysis and Analytic Number Theory

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Infinite Dimensional Analysis and Analytic Number Theory

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Dedicated to Professor Takeyuki Hida on the occasion of his 70th birthday

Abstract

We consider arithmetical aspects of analysis on Fock spaces (Boson Fock space, Fermion Fock space, and Boson-Fermion Fock space) with applications to analytic number theory.

1 Introduction

In recent years, connections between number theory and physics have been noted and discussed (e.g., [25, 33]). From an arithmetic point of view, "statistical mechanics" of numbers may be particularly interesting, because it is related in a direct way to the Riemann zeta function and may give a key to solve the Riemann hypothesis ([18, 19, 21, 23, 24, 29, 30, 31] and references therein). Spector [30] pointed out intriguing relationships between analytic number theory and a free supersymmetric quantum field theory, and further discussed these aspects with notions of partial supersymmetry and "duality"[31].

On the other hand, general mathematical structures and aspects of some models in supersymmetric quantum field theory have been studied as a subject of infinite dimensional analysis ([3]–[12], [15]–[17]) (cf. also [22]). Motivated by the work of Spector mentioned above, we are interested in developing analytic number theory *as a field of infinite dimensional analysis.* In this paper we start this program with reviewing fundamental aspects of relationships between analytic number theory and analysis on Fock spaces (Boson Fock space, Fermion Fock space and Boson-Fermion Fock space).

The present paper, which is intended to be of review nature, is organized as follows. In Section 2, we discuss relationships between analysis on Boson Fock space and analytic number theory. Statistical mechanical partition functions are defined in an abstract level and their arithmetical structures are analyzed. One of new aspects here is an introduction of a graded partition function which is associated with a graded structure of the abstract Boson Fock space. We apply the abstract results to a concrete case where arithmetical functions (the Riemann zeta function, the Liouville function, Dirichlet series etc.) are described in terms of Fock space langauge. In Section 3 analysis similar to that of Section 2 is made on Fermion Fock spaces. We prove duality relations between bosonic and fermionic partition functions. One of the relevant arithmetical functions on a Fermion Fock space is the Möbius function. As applications of the abstract results, we rederive some known formulas on the Möbius function and completely multiplicative functions. We also derive a Fock space expression of Jordan's totient function. Section 4 is devoted to a brief review of fundamental aspects of analysis on the abstract Boson-Fermion Fock space developed by the present author [7]. The main objects in this analysis are infinite dimensional Dirac and Laplace-Beltrami operators. In the last section, we discuss arithmetical aspects of analysis of Boson-Fermion Fock spaces, generalizing ideas in [30, 31].

2 Boson Fock Spaces and Arithmetical Functions

2.1 Partition functions and correlation functions

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space with inner product $(\cdot\,,\,\cdot)_{\mathcal{H}}$ (complex linear in the second variable) and $\otimes_{\rm s}^n\mathcal{H}$ be the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$; $\otimes^0\mathcal{H} := \mathbf{C}$). Then the Boson Fock space over $\mathcal{H}$ is defined by

$$\mathcal{F}_{\rm B}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\rm s}^n \mathcal{H} \tag{2.1}$$

(e.g, [20, §5.2], [26, §II.4]).

We denote by $a_{\mathcal{H}}(f)$ ($f \in \mathcal{H}$) the annihilation operator on $\mathcal{F}_{\rm B}(\mathcal{H})$ (e.g., [20, §5.2], [27, §X.7]) ($a_{\mathcal{H}}(f)$ is antilinear in f). The set $\{a_{\mathcal{H}}(f), a_{\mathcal{H}}(f)^* | f \in \mathcal{H}\}$ satisfies the canonical commutation relations

$$[a_{\mathcal{H}}(f), a_{\mathcal{H}}(g)^*] = (f, g)_{\mathcal{H}}, \quad [a_{\mathcal{H}}(f), a_{\mathcal{H}}(g)] = 0, \quad [a_{\mathcal{H}}(f)^*, a_{\mathcal{H}}(g)^*] = 0, \quad f, g \in \mathcal{H}, \tag{2.2}$$

on

$$\mathcal{F}_{\rm B,0}(\mathcal{H}) := \{\Psi = \{\Psi_n\}_{n=1}^{\infty} \in \mathcal{F}_{\rm B}(\mathcal{H}) | \Psi_n = 0 \text{ for all but finitely many } n\text{'s}\}, \tag{2.3}$$

the space of finite particle vectors in $\mathcal{F}_{\rm B}(\mathcal{H})$. We denote by $\Omega_{\mathcal{H}} := \{1, 0, 0, \cdots\}$ the Fock vacuum in $\mathcal{F}_{\rm B}(\mathcal{H})$. We have

$$a_{\mathcal{H}}(f)\Omega_{\mathcal{H}} = 0, \quad f \in \mathcal{H}. \tag{2.4}$$

Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ and $d\Gamma_{\rm B}(A)$ the second quantization of A on $\mathcal{F}_{\rm B}(\mathcal{H})$:

$$d\Gamma_{\rm B}(A) := \oplus_{n=0}^{\infty} d\Gamma_{\rm B}^{(n)}(A) \tag{2.5}$$

with $d\Gamma_{\rm B}^{(0)}(A) := 0$ and

$$d\Gamma_{\rm B}^{(n)}(A) := I \otimes \cdots \otimes I \otimes \overbrace{A}^{n\text{th}} \otimes I \otimes \cdots \otimes I,$$

where I denotes identity operator (e.g., [20, §5.2], [27, p. 302, Example 2]). Then $d\Gamma_{\rm B}(A)$ is nonegative and self-adjoint. We set

$$H_{\rm B}(A) := d\Gamma_{\rm B}(A). \tag{2.6}$$

Remark 2.1 In quantum field theory, $H_{\rm B}(A)$ describes a *free Hamiltonian* of a quantized Bose field with A being its one particle Hamiltonian.

Let C be a densely defined closed linear operator on $\mathcal{H}$ and $\otimes^n C$ the n-fold tensor product of C on the n-fold tensor product Hilbert space $\otimes^n \mathcal{H}$ of $\mathcal{H}$ ($\otimes^0 C := 1$). Then

$$\Gamma(C) := \oplus_{n=0}^{\infty} \otimes^n C \tag{2.7}$$

is a densely defined closed linear operator on the full Fock space

$$\mathcal{F}_{\rm full}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes^n \mathcal{H} \tag{2.8}$$

over $\mathcal{H}$. It is well known (or easy to see) that, if C is a contraction operator, then so is $\Gamma(C)$. In particular, if C is unitary, then so is $\Gamma(C)$ (cf. [27, §X.7]).

The operator $\Gamma(C)$ is reduced by $\mathcal{F}_{\rm B}(\mathcal{H})$. We denote its reduced part by $\Gamma_{\rm B}(C)$. Then we have

$$\Gamma_{\rm B}(e^{itA}) = e^{itH_{\rm B}(A)}, \quad t \in \mathbf{R}. \tag{2.9}$$

For a self-adjoint operator T, we denote by $\sigma(T)$ (resp. $\sigma_{\rm d}(T)$) the spectrum (resp. the discrete spectrum) of T.

We denote by $\mathbf{N} := \{1, 2, \cdots,\}$ the set of natural numbers.

The following lemma is easily proven (e.g., [14, Lemma 3.25]).

Lemma 2.1 *Suppose that A is strictly positive and the spectrum of A is purely discrete with*

$$\sigma(A) = \sigma_{\rm d}(A) = \{E_n(A)\}_{n=1}^{\infty}, \tag{2.10}$$

$0 < E_1(A) \leq E_2(A) \leq \cdots$, $E_n(A) \to \infty\, (n \to \infty)$, *counted with algebraic multiplicity. Then the spectrum of $H_{\rm B}(A)$ is puerly discrete with*

$$\sigma(H_{\rm B}(A)) = \sigma_{\rm d}(H_{\rm B}(A)) = \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} k_j E_j(A) \middle| k_j \in \{0\} \cup \mathbf{N} \right\}. \tag{2.11}$$

We denote by $N_{\rm B}$ the number operator on $\mathcal{F}_{\rm B}(\mathcal{H})$:

$$N_{\rm B} := d\Gamma_{\rm B}(I). \tag{2.12}$$

The Boson Fock space $\mathcal{F}_{\rm B}(\mathcal{H})$ is $\mathbf{Z}^2$-graded with

$$\mathcal{F}_{\rm B}(\mathcal{H}) = \mathcal{F}_{\rm B,+}(\mathcal{H}) \oplus \mathcal{F}_{\rm B,-}(\mathcal{H}), \tag{2.13}$$

where

$$\mathcal{F}_{\rm B,+}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\rm s}^{2n} \mathcal{H}, \quad \mathcal{F}_{\rm B,-}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\rm s}^{2n+1} \mathcal{H}. \tag{2.14}$$

The self-adjoint operator $(-1)^{N_{\rm B}}$ is the grading operator of this gradation.

For $s > 0$, we define

$$Z_{\rm B}(s; A) := \mathrm{Tr}\, e^{-sH_{\rm B}(A)}, \tag{2.15}$$

$$\tilde{Z}_{\rm B}(s; A) := \mathrm{Tr} \left\{ (-1)^{N_{\rm B}} e^{-sH_{\rm B}(A)} \right\}, \tag{2.16}$$

provided that $e^{-sH_{\rm B}(A)}$ is trace class on $\mathcal{F}_{\rm B}(\mathcal{H})$, where Tr denotes trace.

Remark 2.2 In statistical mechanics of quantum fields, $Z_{\mathrm{B}}(s;A)$ is called the *partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$ (physically s denotes an *inverse temperature*). The function $\tilde{Z}_{\mathrm{B}}(s;A)$ is not so standard. We call it the *graded partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$. This type of partition function was considered in a concrete case by Spector [31].

To treat the partition functions in a unified way, we introduce a more general partition function

$$Z_{\mathrm{B}}(s,z;A) := \mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}\right) \tag{2.17}$$

with

$$z \in D := \{w \in \mathbf{C} |\, |w| \leq 1\}, \tag{2.18}$$

provided that $e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$. Since $\Gamma_{\mathrm{B}}(1) = I$ and $\Gamma_{\mathrm{B}}(-1) = (-1)^{N_{\mathrm{B}}}$, we have

$$Z_{\mathrm{B}}(s,1;A) = Z_{\mathrm{B}}(s;A), \quad Z_{\mathrm{B}}(s,-1;A) = \tilde{Z}_{\mathrm{B}}(s;A). \tag{2.19}$$

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For each $z \in \mathbf{C}$, we can define an operator $z^{N_{\mathrm{B}}}$ on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ by

$$D(z^{N_{\mathrm{B}}}) := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{B}}(\mathcal{H}) \middle| \sum_{n=0}^{\infty} |z|^{2n} \|\Psi^{(n)}\|^2 < \infty \right\}, \tag{2.20}$$

$$(z^{N_{\mathrm{B}}}\Psi)^{(n)} := z^n \Psi^{(n)}, \quad \Psi \in D(z^{N_{\mathrm{B}}}), \quad n \geq 0. \tag{2.21}$$

It is easy to see that

$$\Gamma_{\mathrm{B}}(z) = z^{N_{\mathrm{B}}}. \tag{2.22}$$

If T is a trace class operator on a Hilbert space, then one can define $\det(I+T)$, the determinant for $I+T$, in an intrinsic way [28, §XIII.17]. It follows that

$$\det(I+T) = \prod_{n=1}^{N(T)} (1 + E_n(T)), \tag{2.23}$$

where $\{E_n(T)\}_{n=1}^{N(T)}$ are the eigenvalues of T counted with algebraic multiplicity [28, Theorem XIII.106]. We set $\det I := 1$.

In what follows, we assume the following.

(A) *The operator A is strictly positive, self-adjoint and, for some $s > 0$, e^{-sA} is trace class on $\mathcal{H}$.*

Remark 2.3 Under Assumption (A), e^{-tA} is trace class on $\mathcal{H}$ for all $t > s$.

Theorem 2.2 *Let $z \in D$. Then the operator $\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ and*

$$Z_{\mathrm{B}}(s,z;A) = \frac{1}{\det\left(1 - ze^{-sA}\right)}. \tag{2.24}$$

In particular,

$$Z_{\mathrm{B}}(s;A) = \frac{1}{\det\left(I - e^{-sA}\right)}, \tag{2.25}$$

$$\tilde{Z}_{\mathrm{B}}(s;A) = \frac{1}{\det\left(I + e^{-sA}\right)}. \tag{2.26}$$

Proof. By the assumption, e^{-sA} is compact. Hence, by the Hilbert-Schmidt theorem, the spectrum of A is purely discrete, satisfying the assumption of Lemma 2.1. Therefore we have

$$\sigma_{\rm d}(\Gamma_{\rm B}(z)e^{-sH_{\rm B}(A)}) = \bigcup_{n=1}^{\infty}\left\{\prod_{j=1}^{n} z^{k_j}e^{-sk_jE_j(A)}|k_j \in \{0\}\cup \mathbf{N}\right\},$$

from which the desired assertion follows. See [14, Theorem 3.26] (cf. also [20, Proposition 5.2.27]). ■

Using Theorem 2.2 and the product law of the determinant det(·), we can derive relations of partition functions at different temperatures:

Theorem 2.3 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$Z_{\rm B}(s,z;A) = \det\left(\sum_{k=0}^{n-1} z^k e^{-ksA}\right) Z_{\rm B}(ns,z^n;A) \tag{2.27}$$

and

$$Z_{\rm B}(s,z;A)Z_{\rm B}(s,-z;A) = Z_{\rm B}(2s,z^2;A). \tag{2.28}$$

In particular,

$$Z_{\rm B}(s;A) = \det\left(\sum_{k=0}^{n-1} e^{-ksA}\right) Z_{\rm B}(ns;A), \tag{2.29}$$

$$Z_{\rm B}(ns;A)\widetilde{Z}_{\rm B}(ns;A) = Z_{\rm B}(2ns;A). \tag{2.30}$$

Proof. By (2.24),

$$\frac{Z_{\rm B}(s,z;A)}{Z_{\rm B}(ns,z^n;A)} = \frac{\det(1-z^n e^{-nsA})}{\det(1-ze^{-sA})}.$$

Note that

$$1 - z^n e^{-nsA} = (1-ze^{-sA})\left(\sum_{k=0}^{n-1} z^k e^{-ksA}\right).$$

Hence, by the product law of det(·) [28, Theorem XIII.105],

$$\det(1-z^n e^{-nsA}) = \det(1-ze^{-sA})\det\left(\sum_{k=0}^{n-1} z^k e^{-ksA}\right).$$

Thus (2.27) follows. ■

Remark 2.4 In general, relationships among theories at different coupling constants are referred to as "duality" [31]. Eq.(2.30) is a duality relation, where the coupling constant is the inverse temperature.

In statistical mechanics, *correlation functions* are also important objects. Let $f \in D(A^{-1/2})$ and $g \in D(A^{-1/2}) \cap D(A)$. Then, using the well known estimates

$$\|a(f)\Psi\|_{\mathcal{F}_{\mathrm{B}}(\mathcal{H})} \leq \|A^{-1/2}f\|_{\mathcal{H}}\|H_{\mathrm{B}}(A)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{B}}(\mathcal{H})}, \tag{2.31}$$
$$\|a(f)^*\Psi\|_{\mathcal{F}_{\mathrm{B}}(\mathcal{H})} \leq \|A^{-1/2}f\|_{\mathcal{H}}\|H_{\mathrm{B}}(A)^{1/2}\Psi\|_{\mathcal{F}_{\mathrm{B}}(\mathcal{H})} + \|f\|_{\mathcal{H}}\|\Psi\|_{\mathcal{F}_{\mathrm{B}}(\mathcal{H})} \tag{2.32}$$

and commutation properties of the annihilation and creation operators with $H_{\mathrm{B}}(A)$, one can show that $a(f)^*a(g)$ is $H_{\mathrm{B}}(A)$-bounded (cf. [5, §II]). Hence, for all $t > 0$, $a(f)^*a(g)e^{-tH_{\mathrm{B}}(A)}$ is bounded. Let $t > s$. Then, by the identity

$$a(f)^*a(g)e^{-tH_{\mathrm{B}}(A)} = a(f)^*a(g)e^{-(t-s)H_{\mathrm{B}}(A)}e^{-sH_{\mathrm{B}}(A)},$$

$a(f)^*a(g)e^{-tH_{\mathrm{B}}(A)}$ is trace class. Hence we can define

$$R_{\mathrm{B}}(t,z;f,g;A) := \frac{\mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)a_{\mathcal{H}}(f)^*a_{\mathcal{H}}(g)e^{-tH_{\mathrm{B}}(A)}\right)}{Z_{\mathrm{B}}(t,z;A)}, \quad z \in D. \tag{2.33}$$

This is called a *two-point correlation function*. In the same manner as in [20, Proposition 5.2.28], we can show that

$$R_{\mathrm{B}}(t,z;f,g;A) = (g, ze^{-tA}(1-ze^{-tA})^{-1}f)_{\mathcal{H}},. \tag{2.34}$$

Remark 2.5 Correlation functions of the form

$$\mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)a_{\mathcal{H}}(f_1)^*\cdots a_{\mathcal{H}}(f_n)^*a_{\mathcal{H}}(g_1)\cdots a_{\mathcal{H}}(g_m)e^{-tH_{\mathrm{B}}(A)}\right)/Z_{\mathrm{B}}(t,z;A)$$

can be defined for f_j, g_k in a dense subspace in $\mathcal{H}$. These are computed in terms of two-point correlation functions [20, §5.2]. Moreover, we can consider a perturbation of $H_{\mathrm{B}}(A)$ by a symmetric operator V on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ such that $H := H_{\mathrm{B}}(A) + V$ defines a self-adjoint operator in the sense of sequilinear form and derive trace formulas for the heat semi-group e^{-sH} ($s > 0$) [13].

2.2 Arithmetical aspects

Let the spectrum of A be as in (2.10) and denote by ϕ_n a normalized eigenvector of A with eigenvalue $E_n(A)$:

$$A\phi_n = E_n(A)\phi_n \tag{2.35}$$

such that $\{\phi_n\}_{n=1}^\infty$ is an orthonormal system of $\mathcal{H}$. Then $\{\phi_n\}_{n=1}^\infty$ is complete in $\mathcal{H}$. We set

$$a_n := a_{\mathcal{H}}(\phi_n) \tag{2.36}$$

Then we have

$$[a_n, a_m^*] = \delta_{mn}, \quad [a_n, a_m] = 0, \quad [a_n^*, a_m^*] = 0, \quad n, m \geq 1, \tag{2.37}$$

on $\mathcal{F}_{\mathrm{B},0}(\mathcal{H})$.

We denote by

$$\mathcal{P} := \{p_n\}_{n=1}^\infty \tag{2.38}$$

the set of all prime numbers with $p_n < p_{n+1}$, $n \geq 1$ ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \cdots$).

By definition, an arithmetical function is a complex-valued function on $\mathbf{N}$. An arithmetical function f is called *completely multiplicative* if it satisfies

$$f(1) = 1, \quad f(mn) = f(m)f(n), \quad m, n \in \mathbf{N}.$$

Let $N \geq 2$ be a natural namber. Then, by the fundamental theorem of arithmetic, there exists a unique set $\{i_1, \cdots, i_n, \alpha_1, \cdots, \alpha_n\} \subset \mathbf{N}$ ($i_1 < \cdots < i_n$) such that

$$N = (p_{i_1})^{\alpha_1} \cdots (p_{i_n})^{\alpha_n}. \tag{2.39}$$

Then we define an arithmetical function $\gamma(N)$ by

$$\gamma(N) := \sum_{k=1}^{n} \alpha_k \tag{2.40}$$

and $\gamma(1) := 0$.

The arithmetical function defined by $\lambda(1) := 1$ and

$$\lambda(N) := (-1)^{\gamma(N)} \tag{2.41}$$

is called the *Liouville fucntion* [1, §2.12]. This function is completely multiplicative.

Using the representation (2.39) of N, we can define a vector $\Psi_N \in \mathcal{F}_{\mathrm{B}}(\mathcal{H})$ by

$$\Psi_N := C_N (a_{i_1}^*)^{\alpha_1} \cdots (a_{i_n}^*)^{\alpha_n} \Omega_{\mathcal{H}}, \tag{2.42}$$

where

$$C_N := \frac{1}{\sqrt{\alpha_1! \cdots \alpha_n!}}$$

is a normalization constant so that $\|\Psi_N\| = 1$. We set

$$\Psi_1 := \Omega_{\mathcal{H}}. \tag{2.43}$$

Lemma 2.4 *The set $\{\Psi_N\}_{N=1}^{\infty}$ is a complete orthonormal system (CONS) of $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$.*

Proof. This is due to the fact that $\mathcal{D} := \{\Omega_{\mathcal{H}},\ a_{j_1}^* \cdots a_{j_n}^* \Omega_{\mathcal{H}} | n \geq 1, j_1 \leq j_2 \leq \cdots \leq j_n, j_k \geq 1, k = 1, \cdots, n\}$ is a complete orthogonal system of $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$. Note that $\{\Psi_N\}_{N=1}^{\infty}$ is just obtained by relabeling and normalizing vectors in $\mathcal{D}$. ∎

Remark 2.6 The set $\{\Psi_N\}_{n=1}^{\infty}$ of vectors was introduced in [30].

Lemma 2.5 *For all $N \in \mathbf{N}$, Ψ_N is a unique eigenvector (up to constant multiples) of $\Gamma_{\mathrm{B}}(z)$ with eigenvalue $z^{\gamma(N)}$.*

Proof. Let N be as in (2.39). Then $N_{\mathrm{B}}\Psi_N = \gamma(N)\Psi_N$, which, combined with (2.22) implies the desired assertion. ∎

We introduce a function $F_A : \mathbf{N} \to (0, \infty)$ as follows: $F_A(1) := 1$ and if $N \geq 2$ is represented as (2.39), then

$$F_A(N) := \prod_{k=1}^{n} e^{\alpha_k E_{i_k}(A)}. \tag{2.44}$$

It is easy to see that F_A is compeltely multiplicative.

Lemma 2.6 *For all $N \in \mathbf{N}$, Ψ_N is a unique eigenvector (up to constant multiples) of $H_{\mathrm{B}}(A)$ with eigenvalue $\log F_A(N)$.*

Proof. Let N be given by (2.39). Then

$$H_{\mathrm{B}}(A)\Psi_N = \left(\sum_{k=1}^{n} \alpha_k E_{i_k}(A)\right) \Psi_N = (\log F_A(N))\Psi_N. \tag{2.45}$$

Hence the desired assertion follows. ■

By Lemmas 2.5 and 2.6, we have

$$Z_{\mathrm{B}}(s,z;A) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s}, \quad z \in D. \tag{2.46}$$

Theorem 2.7 *For all $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s} = \frac{1}{\prod_{n=1}^{\infty}(1 - ze^{-sE_n(A)})}. \tag{2.47}$$

In particular,

$$\sum_{N=1}^{\infty} \frac{1}{F_A(N)^s} = \frac{1}{\prod_{n=1}^{\infty}(1 - e^{-sE_n(A)})}, \tag{2.48}$$

$$\sum_{N=1}^{\infty} \frac{\lambda(N)}{F_A(N)^s} = \frac{1}{\prod_{n=1}^{\infty}(1 + e^{-sE_n(A)})}. \tag{2.49}$$

Proof. By (2.24), we have

$$Z_{\mathrm{B}}(s,z;A) = \frac{1}{\prod_{n=1}^{\infty}(1 - ze^{-sE_n(A)})},$$

which, combined with (2.46), implies (2.47). ■

Remark 2.7 Formulas (2.47) may be regarded as a general form unifying arithmetical formulas known under the name of *Euler products* [1, Chapter 11]. See Section 2.3 below.

We introduce a function $\varrho(N,m) : \mathbf{N} \times \mathbf{N} \to \{0\} \cup \mathbf{N}$ by

$$\varrho(1,m) := 0, \tag{2.50}$$

$$\varrho(N,m) := \sum_{k=1}^{n} \alpha_k \delta_{i_k m} \tag{2.51}$$

if $N \geq 2$ is expressed as (2.39) $(N, m \in \mathbf{N})$.

Theorem 2.8 *Let $t > s$. Then, for all $m \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}\varrho(N,m)}{F_A(N)^t} = \frac{z}{e^{tE_m(A)} - z} Z_{\mathrm{B}}(t,z;A). \tag{2.52}$$

Proof. Putting $f = g = \phi_m$ in (2.34), we have $R_{\rm B}(t,z;\phi_m,\phi_m;A) = z/(e^{tE_m(A)} - z)$. It is easy to see that

$$\mathrm{Tr}\left(\Gamma_{\rm B}(z)a_m^* a_m e^{-tH_{\rm B}(A)}\right) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}\varrho(N,m)}{F_A(N)^t}.$$

Thus (2.52) follows. ∎

Let $N \geq 2$ be given as (2.39). Then, each divisor m of N is of the form

$$m = p_{i_1}^{r_1} \cdots p_{i_n}^{r_n} \tag{2.53}$$

with $0 \leq r_j \leq \alpha_j$, $j = 1, \cdots, n$. We define a vector $\Psi_{N,m} \in \mathcal{F}_{\rm B}(\mathcal{H})$ by

$$\Psi_{N,m} := C_{N,m} a_{i_1}^{*\,r_1} \cdots a_{i_n}^{*\,r_n} \Omega_{\mathcal{H}}, \tag{2.54}$$

where $C_{N,m} > 0$ is a normalization constant. For an $m \in \mathbf{N}$ and $N \in \mathbf{N}$, we mean by $m|N$ that m is a divisor of N. The set $\{\Psi_{N,m}\}_{m|N}$ of vectors is orthonormal. We introduce

$$\mathcal{F}_{\rm B}^{(N)}(\mathcal{H}) := \mathcal{L}\{\Psi_{N,m}\}_{m|N}, \tag{2.55}$$

where $\mathcal{L}\{\cdot\}$ means the subspace spanned algebraically by the vectors in the set $\{\cdot\}$. We set $\mathcal{F}_{\rm B}^{(1)} := \{\alpha\Omega_{\mathcal{H}} | \alpha \in \mathbf{C}\}$. We denote by P_N the orthogonal projection from $\mathcal{F}_{\rm B}(\mathcal{H})$ onto $\mathcal{F}_{\rm B}^{(N)}(\mathcal{H})$.

Proposition 2.9 *Let $z \in D$. Then, for all N,*

$$\mathrm{Tr}\left(P_N \Gamma_{\rm B}(z) e^{-sH_{\rm B}(A)} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{F_A(m)^s}. \tag{2.56}$$

Proof. We have $N_{\rm B}\Psi_{N,m} = \gamma(m)\Psi_{N,m}$ and $H_{\rm B}(A)\Psi_{N,m} = \log F_A(m)\Psi_{N,m}$. From these facts the desired formula follows. ∎

2.3 Applications to analytic number theory

A basic object in analytic number theory is the *Dirichlet series*

$$D(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{2.57}$$

for an arithmetical function f and $s \in \mathbf{C}$, provided that the infinite series converges. The *Riemann zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1, \tag{2.58}$$

is a special case of $D(s,f)$, i.e., the case $f \equiv 1$. We first show that $\zeta(s)$ and $D(s,\lambda)$ can be represented as partition functions of $H_{\mathrm{B}}(A)$ with a suitable A. For this purpose, we consider the case where $\mathcal{H}$ is given by

$$\ell^2 := \oplus_{n=1}^{\infty}\mathbf{C} = \left\{\psi = \{\psi_n\}_{n=1}^{\infty} \middle| \psi_n \in \mathbf{C}, n \geq 1, \sum_{n=1}^{\infty} |\psi_n|^2 < \infty\right\}. \tag{2.59}$$

On this Hilbert space we define an operator $\omega_{\mathcal{P}}$ as follows:

$$D(\omega_{\mathcal{P}}) = \left\{\psi = \{\psi_n\}_{n=1}^{\infty} \in \ell^2 \middle| \sum_{n=1}^{\infty} |(\log p_n)\psi_n|^2 < \infty\right\}, \tag{2.60}$$

$$(\omega_{\mathcal{P}}\psi)_n = (\log p_n)\psi_n, \quad \psi \in D(\omega_{\mathcal{P}}),\ n \geq 1. \tag{2.61}$$

Then $\omega_{\mathcal{P}}$ is strictly positive and self-adjoint. Moreover, the spectrum of $\omega_{\mathcal{P}}$ is purely discrete with

$$\sigma(\omega_{\mathcal{P}}) = \sigma_{\mathrm{d}}(\omega_{\mathcal{P}}) = \{\log p_n\}_{n=1}^{\infty} \tag{2.62}$$

with the multiplicity of each eigenvalue $\log p_n$ being one. A normalized eigenvector of $\omega_{\mathcal{P}}$ with eigenvalue $\log p_n$ is given by

$$e_n := \{\delta_{nj}\}_{j=1}^{\infty} \in \ell^2. \tag{2.63}$$

Theorem 2.10 *For all $s > 1$ and $z \in D$,*

$$Z_{\mathrm{B}}(s,z;\omega_{\mathcal{P}}) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s}. \tag{2.64}$$

In particular,

$$\zeta(s) = Z_{\mathrm{B}}(s;\omega_{\mathcal{P}}), \tag{2.65}$$

$$D(s,\lambda) = \widetilde{Z}_{\mathrm{B}}(s;\omega_{\mathcal{P}}). \tag{2.66}$$

Proof. One can easily show that

$$F_{\omega_{\mathcal{P}}}(N) = N. \tag{2.67}$$

Hence, by (2.46), we obtain (2.64). ∎

Applying Theorem 2.7 with $A = \omega_{\mathcal{P}}$, we obtain the following.

Corollary 2.11 *For all $s > 1$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s} = \frac{1}{\prod_{p\in\mathcal{P}}(1 - zp^{-s})}. \tag{2.68}$$

In particular,

$$\zeta(s) = \frac{1}{\prod_{p\in\mathcal{P}}(1 - p^{-s})}, \tag{2.69}$$

$$D(s,\lambda) = \frac{1}{\prod_{p\in\mathcal{P}}(1 + p^{-s})}. \tag{2.70}$$

An application of Theprem 2.8 gives the following.

Corollary 2.12 *For all $s > 1$, $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}\varrho(N,n)}{N^s} = \frac{z}{p_n^s - z} Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}). \tag{2.71}$$

In particular,

$$\sum_{N=1}^{\infty} \frac{\varrho(N,n)}{N^s} = \frac{\zeta(s)}{p_n^s - 1}, \tag{2.72}$$

$$\sum_{N=1}^{\infty} \frac{\lambda(N)\varrho(N,n)}{N^s} = -\frac{D(s,\lambda)}{p_n^s + 1}. \tag{2.73}$$

Moreover, (2.30) yields the following.

Corollary 2.13 *Let $s > 1$. Then*

$$D(s,\lambda) = \frac{\zeta(2s)}{\zeta(s)}. \tag{2.74}$$

The operator $\omega_{\mathcal{P}}$ may be regarded as as a special case of a more general operator associated with a completely multiplicative function. Let f be a completely multiplicative function such that $0 < f(n) < 1$ for all $n \geq 2$ and

$$\sum_{n=1}^{\infty} f(p_n) < \infty, \tag{2.75}$$

and define an operator A_f on ℓ^2 by

$$D(A_f) = \left\{ \psi = \{\psi_n\}_{n=1}^{\infty} \middle| \sum_{n=1}^{\infty} |\log f(p_n)|^2 |\psi_n|^2 < \infty \right\}, \tag{2.76}$$

$$(A_f\psi)_n = [-\log f(p_n)]\psi_n, \quad \psi \in D(A_f),\ n \geq 1. \tag{2.77}$$

Then A_f is a strictly positive self-adjoint operator. Since

$$\sum_{n=1}^{\infty} e^{\log f(p_n)} = \sum_{n=1}^{\infty} f(p_n) < \infty,$$

the self-adjoint operator e^{-A_f} is trace class on ℓ^2. It is easy to see that

$$F_{A_f}(N) = \frac{1}{f(N)}, \quad N \in \mathbf{N}. \tag{2.78}$$

Hence we have

$$Z_{\mathrm{B}}(1, z; A_f) = \sum_{n=1}^{\infty} z^{\gamma(n)} f(n), \quad z \in D. \tag{2.79}$$

In particular,

$$Z_{\rm B}(1;A_f) = \sum_{n=1}^{\infty} f(n), \tag{2.80}$$

$$\widetilde{Z}_{\rm B}(1;A_f) = \sum_{n=1}^{\infty} f(n)\lambda(n). \tag{2.81}$$

Applying Theorem 2.7, we obtain the following fact.

Corollary 2.14 *Let f be as above. Then, for all $z \in D$,*

$$\sum_{n=1}^{\infty} z^{\gamma(n)} f(n) = \frac{1}{\prod_{p\in\mathcal{P}}(1 - zf(p))}. \tag{2.82}$$

In particular,

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{\prod_{p\in\mathcal{P}}(1 - f(p))}, \tag{2.83}$$

$$\sum_{n=1}^{\infty} f(n)\lambda(n) = \frac{1}{\prod_{p\in\mathcal{P}}(1 + f(p))}. \tag{2.84}$$

(2.85)

Theorem 2.8 gives the following.

Corollary 2.15 *Let f be as above. Then, for all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} z^{\gamma(N)} \varrho(N,n) f(N) = \frac{zf(p_n)}{1 - zf(p_n)} Z_{\rm B}(1,z;A_f). \tag{2.86}$$

In particular,

$$\sum_{N=1}^{\infty} \varrho(N,n) f(N) = \frac{f(p_n)}{1 - f(p_n)} Z_{\rm B}(1;A_f), \tag{2.87}$$

$$\sum_{N=1}^{\infty} \varrho(N,n)\lambda(N) f(N) = -\frac{f(p_n)}{1 + f(p_n)} \widetilde{Z}_{\rm B}(1;A_f). \tag{2.88}$$

Applying Proposition 2.9, we have for all $s > 1$

$$\mathrm{Tr}\left(P_N z^{N_{\rm B}} e^{-sH_{\rm B}(\omega_{\mathcal{P}})} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{m^s}, \quad z \in D. \tag{2.89}$$

In particular,

$$\mathrm{Tr}\left(P_N e^{-sH_{\rm B}(\omega_{\mathcal{P}})} P_N\right) = \sum_{m|N} \frac{1}{m^s}, \tag{2.90}$$

$$\mathrm{Tr}\left(P_N (-1)^{N_{\rm B}} e^{-sH_{\rm B}(\omega_{\mathcal{P}})} P_N\right) = \sum_{m|N} \frac{\lambda(m)}{m^s}. \tag{2.91}$$

3 Fermion Fock Spaces and Arithmetical Functions

3.1 Partition functions and correlation functions

Let $\mathcal{K}$ be a separable infinite dimensional Hilbert space and $\otimes_{\rm as}^n \mathcal{K}$ be the n-fold antisymmetric tensor product Hilbert space of $\mathcal{K}$ ($n = 0, 1, 2, \cdots$; $\otimes^0 \mathcal{K} := \mathbf{C}$). Then the Fermion Fock space over $\mathcal{K}$ is defined by

$$\mathcal{F}_{\rm F}(\mathcal{K}) := \oplus_{n=0}^{\infty} \otimes_{\rm as}^n \mathcal{K} \tag{3.1}$$

(e.g., [20, §5.2], [26, §II.4]).

We denote by $b_{\mathcal{K}}(u)$ ($u \in \mathcal{K}$) the annihilation operator on $\mathcal{F}_{\rm F}(\mathcal{K})$ (e.g., [20, §5.2]). $b_{\mathcal{K}}(u)$ is antilinear in u and bounded with $\|b(u)\| = \|u\|_{\mathcal{K}}$. The set $\{b_{\mathcal{K}}(u), b_{\mathcal{K}}(u)^* | u \in \mathcal{K}\}$ satisfies the canonical anti-commutation relations

$$\{b_{\mathcal{K}}(u), b_{\mathcal{K}}(v)^*\} = (u, v)_{\mathcal{K}}, \tag{3.2}$$

$$\{b_{\mathcal{K}}(u), b_{\mathcal{K}}(v)\} = 0, \quad \{b_{\mathcal{K}}(u)^*, b_{\mathcal{K}}(v)^*\} = 0, \quad u, v \in \mathcal{K}, \tag{3.3}$$

where $\{X, Y\} := XY + YX$. We denote by $\Omega_{\mathcal{K}} := \{1, 0, 0, \cdots\}$ the Fock vacuum in $\mathcal{F}_{\rm F}(\mathcal{K})$. We have

$$b_{\mathcal{K}}(u)\Omega_{\mathcal{K}} = 0, \quad u \in \mathcal{K}. \tag{3.4}$$

Let T be a nonnegative self-adjoint operator on $\mathcal{K}$ and $d\Gamma_{\rm F}(T)$ the second quantization of T in $\mathcal{F}_{\rm F}(\mathcal{K})$:

$$d\Gamma_{\rm F}(T) := \oplus_{n=0}^{\infty} d\Gamma_{\rm F}^{(n)}(T) \tag{3.5}$$

with $d\Gamma_{\rm F}^{(0)}(T) := 0$ and

$$d\Gamma_{\rm F}^{(n)}(T) := I \otimes \cdots \otimes I \otimes \overset{n\text{th}}{\breve{T}} \otimes I \otimes \cdots \otimes I.$$

Then $d\Gamma_{\rm F}(T)$ is nonnegative and self-adjoint.

We set

$$H_{\rm F}(T) := d\Gamma_{\rm F}(T). \tag{3.6}$$

Lemma 3.1 *Suppose that T is strictly positive and the spectrum of T is purely discrete with*

$$\sigma(T) = \sigma_{\rm d}(T) = \{E_n(T)\}_{n=1}^{\infty}, \tag{3.7}$$

$0 < E_1(T) \leq E_2(T) \leq \cdots$, $E_n(T) \to \infty \, (n \to \infty)$, *counted with algebraic multiplicity. Then the spectrum of $H_{\rm F}(T)$ is puerly discrete with*

$$\sigma(H_{\rm F}(T)) = \sigma_{\rm d}(H_{\rm F}(T)) = \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} k_j E_j(T) | k_j = 0, 1 \right\}. \tag{3.8}$$

Proof. See, e.g., [14, Lemma 4.3]. ■

Remark 3.1 Note the difference between the spectral property of $H_{\rm B}(A)$ (Lemma 2.1) and that of $H_{\rm F}(T)$.

We set

$$N_{\rm F} := d\Gamma_{\rm F}(I), \tag{3.9}$$

the number operator on $\mathcal{F}_{\rm F}(\mathcal{K})$.

As in the Boson Fock space, $\mathcal{F}_{\rm F}(\mathcal{K})$ is $\mathbf{Z}_2$-graded with

$$\mathcal{F}_{\rm F}(\mathcal{K}) = \mathcal{F}_{{\rm F},+}(\mathcal{K}) \oplus \mathcal{F}_{{\rm F},-}(\mathcal{K}), \tag{3.10}$$

where

$$\mathcal{F}_{{\rm F},+}(\mathcal{K}) := \oplus_{n=0}^{\infty} \otimes_{\rm as}^{2n} \mathcal{K}, \quad \mathcal{F}_{{\rm F},-}(\mathcal{K}) := \oplus_{n=0}^{\infty} \otimes_{\rm as}^{2n+1} \mathcal{K}. \tag{3.11}$$

The self-adjoint operator $(-1)^{N_{\rm F}}$ is the grading operator of this gradation.

For all $z \in D$, the operator $\Gamma(z)$ on the full Fock space $\mathcal{F}_{\rm full}(\mathcal{K})$ over $\mathcal{K}$ is reduced by $\mathcal{F}_{\rm F}(\mathcal{K})$. We denote by $\Gamma_{\rm F}(z)$ its reduced part. As in the case of the Boson Fock space, we have

$$\Gamma_{\rm F}(z) = z^{N_{\rm F}}. \tag{3.12}$$

Let $s > 0$, $z \in D$ and

$$Z_{\rm F}(s, z; T) := {\rm Tr}\left(\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)}\right), \tag{3.13}$$

provided that $e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{H})$. In particular, we define

$$Z_{\rm F}(s;T) := Z_{\rm F}(s,1;T) = {\rm Tr}\, e^{-sH_{\rm F}(T)}, \tag{3.14}$$

$$\tilde{Z}_{\rm F}(s;T) := Z_{\rm F}(s,-1;T) = {\rm Tr}\left\{(-1)^{N_{\rm F}} e^{-sH_{\rm F}(T)}\right\}. \tag{3.15}$$

In what follows, we assume the following.

(T) *For some $s > 0$, e^{-sT} is trace class on $\mathcal{K}$.*

Theorem 3.2 *For all $z \in D$, $\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{K})$ and*

$$Z_{\rm F}(s, z; T) = \det(I + z e^{-sT}). \tag{3.16}$$

In particular,

$$Z_{\rm F}(s;T) = \det(I + e^{-sT}), \tag{3.17}$$

$$\tilde{Z}_{\rm F}(s;T) = \det(I - e^{-sT}). \tag{3.18}$$

Proof. We have

$$\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)} = \Gamma(z e^{-sT}) | \mathcal{F}_{\rm F}(\mathcal{K}). \tag{3.19}$$

Hence, by the definition of $\det(I + \cdot)$ [28, p.323], we obtain (3.16). ∎

Remark 3.2 By (3.8) and the functional calculus, we have

$$\sigma_{\rm d}(\Gamma_{\rm F}(z)e^{-sH_{\rm F}(T)}) = \bigcup_{n=1}^{\infty}\left\{\prod_{j=1}^{n} z^{k_j}e^{-sk_jE_j(T)}|k_j = 0,1\right\}.$$

One can use this relation to prove (3.16)(e.g., [14, Theorem 4.4]).

Remark 3.3 In Theorem 3.2, we do not need assume that T is *strictly* positive.

By Theorems 2.2 and 3.2, we have interesting relations between bosonic and fermionic partition functions:

Corollary 3.3 *Consider the case $\mathcal{H} = \mathcal{K}$ and A be an operator on $\mathcal{H}$ obeying Assumption (A) in Section 2. Then, for all $z \in D$,*

$$Z_{\rm B}(s,-z;A) = \frac{1}{Z_{\rm F}(s,z;A)}. \tag{3.20}$$

In particular,

$$Z_{\rm B}(s;A) = \frac{1}{\tilde{Z}_{\rm F}(s;A)}, \quad \tilde{Z}_{\rm B}(s;A) = \frac{1}{Z_{\rm F}(s;A)}. \tag{3.21}$$

In the same way as in Theorem 2.3, we can prove the following.

Theorem 3.4 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$Z_{\rm F}(ns,-z^n;T) = \det\left(\sum_{k=1}^{n-1} z^k e^{-skT}\right) Z_{\rm F}(s,-z;T), \tag{3.22}$$

$$Z_{\rm F}(s,-z;T)Z_{\rm F}(s,z;T) = Z_{\rm F}(2s,-z^2;T). \tag{3.23}$$

In particular,

$$\tilde{Z}_{\rm F}(ns;T) = \det\left(\sum_{k=0}^{n-1} e^{-ksT}\right)\tilde{Z}_{\rm F}(s;T), \tag{3.24}$$

$$\tilde{Z}_{\rm F}(s;T)Z_{\rm F}(s;T) = \tilde{Z}_{\rm F}(2s;T). \tag{3.25}$$

Remark 3.4 If T is strictly positive, these relations follow from Theorem 2.3 and (3.21).

Remark 3.5 Relation (3.23) is a form of *duality* of fermionic partition functions. A special case is discussed in [31].

Corollary 3.5 *Consider the case $\mathcal{H} = \mathcal{K}$ and A be an operator on $\mathcal{H}$ obeying (A). Then*

$$Z_{\rm B}(2s,z^2;A)Z_{\rm F}(s,z;A) = Z_{\rm B}(s,z;A) \tag{3.26}$$

Proof. This follows from (3.23) and Corollary 3.3. ■

Remark 3.6 Relation (3.26) is also a form of *duality* of fermionic and bosonic partition functions. For a special case, see [31].

Let $u, v \in \mathcal{K}$ and $z \in D$. Then a *fermionic two-point correlation function* is defined by

$$R_{\mathrm{F}}(s, z; u, v; T) := \frac{\mathrm{Tr}\left(z^{N_{\mathrm{F}}} e^{-sH_{\mathrm{F}}(T)} b_{\mathcal{K}}(u)^* b_{\mathcal{K}}(v)\right)}{Z_{\mathrm{F}}(s, z; T)}. \tag{3.27}$$

It is easy to see (e.g., cf. [20]) that

$$R_{\mathrm{F}}(s, z; u, v; T) = (v, ze^{-sT}(1 + ze^{-sT})^{-1}u)_{\mathcal{K}}. \tag{3.28}$$

Remark 3.7 Correlation functions of the form

$$\mathrm{Tr}\left(z^{N_{\mathrm{F}}} e^{-sH_{\mathrm{F}}(T)} b_{\mathcal{K}}(u_1)^* \cdots b_{\mathcal{K}}(u_n)^* b_{\mathcal{K}}(v_1) \cdots b_{\mathcal{K}}(v_m)\right) / Z_{\mathrm{F}}(s, z; T)$$

$(u_j, v_k \in \mathcal{K})$ are defined. These are computed in terms of two-point correlation functions ([20, §5.2], [3, 7]).

3.2 Arithmetical aspects

Let the spectrum of T be as in (3.7) and denote by u_n a normalized eigenvector of T with eigenvalue $E_n(T)$:

$$Tu_n = E_n(T)u_n \tag{3.29}$$

such that $\{u_n\}_{n=1}^{\infty}$ is an orthonormal system. Then $\{u_n\}_{n=1}^{\infty}$ is complete in $\mathcal{K}$. We set

$$b_n := b_{\mathcal{K}}(u_n). \tag{3.30}$$

Then we have

$$\{b_n, b_m^*\} = \delta_{mn}, \quad \{b_n, b_m\} = 0, \quad \{b_n^*, b_m^*\} = 0, \quad n, m \geq 1. \tag{3.31}$$

In particular, $b_n^2 = 0,\ b_n^{*2} = 0,\ n \in \mathbf{N}$.

For $N \in \mathbf{N}$ we define $\nu(N)$ by $\nu(1) := 1$ and

$$\nu(N) = n, \quad N \geq 2, \tag{3.32}$$

if N is represented as (2.39) [1, p.247].

A natural number $m \geq 2$ is called *square free* if it is written as a product of mutually different prime numbers. As a convention, 1 is defined to be square free. We denote by $\mathcal{S}_0$ the set of square free elements in $\mathbf{N}$:

$$\mathcal{S}_0 := \{m \in \mathbf{N} | m \text{ is square free}\}. \tag{3.33}$$

For each $N \in \mathbf{N}$, we define a set $\mathcal{S}_0(N)$ as follows:

$$\mathcal{S}_0(1) := \{1\}, \tag{3.34}$$

$$\mathcal{S}_0(N) := \{m \in \mathcal{S}_0 | m \text{ is a divisor of } N\}, \quad N \geq 2. \tag{3.35}$$

Let $N \geq 2$ be given as (2.39). Then each element m of $\mathcal{S}_0(N)$ is of the form

$$m = p_{i_1}^{q_1} \cdots p_{i_n}^{q_n}, \tag{3.36}$$

where $q_j = 0$ or $q_j = 1$ $(j = 1, \cdots, n)$. Corresponding to this, we define a vector $\Phi_{N,m}$ by

$$\Phi_{N,m} := b_{i_1}^{*}{}^{q_1} \cdots b_{i_n}^{*}{}^{q_n} \Omega_{\mathcal{K}}. \tag{3.37}$$

Let

$$\mathcal{F}_{\mathrm{F}}^{(1)}(\mathcal{K}) := \{c\Omega_{\mathcal{K}} | c \in \mathbf{C}\}, \quad \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) := \mathcal{L}\{\Phi_{N,m} | m \in \mathcal{S}_0(N)\}, \quad N \geq 2. \tag{3.38}$$

Then $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$ is finite dimensional with $\dim \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) = 2^{\nu(N)}$. We denote by R_N the orthogonal projection from $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ onto $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$.

Let $N \geq 2$ be of the form (2.39),

$$\mathcal{K}_N := \mathcal{L}\{u_{i_k} | k = 1, \cdots, n\} \tag{3.39}$$

and T_N be the restriction of T to $\mathcal{K}_N$. Then we can show that

$$\mathrm{Tr}\left(R_N z^{N_{\mathrm{F}}} e^{-sH_{\mathrm{F}}(T)} R_N\right) = \det(1 + ze^{-sT_N}). \tag{3.40}$$

Let $m \in \mathcal{S}_0, m \geq 2$ and

$$m = p_{i_1} \cdots p_{i_r} \tag{3.41}$$

be its factorization in prime numbers $(i_j \neq i_k,\ j \neq k)$. Then we define a vector Φ_m in $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ by

$$\Phi_m := b_{i_1}^* \cdots b_{i_r}^* \Omega_{\mathcal{K}}. \tag{3.42}$$

For $m = 1$, we set

$$\Phi_1 := \Omega_{\mathcal{K}}. \tag{3.43}$$

For $m \notin \mathcal{S}_0$, we define

$$\Phi_m := 0. \tag{3.44}$$

Lemma 3.6 *The set $\{\Phi_m\}_{m \in \mathcal{S}_0}$ is a CONS of $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$.*

Proof. By using (3.31) and (3.4), one easily sees that $\{\Phi_m\}_{m \in \mathcal{S}_0}$ is an orthonormal system of $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$. It is well known that $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ is generated by $\Omega_{\mathcal{K}}$ and the vectors of the form $b_{i_1}^* \cdots b_{i_r}^* \Omega_{\mathcal{K}}$ with $r \in \mathbf{N}$, $i_j \in \mathbf{N}$, $i_j \neq i_k$, $j \neq k$. Thus the assertion follows. ■

Remark 3.8 The set $\{\Phi_m\}_m$ was introduced in [30].

The *Möbius function* $\mu : \mathbf{N} \to \{0, \pm 1\}$ is defined as follows: $\mu(1) := 1$, $\mu(m) := 0$ if $m \notin \mathcal{S}_0$ and $\mu(m) := (-1)^r$ if m is written as the product of mutually different r prime numbers. We have

$$\mu(m) = (-1)^{\gamma(m)}, \quad m \in \mathcal{S}_0. \tag{3.45}$$

Lemma 3.7 *For all $m \in \mathcal{S}_0$, Φ_m is an eigenvector of N_{F} with eigenvalue $\gamma(m)$. In particular, for all $m \in \mathcal{S}_0$ and $z \in D$,*

$$z^{N_{\mathrm{F}}}\Phi_m = z^{\gamma(m)}\Phi_m, \quad (-1)^{N_{\mathrm{F}}}\Phi_m = \mu(m)\Phi_m. \tag{3.46}$$

Proof. Let $m \in \mathcal{S}_0$ be as in (3.41). Then $N_{\mathrm{F}}\Phi_m = r\Phi_m = \gamma(m)\Phi_m$, which implies that $z^{N_{\mathrm{F}}}\Phi_m = z^{\gamma(m)}\Phi_m$. ■

Lemma 3.8 *For all $m \in \mathcal{S}_0$, Φ_m is an eigenvector of $H_{\mathrm{F}}(T)$ with eigenvalue $\log F_T(m)$, where F_T is defined by (2.44) with $A = T$.*

Proof. Let $m \in \mathcal{S}_0$ be expressed as (3.41). Then

$$H_{\mathrm{F}}(T)\Phi_m = (E_{i_1}(T) + \cdots + E_{i_r}(T))\Phi_m = (\log F_T(m))\Phi_m.$$

Hence the desired assertion follows. ■

It follows from Lemmas 3.7 and 3.8 that

$$Z_{\mathrm{F}}(s, z; T) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s}, \quad z \in D, \tag{3.47}$$

where we have used that $\mu(m) = 0$ for all $m \notin \mathcal{S}_0$ and $|\mu(m)| = 1$ for all $m \in \mathcal{S}_0$. In particular,

$$Z_{\mathrm{F}}(s; T) = \sum_{m=1}^{\infty} \frac{|\mu(m)|}{F_T(m)^s}, \tag{3.48}$$

$$\tilde{Z}_{\mathrm{F}}(s; T) = \sum_{m=1}^{\infty} \frac{\mu(m)}{F_T(m)^s}, \tag{3.49}$$

By (3.47) and Theorem 3.2, we obtain the following.

Theorem 3.9 *Let $z \in D$. Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \prod_{n=1}^{\infty}\left(1 + z e^{-sE_n(T)}\right). \tag{3.50}$$

In particular,

$$\sum_{m=1}^{\infty} \frac{|\mu(m)|}{F_T(m)^s} = \prod_{n=1}^{\infty}(1 + e^{-sE_n(T)}), \tag{3.51}$$

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{F_T(m)^s} = \prod_{n=1}^{\infty}(1 - e^{-sE_n(T)}). \tag{3.52}$$

Theorems 3.9 and 2.7 imply the following.

Corollary 3.10 *Let $z \in D$. Then,*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \frac{1}{\sum_{n=1}^{\infty} \frac{(-z)^{\gamma(n)}}{F_T(n)^s}}. \tag{3.53}$$

In particular,

$$\sum_{m=1}^{\infty} \frac{|\mu(m)|}{F_T(m)^s} = \frac{1}{\sum_{n=1}^{\infty} \frac{\lambda(n)}{F_T(n)^s}}, \tag{3.54}$$

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{F_T(m)^s} = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{F_T(n)^s}}. \tag{3.55}$$

We introduce a function η on $\mathbf{N} \times \mathbf{N}$ by

$$\eta(1,n) := 0, \tag{3.56}$$

$$\eta(m,n) := \sum_{k=1}^{r} (-1)^{k-1} \delta_{i_k n} \tag{3.57}$$

if $m \in \mathcal{S}_0$ is expressed as (3.41). If $m \notin \mathcal{S}_0$, then $\eta(m,n) := 0$ for all $n \in \mathbf{N}$.

Theorem 3.11 *Let $z \in D$ and $n \in \mathbf{N}$. Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}\eta(m,n)}{F_T(m)^s} = \frac{z}{e^{sE_n(T)} + z} Z_{\mathrm{F}}(s,z;T). \tag{3.58}$$

In particular,

$$\sum_{m=1}^{\infty} \frac{\eta(m,n)}{F_T(m)^s} = \frac{Z_{\mathrm{F}}(s;T)}{e^{sE_n(T)} + 1}, \tag{3.59}$$

$$\sum_{m=1}^{\infty} \frac{\mu(m)\eta(m,n)}{F_T(m)^s} = -\frac{\widetilde{Z}_{\mathrm{F}}(s;T)}{e^{sE_n(T)} - 1}. \tag{3.60}$$

Proof. Similar to the proof of Theorem 2.8 [use (3.28)]. ■

The left hand side of (3.40) is equal to $\sum_{m \in \mathcal{S}_0(N)} z^{\gamma(m)}/F_T(m)^s$. Hence we obtain

$$\sum_{m|N} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \det\left(1 + ze^{-sT_N}\right). \tag{3.61}$$

3.3 Applicaitons to analytic number theory

Consider the case where $\mathcal{H} = \ell^2$ and $T = \omega_{\mathcal{P}}$. Let $z \in D$ and $s > 1$. Then, by (2.67), we have

$$Z_{\mathrm{F}}(s,z;\omega_{\mathcal{P}}) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{m^s}. \tag{3.62}$$

In particular,

$$Z_{\mathrm{F}}(s;\omega_{\mathcal{P}}) = \sum_{m=1}^{\infty} \frac{|\mu(m)|}{m^s}, \tag{3.63}$$

$$\widetilde{Z}_{\mathrm{F}}(s;\omega_{\mathcal{P}}) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}. \tag{3.64}$$

Let f be a completely multiplicative function as in Section 2.3 and $z \in D$. Then, by (2.78), we have

$$Z_{\mathrm{F}}(1,z;A_f) = \sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m). \tag{3.65}$$

In particular,

$$Z_{\mathrm{F}}(1;A_f) = \sum_{m=1}^{\infty} |\mu(m)|f(m), \tag{3.66}$$

$$\widetilde{Z}_{\mathrm{F}}(1;A_f) = \sum_{m=1}^{\infty} \mu(m)f(m). \tag{3.67}$$

By Theorem 3.9, we obtain the following.

Corollary 3.12 *For all $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m) = \prod_{p\in\mathcal{P}}(1+zf(p)). \tag{3.68}$$

In particular,

$$\sum_{m=1}^{\infty} |\mu(m)|f(m) = \prod_{p\in\mathcal{P}}(1+f(p)), \tag{3.69}$$

$$\sum_{m=1}^{\infty} \mu(m)f(m) = \prod_{p\in\mathcal{P}}(1-f(p)). \tag{3.70}$$

Theorem 3.11 gives the following.

Corollary 3.13 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}\eta(m,n)f(m) = \frac{zf(p_n)}{1+zf(p_n)} Z_{\mathrm{F}}(1,z;A_f). \tag{3.71}$$

In particular,

$$\sum_{m=1}^{\infty} \eta(m,n)f(m) = \frac{f(p_n)}{f(p_n)+1} Z_{\mathrm{F}}(1;A_f) \tag{3.72}$$

$$\sum_{m=1}^{\infty} \mu(m)\eta(m,n)f(m) = \frac{f(p_n)}{f(p_n)-1} \widetilde{Z}_{\mathrm{F}}(1;A_f). \tag{3.73}$$

Jordan's totient function $J_s(N)$ ($s \geq 0, N \in \mathbf{N}$) is defined by $J_s(1) := 1$ and, for $N \geq 2$,

$$J_s(N) = N^s \prod_{p|N; p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right) \tag{3.74}$$

[1, p.48]. The special case

$$\varphi(N) = J_1(N) \tag{3.75}$$

is *Euler's totient function* [1, p.25, p.27]. We have

$$\det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right) = \prod_{p|N; p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right), \quad s \geq 0,\ N \geq 2. \tag{3.76}$$

Hence we obtain

$$J_s(N) = N^s \det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right), \quad s \geq 0,\ N \geq 2, \tag{3.77}$$

which, together with (3.40), implies that

$$J_s(N) = N^s \mathrm{Tr}\left(R_N(-1)^{N_{\mathrm{F}}} e^{-sH_{\mathrm{F}}(\omega_{\mathcal{P}})} R_N\right), \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.78}$$

This gives an expression of Jordan's totient function in terms of Fock space objects. Formula (3.61) implies the well known identity [1, p.48]:

$$J_s(N) = \sum_{m|N} \mu(m) \left(\frac{N}{m}\right)^s, \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.79}$$

4 Dirac and Laplace-Beltrami Operators on the Abstract Boson-Fermion Fock Space

In this section we review basic results of analysis on the abstract Boson-Fermion Fock space [3, 6, 7, 16].

4.1 Exterior differential operators

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then the Boson-Fermion Fock space associated with the pair $\langle \mathcal{H}, \mathcal{K} \rangle$ is defined by the tensor product Hilbert space

$$\mathcal{F}_{\mathrm{BF}}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_{\mathrm{B}}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{F}}(\mathcal{K}). \tag{4.1}$$

We define

$$\Omega := \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}} \in \mathcal{F}_{\mathrm{BF}}(\mathcal{H}, \mathcal{K}), \tag{4.2}$$

the *vacuum* in $\mathcal{F}_{\mathrm{BF}}(\mathcal{H}, \mathcal{K})$.

The annihilation operators $a_{\mathcal{H}}(f)$ and $b_{\mathcal{K}}(u)$ ($f \in \mathcal{H}, u \in \mathcal{K}$) can be extended to operators on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H}, \mathcal{K})$ as

$$a(f) := a_{\mathcal{H}}(f) \otimes I, \quad b(u) := I \otimes b_{\mathcal{K}}(u). \tag{4.3}$$

We denote by $\mathsf{C}(\mathcal{H},\mathcal{K})$ the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. For each $S \in \mathsf{C}(\mathcal{H},\mathcal{K})$, we define a subspace

$$\mathcal{D}_S := \mathcal{L}\Big\{a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^*\Omega \Big| n,m \geq 0, f_j \in D(S), \\ j=1,\cdots,n,\ u_k \in D(S^*), k=1,\cdots,m\Big\}. \tag{4.4}$$

Since $D(S)$ and $D(S^*)$ are dense in $\mathcal{H}$ and $\mathcal{K}$ respectively, it follows that $\mathcal{D}_S$ is dense in $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$.

Proposition 4.1 [7] *For each $S \in \mathsf{C}(\mathcal{H},\mathcal{K})$, there exists a unique densely defined closed linear operator d_S on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ with the following properties:*

(i) *$\mathcal{D}_S$ is a core of d_S.*

(ii) *For each vector $\Psi \in \mathcal{D}_S$ of the form*

$$\Psi = a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^*\Omega, \tag{4.5}$$

d_S acts as

$$d_S\Psi = 0 \quad \text{for } n=0,$$
$$d_S\Psi = \sum_{j=1}^{n} a(f_1)^* \cdots \widehat{a(f_j)}^* \cdots a(f_n)^* b(Sf_j)^* b(u_1)^* \cdots b(u_m)^*\Omega \quad \text{for } n \geq 1,$$

where $\widehat{T}$ indicates the omission of T.

We call the operator d_S an *exterior differential operator* on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$. Fundamental properties of the operator d_S are given in the following proposition.

Proposition 4.2 [7] *Let $S \in \mathsf{C}(\mathcal{H},\mathcal{K})$. Then the following* (i)-(iv) *hold.*

(i) $d_S^2 = 0$.

(ii) *For each CONS $\{u_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ with $u_n \in D(S^*)$,*

$$d_S\Psi = \sum_{n=1}^{\infty} a(S^*u_n)b(u_n)^*\Psi, \quad \Psi \in \mathcal{D}_S,$$

where the convergence is taken in the strong topology of $\mathcal{F}_{\mathrm{FB}}(\mathcal{H},\mathcal{K})$.

(iii) *For each CONS $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ with $\phi_n \in D(S)$, we have*

$$(\Phi, d_S\Psi) = \lim_{N\to\infty}\left(\Phi, \sum_{n=1}^{N} a(\phi_n)b(S\phi_n)^*\Psi\right), \quad \Phi,\Psi \in \mathcal{D}_S.$$

(iv) $\mathcal{D}_S \subset D(d_S^*)$ *and, for all vectors* Ψ *of the form (4.5) with* $m \geq 1$,

$$d_S^*\Psi = \sum_{k=1}^{m}(-1)^{k-1}a(S^*u_k)^*a(f_1)^*\cdots a(f_n)^*b(u_1)^*\cdots \widehat{b(u_k)}^*\cdots b(u_m)^*\Omega.$$

If $m = 0$, *then* $d_S^*\Psi = 0$.

The Boson-Fermion Fock space $\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K})$ is $\mathbf{Z}_2$-graded:

$$\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K}) = \mathcal{F}_+(\mathcal{H},\mathcal{K}) \oplus \mathcal{F}_-(\mathcal{H},\mathcal{K}) \tag{4.6}$$

with

$$\mathcal{F}_\pm(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\rm B}(\mathcal{H}) \otimes \mathcal{F}_{{\rm F},\pm}(\mathcal{K}). \tag{4.7}$$

The grading operator of this $\mathbf{Z}_2$-gradation is given by

$$\Gamma_{\rm F} := (-1)^{I\otimes N_{\rm F}} = I \otimes (-1)^{N_{\rm F}}. \tag{4.8}$$

Let

$$\mathcal{F}_n(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\rm B}(\mathcal{H}) \otimes (\otimes_{\rm as}^n \mathcal{K}). \tag{4.9}$$

Then

$$\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K}) = \oplus_{n=0}^{\infty}\mathcal{F}_n(\mathcal{H},\mathcal{K}), \tag{4.10}$$

$$\mathcal{F}_+(\mathcal{H},\mathcal{K}) = \oplus_{n=0}^{\infty}\mathcal{F}_{2n}(\mathcal{H},\mathcal{K}), \quad \mathcal{F}_-(\mathcal{H},\mathcal{K}) = \oplus_{n=0}^{\infty}\mathcal{F}_{2n+1}(\mathcal{H},\mathcal{K}). \tag{4.11}$$

We denote by Π_n the orthogonal projection from $\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K})$ onto $\mathcal{F}_n(\mathcal{H},\mathcal{K})$. The following fact is easily proven.

Proposition 4.3 *For all* $n \geq 0$, $\Pi_{n+1}d_S \subset d_S\Pi_n$. *In particular,* d_S *maps* $D(d_S) \cap \mathcal{F}_n(\mathcal{H},\mathcal{K})$ *to* $\mathcal{F}_{n+1}(\mathcal{H},\mathcal{K})$.

By this proposition, we can define, for each $n \in \mathbf{N}$, a densely define closed linear operator $d_{S,n}$ from $\mathcal{F}_n(\mathcal{H},\mathcal{K})$ to $\mathcal{F}_{n+1}(\mathcal{H},\mathcal{K})$ by

$$D(d_{S,n}) := D(d_S) \cap \mathcal{F}_n(\mathcal{H},\mathcal{K}), \quad d_{S,n}\Psi := d_S\Psi, \quad \Psi \in D(d_{S,n}). \tag{4.12}$$

We have

$$d_{S,n+1}d_{S,n} = 0, \quad n \geq 0. \tag{4.13}$$

4.2 Dirac and Laplace-Beltrami operators

In what follows, we fix $S \in \mathsf{C}(\mathcal{H},\mathcal{K})$. We define

$$Q_S := d_S + d_S^* \tag{4.14}$$

with $D(Q_S) = D(d_S) \cap D(d_S^*)$. We call it a *Dirac operator* on $\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K})$.

Since S^*S and SS^* are nonnegative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively, we can define

$$L_S := H_{\rm B}(S^*S) \otimes I + I \otimes H_{\rm F}(SS^*) \tag{4.15}$$

acting in $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$, which is nonnegative and self-adjoint (cf. [26, §VIII.10, Corollary]).

For a linear operator A on a Hilbert space, we set

$$C^\infty(A) := \cap_{n=1}^\infty D(A^n).$$

Since $C^\infty(S^*S)$ and $C^\infty(SS^*)$ are dense in $\mathcal{H}$ and $\mathcal{K}$ respectively, the set

$$\begin{aligned}\mathcal{D}_S^\infty \; := \; & \mathcal{L}\Big\{a(f_1)^*\cdots a(f_n)^* b(u_1)^*\cdots b(u_m)^*\Omega \Big| n,m\geq 0, f_j\in C^\infty(S^*S), \\ & \qquad j=1,\cdots,n,\; u_k\in C^\infty(SS^*), k=1,\cdots,m\Big\} \end{aligned} \tag{4.16}$$

is dense in $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$.

Theorem 4.4 [7]

(i) *The operator Q_S is self-adjoint, and essentially self-adjoint on every core of L_S. In particular, Q_S is essentially self-adjoint on $\mathcal{D}_S^\infty$.*

(ii) *The operator Γ_{F} leaves $D(Q_S)$ invariant and*

$$\Gamma_{\mathrm{F}} Q_S + Q_S \Gamma_{\mathrm{F}} = 0$$

on $D(Q_S)$.

(iii) *The following operator equations hold :*

$$L_S = Q_S^2 = d_S^* d_S + d_S d_S^*.$$

Remark 4.1 (i) The operators d_S and d_S^* leave $\mathcal{D}_S^\infty$ invariant and so does Q_S.

(ii) Part (ii) of Theorem 4.4 justifies calling Q_S a Dirac type operator (cf. [32, Chapter 5]).

(iii) Because of part (iii) of Theorem 4.4, we call the operator L_S the *Laplace-Beltrami operator* associated with the exterior differential operator d_S.

(iv) For all $z\in\mathbf{C}$ with $|z|=1$, $L_{zS}=L_S$. Hence $L_S = Q_{zS}^2$.

4.3 Analytical indices of Dirac operators and partition functions

By Theorem 4.4(i) and (ii), there exists a unique densely defined closed linear operator $Q_{S,+}$ from $\mathcal{F}_+(\mathcal{H},\mathcal{K})$ to $\mathcal{F}_-(\mathcal{H},\mathcal{K})$ such that

$$Q_S = \begin{pmatrix} 0 & Q_{S,+}^* \\ Q_{S,+} & 0 \end{pmatrix}, \tag{4.17}$$

where the matrix representation is relative to the orthogonal decomposition (4.6).

In general, for a densely defined closed linear operator T from a Hilbert space to a Hilbert space, its analytical index is defined by

$$\mathrm{ind}(T) := \dim\ker T - \dim\ker T^*$$

provided that at least one of $\dim\ker T$ and $\dim\ker T^*$ is finite.

The following theorem is concerned with Fredholm property of $Q_{S,+}$.

Theorem 4.5 [7]

(i) *If S is Fredholm with* $\ker S = \{0\}$, *then* $Q_{S,+}$ *is Fredholm with*

$$\mathrm{ind}(Q_{S,+}) = \delta_{0,\dim\ker S^*}.$$

(ii) *If S is semi-Fredholm with* $\dim\ker S \geq 1$ *and* $\ker S^* = \{0\}$, *then* $Q_{S,+}$ *is semi-Fredholm with*

$$\mathrm{ind}(Q_{S,+}) = \dim\ker Q_{S,+} = +\infty.$$

Remark 4.2 Let V be a symmetric operator on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ and $Q_S(V) := Q_S + V$. For a class of V such that $\{\Gamma_{\mathrm{F}}, Q_S(V)\} = 0$ on $D(Q_S(V))$, an index formula of $Q_S(V)_+ := Q_S(V)|\mathcal{F}_+(\mathcal{H},\mathcal{K})$ is established in terms of a functional integral representation [3, 7].

As for the heat semi-group e^{-sL_S} $(s > 0)$, the following theorem holds.

Theorem 4.6 [7] *Let $s > 0$. Suppose that e^{-sS^*S} and e^{-sSS^*} are trace class on $\mathcal{H}$ and $\mathcal{K}$ respectively with* $\ker S = \{0\}$. *Then e^{-sL_S} is trace class on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ and*

$$\mathrm{Tr}\left(\Gamma_{\mathrm{F}} e^{-sL_S}\right) = \mathrm{ind}(Q_{S,+}) = \delta_{0,\dim\ker S^*}. \tag{4.18}$$

We have another orthogonal decomposition

$$\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}) = \mathcal{G}_+(\mathcal{H},\mathcal{K}) \oplus \mathcal{G}_-(\mathcal{H},\mathcal{K}) \tag{4.19}$$

with

$$\mathcal{G}_\pm(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\mathrm{B},\pm}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{F}}(\mathcal{K}). \tag{4.20}$$

The grading operator of this gradation is given by

$$\Gamma_{\mathrm{B}} := (-1)^{N_{\mathrm{B}}\otimes I} = (-1)^{N_{\mathrm{B}}} \otimes I. \tag{4.21}$$

It is easy to see that

$$\Gamma_{\mathrm{B}} Q_S \subset -Q_S \Gamma_{\mathrm{B}}. \tag{4.22}$$

Hence, as in the case of the decomposition (4.6), there exists a unique densely defined closed linear operator $\tilde{Q}_{S,+}$ from $\mathcal{G}_+(\mathcal{H},\mathcal{K})$ to $\mathcal{G}_-(\mathcal{H},\mathcal{K})$ such that

$$Q_S = \begin{pmatrix} 0 & \tilde{Q}_{S,+}^* \\ \tilde{Q}_{S,+} & 0 \end{pmatrix}, \tag{4.23}$$

where the matrix representation is relative to the orthogonal decomposition (4.19).

Theorem 4.7 (i) *If S is Fredholm with* $\ker S = \{0\}$ *and* $\dim\ker S^* < \infty$, *then* $\tilde{Q}_{S,+}$ *is Fredholm with*

$$\mathrm{ind}(\tilde{Q}_{S,+}) = \dim\ker \tilde{Q}_{S,+} = 2^{\dim\ker S^*}.$$

(ii) *If S is semi-Fredholm with* $\dim\ker S \geq 1$ *and* $\ker S^* = \{0\}$, *then* $\tilde{Q}_{S,+}$ *is semi-Fredholm with*

$$\mathrm{ind}(\tilde{Q}_{S,+}) = \dim\ker \tilde{Q}_{S,+} = +\infty.$$

(iii) *If S is semi-Fredholm with* $\ker S = \{0\}$ *and* $\dim\ker S^* = +\infty$, *then* $\tilde{Q}_{S,+}$ *is semi-Fredholm with*

$$\operatorname{ind}(\tilde{Q}_{S,+}) = \dim\ker \tilde{Q}_{S,+} = +\infty.$$

Proof. Similar to the proof of Theorem 4.5. ∎

Theorem 4.8 *Under the same assumption as in Theorem 4.6,* e^{-sL_S} *is trace class on* $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ *and*

$$\operatorname{Tr}\left(\Gamma_{\mathrm{B}} e^{-sL_S}\right) = \operatorname{ind}(\tilde{Q}_{S,+}) = 2^{\dim\ker S^*}. \tag{4.24}$$

Proof. Similar to the proof of Theorem 4.6. ∎

4.4 Connection with supersymmetry

Let $\mathcal{X}$ be a Hilbert space and $H, \{Q_j\}_{j=1}^N, \tau$, be self-adjoint operators on $\mathcal{X}$. Then the quadruple $\{\mathcal{X}, H, \{Q_j\}_{j=1}^N, \tau\}$ is called a *supersymmetric quantum mechanics* (SSQM) with N-supersymmetry if the following (S.1)–(S.4) are satisfied:

(S.1) τ is bounded and $\tau^2 = I$.

(S.2) For all $j = 1, \cdots, N$, $H = Q_j^2$.

(S.3) For each $j = 1, \cdots, N$, the operator τ leaves $D(Q_j)$ invariant and $\{\tau, Q_j\} = 0$ on $D(Q_j)$.

(S.4) For all $j, k = 1, \cdots, N$ with $j \neq k$,

$$(Q_j\Psi, Q_k\Phi) + (Q_k\Psi, Q_j\Phi) = 0, \quad \Psi, \Phi \in D(Q_j) \cap D(Q_k).$$

The operators Q_j and H are called a *supercharge* and a *supersymmetric Hamiltonian*, respectively.

For mathematical discussions of SSQM, see, e.g., [2, 22, 32].

It is easy to see that, for all $z \in \mathbf{C}$ with $|z| = 1$, the quadruple $\{\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}), L_S, \{Q_{zS}, Q_{izS}\}, \Gamma_{\mathrm{F}}\}$ is a SSQM. This SSQM produces various supersymmetric quantum field models in concrete realizations [7, 12]. For mathematical analysis of models in supersymmetric quantum field theory, see [4, 8, 9, 12, 22].

The quadruple $\{\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}), L_S, \{Q_{zS}, Q_{izS}\}, \Gamma_{\mathrm{B}}\}$ ($|z| = 1$) also is a SSQM.

4.5 Other aspects

Decomposition theorems of De Rham-Hodge-Kodaira type on the exterior differential operators $d_{S,n}$ are established in [7, 16]. Fundamental spaces associated with the Laplace-Beltrami operator L_S are introduced in [17] and their structures are investigated. The self-adjointness of a perturbed Dirac operator $Q_S(V) = Q_S + V$ with some symmetric operator V is discussed in [10]. The strong anticommutativity of two Dirac operators Q_S and Q_T ($S, T \in \mathsf{C}(\mathcal{H},\mathcal{K})$) and its applications to representations of a supersymmetry algebra are studied in [15](cf. also [11]).

5 Arithmetical Aspects of Boson-Fermion Fock Spaces

5.1 Some general aspects

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and A and T be nonnegative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then the operator

$$H(A,T) := H_{\rm B}(A) \otimes I + I \otimes H_{\rm F}(T) \tag{5.1}$$

on $\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K})$ is nonnegative and self-adjoint.

We assume the following.

(AT) *The operators A and T satisfy (A) in Section 2 and (T) in Section 3 respectively.*

Under this assumption, $e^{-sH(A,T)}$ is trace class and we can define a partition function

$$Z(s,z,w;A,T) := {\rm Tr}\left(\Gamma_{\rm B}(z) \otimes \Gamma_{\rm F}(w) e^{-sH(A,T)}\right), \quad z,w \in D. \tag{5.2}$$

Let

$$N_{\rm BF} := N_{\rm B} \otimes I + I \otimes N_{\rm F}, \tag{5.3}$$

the number operator on $\mathcal{F}_{\rm BF}(\mathcal{H},\mathcal{K})$, and set

$$\Gamma_{\rm BF} := (-1)^{N_{\rm BF}} = \Gamma_{\rm B}\Gamma_{\rm F} = \Gamma_{\rm F}\Gamma_{\rm B}. \tag{5.4}$$

As special cases of $Z(s,z,w;A,T)$, we define the following partition functions:

$$\begin{aligned}
Z(s;A,T) &:= Z(s,1,1;A,T) = {\rm Tr}\, e^{-sH(A,T)}, && (5.5)\\
\widetilde{Z}(s;A,T) &:= Z(s,-1,-1;A,T) = {\rm Tr}\left(\Gamma_{\rm BF}\, e^{-sH(A,T)}\right), && (5.6)\\
\widetilde{Z}_{\rm B}(s;A,T) &:= Z(s,-1,1;A,T) = {\rm Tr}\left(\Gamma_{\rm B}\, e^{-sH(A,T)}\right), && (5.7)\\
\widetilde{Z}_{\rm F}(s;A,T) &:= Z(s,1,-1;A,T) = {\rm Tr}\left(\Gamma_{\rm F}\, e^{-sH(A,T)}\right). && (5.8)
\end{aligned}$$

We have

$$Z(s,z,w;A,T) = Z_{\rm B}(s,z;A) Z_{\rm F}(s,w;T), \quad z,w \in D. \tag{5.9}$$

In particular,

$$\begin{aligned}
Z(s;A,T) &= Z_{\rm B}(s;A) Z_{\rm F}(s;T), && (5.10)\\
\widetilde{Z}(s;A,T) &:= \widetilde{Z}_{\rm B}(s;A) \widetilde{Z}_{\rm F}(s;T), && (5.11)\\
\widetilde{Z}_{\rm B}(s;A,T) &:= \widetilde{Z}_{\rm B}(s;A) Z_{\rm F}(s;T), && (5.12)\\
\widetilde{Z}_{\rm F}(s;A,T) &:= Z_{\rm B}(s;A) \widetilde{Z}_{\rm F}(s;T). && (5.13)
\end{aligned}$$

Hence properties of the partition functions of $H(A,T)$ introduced above are reduced to those of $H_{\rm B}(A)$ and $H_{\rm F}(T)$. However, if one can represent the left hand sides on (5.9)–(5.13) in various ways, (5.9)–(5.13) may produce nontrivial arithmetical relations for eigenvalues of A and T. Moreover, different expressions of ${\rm Tr}\left(X e^{-sH(A,T)}\right)$ with X an operator on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ may yield interesting arithmetical relations. Here we present only an outline of investigations along this line.

We carry over the notation in the preceding sections. Let $N \geq 2$ be of the form (2.39) and $m \in \mathcal{S}_0(N)$. Then we can write

$$m = (p_{i_1})^{q_1}(p_{i_2})^{q_2}\cdots(p_{i_n})^{q_n}, \tag{5.14}$$

where $q_j = 0$ or $q_j = 1$. Based on these factorizations, we define a vector

$$\Omega_{N,m} := C_{N,m}\left[(a_{i_1}^*)^{\alpha_1-q_1}\cdots(a_{i_n}^*)^{\alpha_n-q_n}\Omega_{\mathcal{H}}\right]\otimes\left[(b_{i_1}^*)^{q_1}\cdots(b_{i_n}^*)^{q_n}\Omega_{\mathcal{K}}\right], \tag{5.15}$$

where $C_{N,m} > 0$ is a normalization constant. For $N = 1$ and $m = 1$, we set

$$\Omega_{1,1} := \Omega.$$

Lemma 5.1 *The set $\{\Omega_{N,m}|N \geq 1,\ m \in \mathcal{S}_0(N)\}$ is a CONS of $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$.*

Proof. The subset of vectors of the form

$$\left(a_{j_1}^*\cdots a_{j_n}^*\Omega_{\mathcal{H}}\right)\otimes\left(b_{k_1}^*\cdots b_{k_\ell}^*\Omega_{\mathcal{K}}\right),$$

$j_1 \leq j_2 \leq \cdots$; $k_1 < k_2 < \cdots$ $(j_m, k_i \in \mathbf{N})$, is a complete orthogonal system of $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$. As is easily seen, vectors of this form is a constant multiple of $\Omega_{N,m}$ for some $\langle N, m\rangle$. ■

Remark 5.1 The CONS $\{\Omega_{N,m}\}$ was introduced in [30].

For each $N \in \mathbf{N}$, the subspace

$$\mathcal{F}_{\mathrm{BF}}^{(N)} := \mathcal{L}\{\Omega_{N,m}|m \in \mathcal{S}_0(N)\}. \tag{5.16}$$

is finite dimensional with

$$\dim \mathcal{F}_{\mathrm{BF}}^{(N)} = 2^{\nu(N)}. \tag{5.17}$$

By Lemma 5.1, we have

$$\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}) = \oplus_{N=1}^{\infty}\mathcal{F}_{\mathrm{BF}}^{(N)}. \tag{5.18}$$

The following fact is easily proven.

Lemma 5.2 *Let $N \in \mathbf{N}$, $m \in \mathcal{S}_0(N)$ and $z, w \in D$. Then $\Omega_{N,m}$ is an eigenvector of $\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)$ with eigenvalue $z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}$.*

For each $N \in \mathbf{N}$, we define a function $Y_{A,T}(N,\cdot)$ on $\mathcal{S}_0(N)$ by

$$Y_{A,T}(N,m) := \prod_{k=1}^{n} e^{(\alpha_k - q_k)E_{i_k}(A)+q_kE_{i_k}(T)}, \quad m \in \mathcal{S}_0(N), \tag{5.19}$$

when N and m are represented as (2.39) and (5.14) respectively. Note that

$$Y_{A,T}(N,m) = F_A\left(\frac{N}{m}\right)F_T(m). \tag{5.20}$$

Lemma 5.3 *Let $N \in \mathbf{N}$ and $m \in \mathcal{S}_0(N)$. Then $\Omega_{N,m}$ is an eigenvector of $H(A,T)$ with eigenvalue* $\log Y_{A,T}(N,m)$.

Proof. Let N and m be as in (2.39) and (5.14) respectively. Then

$$\begin{aligned} H(A,T)\Omega_{N,m} &= \left(\sum_{k=1}^{n}\{(\alpha_k - q_k)E_{i_k}(A) + q_k E_{i_k}(T)\}\right)\Omega_{N,m} \\ &= (\log Y_{A,T}(N,m))\Omega_{N,m}. \end{aligned}$$

Hence the desired assertion follows. ■

Theorem 5.4 *Let $z, w \in D$. Then*

$$Z(s,z,w;A,T) = \sum_{N=1}^{\infty}\sum_{m|N}\frac{z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}|\mu(m)|}{Y_{A,T}(N,m)^s}. \tag{5.21}$$

In particular,

$$Z(s;A,T) = \sum_{N=1}^{\infty}\sum_{m|N}\frac{|\mu(m)|}{Y_{A,T}(N,m)^s}, \tag{5.22}$$

$$\widetilde{Z}(s;A,T) := \sum_{N=1}^{\infty}\lambda(N)\sum_{m|N}\frac{|\mu(m)|}{Y_{A,T}(N,m)^s}, \tag{5.23}$$

$$\widetilde{Z}_{\mathrm{B}}(s;A,T) := \sum_{N=1}^{\infty}\lambda(N)\sum_{m|N}\frac{\mu(m)}{Y_{A,T}(N,m)^s}, \tag{5.24}$$

$$\widetilde{Z}_{\mathrm{F}}(s;A,T) := \sum_{N=1}^{\infty}\sum_{m|N}\frac{\mu(m)}{Y_{A,T}(N,m)^s}. \tag{5.25}$$

(5.26)

Proof. For a trace class operator X on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$, we have

$$\mathrm{Tr}X = \sum_{N=1}^{\infty}\sum_{m\in\mathcal{S}_0(N)}(\Omega_{N,m}, X\Omega_{N,m}). \tag{5.27}$$

We need only apply this formula, using Lemmas 5.2 and 5.3. ■

By Theorem 5.4 and (5.10)–(5.13), we obtain the following formulas.

Corollary 5.5 *Let $z, w \in D$. Then*

$$\sum_{N=1}^{\infty}\sum_{m|N}\frac{z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}|\mu(m)|}{Y_{A,T}(N,m)^s} = Z_{\mathrm{B}}(s,z;A)Z_{\mathrm{F}}(s,w;T). \tag{5.28}$$

In particular,

$$\sum_{N=1}^{\infty}\sum_{m|N}\frac{|\mu(m)|}{Y_{A,T}(N,m)^s} = Z_{\rm B}(s;A)Z_{\rm F}(s;T), \tag{5.29}$$

$$\sum_{N=1}^{\infty}\lambda(N)\sum_{m|N}\frac{|\mu(m)|}{Y_{A,T}(N,m)^s} = \tilde{Z}_{\rm B}(s;A)\tilde{Z}_{\rm F}(s;T), \tag{5.30}$$

$$\sum_{N=1}^{\infty}\lambda(N)\sum_{m|N}\frac{\mu(m)}{Y_{A,T}(N,m)^s} = \tilde{Z}_{\rm B}(s;A)Z_{\rm F}(s;T), \tag{5.31}$$

$$\sum_{N=1}^{\infty}\sum_{m|N}\frac{\mu(m)}{Y_{A,T}(N,m)^s} = Z_{\rm B}(s;A)\tilde{Z}_{\rm F}(s;T). \tag{5.32}$$

Remark 5.2 If we put into the right hand sides of (5.28)–(5.32) the formulas established in Sections 2 and 3, then we obtain explicit formulas, which are nontrivial.

Remark 5.3 By rescaling as $T \to tT/s$ $(t > 0)$ in (5.28)–(5.32), we can obtain relations at different temperatures $1/s$ and $1/t$. Hence (5.28)–(5.32) include "duality relations". See Section 5.3 below.

Proposition 5.6 *Let* $N \geq 2$. *Then*

$$\mathrm{Tr}_{\mathcal{F}_{\rm BF}^{(N)}}\Gamma_{\rm F} = 0, \tag{5.33}$$

where $\mathrm{Tr}_{\mathcal{F}_{\rm BF}^{(N)}}$ *means trace restricted to the subspace* $\mathcal{F}_{\rm BF}^{(N)}$.

Proof. Let $N \geq 2$ and m be as in (2.39) and (5.14) respectively. Then the cardinal number of the set $\mathcal{S}_0(N)$ is 2^n, including 1. In the expression (5.14) of m, if there is a j such that $q_j = 1$, then $m' = m/p_{i_j} \in \mathcal{S}_0(N)$. If $q_1 = q_2 = \cdots = q_n = 0$(i.e., $m = 1$), then, $m' := p_{i_1}$ is in $\mathcal{S}_0(N)$. In this case, if $\Gamma_{\rm F}\Omega_{N,m} = \pm\Omega_{N,m}$, then $\Gamma_{\rm F}\Omega_{N,m'} = \mp\Omega_{N,m'}$. Hence $\mathcal{S}_0(N)$ consists of 2^{n-1} pairs (m, m') that satisfy this relation. Using this fact, we obtain (5.33). ■

Corollary 5.7 *Let* $N \geq 2$. *Then*

$$\sum_{m|N}\mu(m) = 0. \tag{5.34}$$

Proof. We have by (5.33)

$$0 = \mathrm{Tr}_{\mathcal{F}_{\rm BF}^{(N)}}\Gamma_{\rm F} = \sum_{m\in\mathcal{S}_0(N)}(\Omega_{N,m}, \Gamma_{\rm F}\Omega_{N,m}) = \sum_{m\in\mathcal{S}_0(N)}\mu(m).$$

Since $\mu(m) = 0$ if $m \notin \mathcal{S}_0$, we obtain (5.34). ■

Remark 5.4 Eq.(5.34) is a well known formula of the Möbius function [1, p.25].

5.2 Partial supersymmetry

A notion of *partial supersymmetry* was introduced in [31] in a physical way. In the context of our general formulation of supersymmetric quantum field theory presented in Section 4, a partial supersymmetry means to consider a "distorted" supersymmetric Hamiltonian

$$H_{\mathrm{PS}}(a,b;S) := L_S + aH_{\mathrm{B}}(S^*S)\otimes I + bI\otimes H_{\mathrm{F}}(SS^*) = L_S + H(aS^*S, bSS^*) \tag{5.35}$$

on the Boson-Fermion Fock space $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ with parameters $a,b\geq 0$, where $S\in \mathsf{C}(\mathcal{H},\mathcal{K})$. Note that

$$H_{\mathrm{PS}}(a,b;S) = H((a+1)S^*S,(b+1)SS^*). \tag{5.36}$$

Hence, in the caes $a=b$, supersymmetry recovers with $H_{\mathrm{PS}}(a,a;S)=(a+1)L_S$.

Let $\beta>0$ and suppose that $e^{-(a+1)\beta S^*S}$ and $e^{-(b+1)\beta SS^*}$ are trace class. Then $e^{-\beta H_{\mathrm{PS}}(a,b;S)}$, $e^{-(a+1)\beta H_{\mathrm{B}}(S^*S)}$ and $e^{-(b+1)\beta H_{\mathrm{F}}(SS^*)}$ are trace class and the following formula holds:

$$\mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) = Z_{\mathrm{B}}((a+1)\beta,z;S^*S)Z_{\mathrm{F}}((b+1)\beta,w;SS^*),\quad z,w\in D. \tag{5.37}$$

In particular,

$$\begin{aligned}
\mathrm{Tr}e^{-\beta H_{\mathrm{PS}}(a,b;S)} &= Z_{\mathrm{B}}((a+1)\beta;S^*S)Z_{\mathrm{F}}((b+1)\beta;SS^*), &(5.38)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{BF}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= \tilde{Z}_{\mathrm{B}}((a+1)\beta;S^*S)\tilde{Z}_{\mathrm{F}}((b+1)\beta;SS^*), &(5.39)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{F}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= Z_{\mathrm{B}}((a+1)\beta;S^*S)\tilde{Z}_{\mathrm{F}}((b+1)\beta;SS^*), &(5.40)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{B}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= \tilde{Z}_{\mathrm{B}}((a+1)\beta;S^*S)Z_{\mathrm{F}}((b+1)\beta;SS^*). &(5.41)
\end{aligned}$$

Note that the left hand sides of these equations may be regarded as correlation functions in the SSQM described by the supersymmetric Hamiltonian L_S:

$$\mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) = \left\langle\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)e^{-\beta H(aS^*S,bSS^*)}\right\rangle_\beta, \tag{5.42}$$

where

$$\langle X\rangle_\beta := \mathrm{Tr}\left(Xe^{-\beta L_S}\right). \tag{5.43}$$

In particular,

$$\begin{aligned}
\mathrm{Tr}e^{-\beta H_{\mathrm{PS}}(a,b;S)} &= \left\langle e^{-\beta H(aS^*S,bSS^*)}\right\rangle_\beta, &(5.44)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{BF}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= \left\langle\Gamma_{\mathrm{BF}}e^{-\beta H(aS^*S,bSS^*)}\right\rangle_\beta, &(5.45)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{F}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= \left\langle\Gamma_{\mathrm{F}}e^{-\beta H(aS^*S,bSS^*)}\right\rangle_\beta, &(5.46)\\
\mathrm{Tr}\left(\Gamma_{\mathrm{B}}e^{-\beta H_{\mathrm{PS}}(a,b;S)}\right) &= \left\langle\Gamma_{\mathrm{B}}e^{-\beta H(aS^*S,bSS^*)}\right\rangle_\beta. &(5.47)
\end{aligned}$$

By computing these correlation functions in various ways, we would obtain from (5.38)–(5.41) nontrivial duality relations. But here we omit the details.

5.3 Applications to analytic number theory

We consider the case where $\mathcal{H} = \mathcal{K} = \ell^2$ and $A = T = \omega_{\mathcal{P}}$. Then we have

$$Y_{\omega_{\mathcal{P}},\omega_{\mathcal{P}}}(N,m) = N. \tag{5.48}$$

Hence Corollary 5.5 gives

$$\sum_{N=1}^{\infty}\sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s} = Z_{\mathrm{B}}(s,z;\omega_{\mathcal{P}}) Z_{\mathrm{F}}(s,w;\omega_{\mathcal{P}}), \quad s > 1. \tag{5.49}$$

In particular, for all $s > 1$,

$$\sum_{N=1}^{\infty} \frac{2^{\nu(N)}}{N^s} = \frac{\zeta(s)}{D(s,\lambda)}, \tag{5.50}$$

$$\sum_{N=1}^{\infty} \frac{\lambda(N) 2^{\nu(N)}}{N^s} = \frac{D(s,\lambda)}{\zeta(s)}. \tag{5.51}$$

Remark 5.5 In the present case, (5.31) and (5.32) imply Corollary 3.3, since Proposition 5.6 holds.

Let f be the completely multiplicative function considered in Section 2.3. Then, taking $S = \sqrt{A_f}$ we have

$$H := H(A_f, A_f) = L_{\sqrt{A_f}}.$$

Hence, by Theorems 4.6 and 4.8, for all $s > 1$,

$$\mathrm{Tr}\left(\Gamma_{\mathrm{F}} e^{-sH}\right) = 1, \tag{5.52}$$

$$\mathrm{Tr}\left(\Gamma_{\mathrm{B}} e^{-sH}\right) = 1. \tag{5.53}$$

These are supersymmetric identities. It is shown [30] that (5.52) implies that

$$\sum_{m=1}^{\infty} \mu(m) f(m) = \frac{1}{\sum_{n=1}^{\infty} f(n)}. \tag{5.54}$$

In the same manner as in the derivation of (5.54) [30], we can show that (5.53) implies that

$$\sum_{m=1}^{\infty} |\mu(m)| f(m) = \frac{1}{\sum_{n=1}^{\infty} \lambda(n) f(n)}. \tag{5.55}$$

We have for all $s, t > 1$

$$Y_{s\omega_{\mathcal{P}},t\omega_{\mathcal{P}}}(N,m) = N^s m^{t-s}. \tag{5.56}$$

Hence, by Corollary 5.5 with rescaling $T \to tT/s$, we obtain

$$\sum_{N=1}^{\infty}\sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s m^{t-s}} = Z_{\mathrm{B}}(s,z;\omega_{\mathcal{P}}) Z_{\mathrm{F}}(t,w;\omega_{\mathcal{P}}), \quad t > s > 1. \tag{5.57}$$

In particular,

$$\sum_{N=1}^{\infty} \frac{1}{N^s} \sum_{m|N} \frac{|\mu(m)|}{m^{t-s}} = \frac{\zeta(s)}{D(t,\lambda)}, \tag{5.58}$$

$$\sum_{N=1}^{\infty} \frac{\lambda(N)}{N^s} \sum_{m|N} \frac{|\mu(m)|}{m^{t-s}} = \frac{D(s,\lambda)}{\zeta(t)}, \tag{5.59}$$

$$\sum_{N=1}^{\infty} \frac{\lambda(N)}{N^s} \sum_{m|N} \frac{\mu(m)}{m^{t-s}} = \frac{D(s,\lambda)}{D(t,\lambda)}, \tag{5.60}$$

$$\sum_{N=1}^{\infty} \frac{1}{N^s} \sum_{m|N} \frac{\mu(m)}{m^{t-s}} = \frac{\zeta(s)}{\zeta(t)}, \quad t \geq s > 1. \tag{5.61}$$

These may be regarded as "duality relations" for the Riemann zeta function $\zeta(s)$ and the Dirichlet series $D(s, \lambda)$.

Using (3.79) to rewrite the left hand sides of (5.60) and (5.61), we obtain

$$\sum_{N=1}^{\infty} \frac{\lambda(N)}{N^t} J_{t-s}(N) = \frac{D(s,\lambda)}{D(t,\lambda)}, \tag{5.62}$$

$$\sum_{N=1}^{\infty} \frac{J_{t-s}(N)}{N^t} = \frac{\zeta(s)}{\zeta(t)}, \quad t \geq s > 1. \tag{5.63}$$

In particular,

$$\sum_{N=1}^{\infty} \frac{\lambda(N)\varphi(N)}{N^s} = \frac{D(s-1,\lambda)}{D(s,\lambda)}, \tag{5.64}$$

$$\sum_{N=1}^{\infty} \frac{\varphi(N)}{N^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad s > 2. \tag{5.65}$$

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