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1 **INFINITE-DIMENSIONAL INPUT-TO-STATE STABILITY AND**
2 **ORLICZ SPACES***

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5 **Abstract.** In this work, the relation between input-to-state stability and integral input-to-
6 state stability is studied for linear infinite-dimensional systems with an unbounded control operator.
7 Although a special focus is laid on the case L^∞ , general function spaces are considered for the inputs.
8 We show that integral input-to-state stability can be characterized in terms of input-to-state stability
9 with respect to Orlicz spaces. Since we consider linear systems, the results can also be formulated
10 in terms of admissibility. For parabolic diagonal systems with scalar inputs, both stability notions
11 with respect to L^∞ are equivalent.

12 **Key words.** Input-to-state stability, integral input-to-state stability, C_0 -semigroup, admissibil-
13 ity, Orlicz spaces

14 **AMS subject classifications.** 93D20, 93C05, 93C20, 37C75

15 **1. Introduction.** In systems and control theory, the question of stability is a
16 fundamental issue. Let us consider the situation where the relation between the input
17 (function) u and the state x is governed by the autonomous equation

18 (1.1)
$$\dot{x} = f(x, u), \quad x(0) = x_0.$$

19 One can then distinguish between *external stability*, that is, stability with respect to
20 the input u , and *internal stability*, i.e. when $u = 0$. For the moment, f is assumed to
21 map from $\mathbb{R}^n \times \mathbb{R}^m$ to \mathbb{R}^n , and to be such that solutions x exist on $[0, \infty)$ for all inputs
22 u in a function space Z . Already from this very general view-point, it seems clear that
23 stability notions may strongly depend on the specific choice of Z (and its norm). The
24 concept of *input-to-state stability* (ISS) combines both external and internal stability
25 in one notion. If Z is chosen to be $L^\infty(0, \infty; U)$, $U = \mathbb{R}^m$, a system is called ISS (with
26 respect to L^∞) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

27
$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\text{ess sup}_{s \in [0, t]} \|u(s)\|_U),$$

28 for all $t > 0$ and $u \in Z$. Here the sets \mathcal{KL} and \mathcal{K} refer to the classic comparison
29 functions from nonlinear systems theory, see Section 2. Introduced by E. Sontag in
30 1989 [27], ISS has been intensively studied in the past decades; see [29] for a survey.
31 A related stability notion is *integral input-to-state stability* (iISS) [28, 2], which means

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⁰The contents of this article emerged based on previous findings of the authors on input-to-state stability for parabolic systems that were published in the proceedings article [7]. However, this article provides a far more general and different approach using Orlicz spaces. This new approach also allowed to extend the theory essentially.

32 that for some $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$,

$$33 \quad (1.2) \quad \|x(t)\| \leq \beta(\|x_0\|, t) + \theta \left(\int_0^t \mu(\|u(s)\|_U) ds \right),$$

34 for all $t > 0$ and $u \in Z = L^\infty(0, \infty; U)$. This property differs from ISS in the sense that
 35 it allows for unbounded inputs u that have “finite energy”, see [28]. Many practically
 36 relevant systems are iISS whereas they are not ISS, see e.g. [19] for a detailed list.
 37 However, for linear systems, i.e., $f(x, u) = Ax + Bu$ with matrices A and B , iISS is
 38 equivalent to ISS. To some extent, this observation marks the starting point of this
 39 work.

40 In contrast to the well-established theory for finite-dimensions, a more intensive
 41 study of (integral) input-to-state stability for infinite-dimensional systems has only
 42 begun recently. We refer to [4, 5, 11, 12, 13, 16, 17, 18, 19, 20]. By nature, in
 43 the infinite-dimensional setting, the stability notions from finite-dimensions are more
 44 subtle. We refer to [21] for a listing of failures of equivalences around ISS known from
 45 finite-dimensional systems. In most of the mentioned infinite-dimensional references,
 46 systems of the form (1.1) with $f: X \times U \rightarrow X$ and Banach spaces X and U are
 47 considered. For linear equations, this setting corresponds to evolution equations of
 48 the form

$$49 \quad (1.3) \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

50 where B is a bounded control operator (note that for fixed t , $x(t) = x(t, \cdot)$ is a function
 51 and \dot{x} denotes the time-derivative). Analogously to finite-dimensions, in this case, ISS
 52 and iISS are known to be equivalent, see e.g., [19, Cor. 2] and Proposition 2.14 below.
 53 However, concerning applications the requirement of bounded control operators B is
 54 rather restrictive. Typical examples for systems which only allow for a formulation
 55 with an unbounded B are boundary control systems. It is clear that such phenomena
 56 cannot occur for linear systems in finite-dimensions.

57 The main point of this paper is to relate and characterize (integral) input-to-state
 58 stability for linear, infinite-dimensional systems with unbounded control operators, i.e.
 59 systems of the form (1.3) with unbounded operators B . This is done by using the
 60 notion of *admissibility*, [25, 31], which also reveals the connection of the mentioned
 61 stability types with the boundedness of the linear mapping

$$62 \quad Z \rightarrow X, \quad u \mapsto x(t)$$

63 (for $x_0 = 0$). It is not surprising that the choice of topology for Z , the space of inputs
 64 u , is crucial here. However, looking at (1.2) for $x_0 = 0$, it is not clear how the right-
 65 hand side could define a norm for general functions μ and θ . The question of the right
 66 norm for Z motivates one to study ISS and iISS with respect to general spaces Z – not
 67 only $Z = L^\infty = L^\infty(0, \infty; U)$. For the precise definition of these notions, we refer to
 68 Section 2. We show that Z -ISS and Z -iISS are equivalent for $Z = L^p = L^p(0, \infty; U)$,
 69 $p \in [1, \infty)$. However, it turns out that this paves the way to characterize L^∞ -iISS
 70 in terms of ISS. More precisely, we will show that L^∞ -iISS is equivalent to ISS with
 71 respect to some *Orlicz space*. This is one of the main results of this work. Orlicz
 72 spaces (or Orlicz–Birbaum spaces) appear naturally as generalizations of L^p -spaces
 73 and ISS with respect to such spaces can thus be seen as a generalization of classical
 74 stability notions. Other choices for general input functions have been made in the
 75 literature – like admissibility with respect to Lorentz spaces [6, 33] or Z -ISS with Z

	Eq. (1.3), B bounded	Eq. (1.3), B unbounded	Eq. (1.1), f nonlinear
$\dim X < \infty$	$\text{ISS} \iff \text{iISS}$	$\text{ISS} \iff \text{iISS}$	$\text{ISS} \xRightarrow{\neq} \text{iISS}$
$\dim X = \infty$	$\text{ISS} \iff \text{iISS}$	$\text{ISS} \begin{matrix} \xleftarrow{?} \\ \xrightarrow{?} \end{matrix} \text{iISS}$	not clear

TABLE 1.1

The relation between ISS and iISS (with respect to L^∞) in various settings.

76 being a Sobolev space [9, 18].

77 As we will see, it is plain that Z -iISS always implies Z -ISS for linear systems. The
 78 converse direction, for $Z = L^\infty$, remains open in general. It is known that ISS is
 79 equivalent to admissibility (together with exponential stability). We will show that
 80 L^∞ -iISS in fact implies *zero-class admissibility* [8, 34], which is slightly stronger than
 81 admissibility, see Proposition 2.13. In Table 1.1, the relation of L^∞ -ISS and L^∞ -iISS
 82 in the various above-mentioned settings is depicted schematically.

83 In Section 2, we will discuss the setting and formally introduce the stability
 84 notions mentioned above. This includes a general abstract definition of ISS, iISS and
 85 admissibility with respect to some function space Z . Furthermore, we will give some
 86 basic facts about their relation.

87 Section 3 deals with the characterization of ISS and iISS in terms of Orlicz-space-
 88 admissibility. As a main result, we show that L^∞ -iISS is equivalent to ISS with
 89 respect to some Orlicz space E_Φ , where Φ denotes a Young function, Theorem 3.16.
 90 Moreover, we show that ISS with respect to an Orlicz space is a natural generalization
 91 of classic L^p -ISS that “interpolates” the notions of L^1 - and L^∞ -ISS, Theorems 3.17
 92 and 3.19.

93 In Section 4, we consider parabolic diagonal systems with scalar input. More
 94 precisely, we assume that A possesses a Riesz basis of eigenvectors with eigenvalues
 95 lying in a sector in the open left half-plane. For this class of systems we show that
 96 L^∞ -ISS implies ISS with respect to some Orlicz space and thus, by the results of
 97 Section 3, the equivalence between iISS and ISS, known in finite dimensions, holds for
 98 this class of systems. Moreover, it turns out that any linear, bounded operator from
 99 U to the extrapolation space X_{-1} is L^∞ -admissible, which yields a characterization of
 100 ISS. The results of this section partially generalize results that were already indicated
 101 in [7].

102 We illustrate the obtained results by examples in Section 5. In particular, we
 103 present a parabolic diagonal system which is L^∞ -ISS, but not L^p -ISS for any $p \in$
 104 $[1, \infty)$. Finally, we conclude by drawing a connection between the question whether
 105 L^∞ -ISS implies L^∞ -iISS and a problem due to G. Weiss.

106 **2. Stability notions for infinite-dimensional systems.**

107 **2.1. The setting and definitions.** In this article we study systems $\Sigma(A, B)$ of
 108 the following form

109 (2.4)
$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0,$$

110 where A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X and B is a linear
 111 and bounded operator from a Banach space U to the extrapolation space X_{-1} . Note
 112 that B is possibly unbounded from U to X . Here X_{-1} is the completion of X with

113 respect to the norm

$$114 \quad \|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X,$$

115 for some $\beta \in \rho(A)$, the resolvent set of A . It can be shown that the semigroup
 116 $(T(t))_{t \geq 0}$ possesses a unique extension to a C_0 -semigroup $(T_{-1}(t))_{t \geq 0}$ on X_{-1} with
 117 generator A_{-1} , which is an extension of A . Thus we may consider equation (2.4)
 118 on the Banach space X_{-1} and therefore for $u \in L^1_{loc}(0, \infty; U)$, the (mild) solution of
 119 (2.4) is given by the variation of parameters formula

$$120 \quad (2.5) \quad x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s) ds, \quad t \geq 0.$$

121 In this paper, we will consider the following types of function spaces.

122 *Assumption 2.1.* For a Banach space U , let $Z \subseteq L^1_{loc}(0, \infty; U)$ be such that for
 123 all $t > 0$

- 124 (a) $Z(0, t; U) := \{f \in Z \mid f|_{[t, \infty)} = 0\}$ becomes a Banach space of functions on
 125 the interval $(0, t)$ with values in U (in the sense of equivalence classes w.r.t.
 126 equality almost everywhere),
 127 (b) $Z(0, t; U)$ is continuously embedded in $L^1(0, t; U)$, that is, there exists $\kappa(t) > 0$
 128 such that for all $f \in Z(0, t; U)$ it holds that $f \in L^1(0, t; U)$ and

$$129 \quad \|f\|_{L^1(0, t; U)} \leq \kappa(t) \|f\|_{Z(0, t; U)}.$$

- 130 (c) For $u \in Z(0, t; U)$ and $s > t$ we have $\|u\|_{Z(0, t; U)} = \|u\|_{Z(0, s; U)}$.
 131 (d) $Z(0, t; U)$ is invariant under the left-shift and reflection, i.e., $S_\tau Z(0, t; U) \subset$
 132 $Z(0, t; U)$ and $R_t Z(0, t; U) \subset Z(0, t; U)$, where

$$S_\tau u = u(\cdot + \tau), \quad R_t u = u(t - \cdot),$$

131 and $\tau > 0$. Furthermore, $\|S_\tau\|_{\mathcal{L}(Z(0, t; U))} \leq 1$ and R_t is isometric.

- 132 (e) For all $u \in Z$ and $0 < t < s$ it holds that $u|_{(0, t)} \in Z(0, t; U)$ and

$$133 \quad \|u|_{(0, t)}\|_{Z(0, t; U)} \leq \|u|_{(0, s)}\|_{Z(0, s; U)}.$$

134 If additionally we have in (b) that

$$135 \quad (B) \quad \kappa(t) \rightarrow 0, \quad \text{as } t \searrow 0,$$

136 then we say that Z satisfies *condition (B)*.

137 For example, $Z = L^p$ refers to the spaces $L^p(0, t; U)$, $t > 0$, for fixed $1 \leq p \leq \infty$ and
 138 U . Other examples can be given by Sobolev spaces and the Orlicz spaces $L_\Phi(0, t; U)$
 139 and $E_\Phi(0, t; U)$, see the appendix. If $p > 1$ (including $p = \infty$) and Φ is a Young
 140 function, then L^p , E_Φ and L_Φ satisfy Condition (B), thanks to Hölder's inequality.
 141 Clearly, L^1 does not satisfy condition (B).

142 In general, the state $x(t)$ given by (2.5) lies in X_{-1} for $u \in L^1_{loc}$ and $t > 0$. The
 143 notion of *admissibility* ensures that indeed $x(t) \in X$.

144 **DEFINITION 2.2.** We call the system $\Sigma(A, B)$ admissible with respect to Z (or
 145 Z -admissible), if

$$146 \quad (2.6) \quad \int_0^t T_{-1}(s)Bu(s) ds \in X$$

147 for all $t > 0$ and $u \in Z(0, t; U)$. If $\Sigma(A, B)$ is admissible with respect to Z , then all
 148 mild solutions (2.5) are in X and by the closed graph theorem there exists a constant
 149 $c(t)$ (take the infimum over all possible constants) such that

$$150 \quad (2.7) \quad \left\| \int_0^t T_{-1}(s)Bu(s) ds \right\| \leq c(t)\|u\|_{Z(0,t;U)}.$$

151 Moreover, it is easy to see that $\Sigma(A, B)$ is admissible if (2.6) holds for one $t > 0$.

152 **DEFINITION 2.3.** *We call the system $\Sigma(A, B)$ infinite-time admissible with respect*
 153 *to Z (or Z -infinite-time admissible), if the system is admissible with respect to Z and*
 154 *$c_\infty := \sup_{t>0} c(t)$ is finite. We call the system $\Sigma(A, B)$ zero-class admissible with*
 155 *respect to Z (or Z -zero-class admissible), if it is admissible with respect to Z and*
 156 *$\lim_{t \rightarrow 0} c(t) = 0$.*

157 **Remark 2.4.** Clearly, zero-class admissibility and infinite-time admissibility imply
 158 admissibility respectively.

159 Since $Z \subseteq L^1_{loc}(0, \infty; U)$, for any $u \in Z$ and any initial value x_0 , the mild solution x
 160 of (2.4) is continuous as function from $[0, \infty)$ to X_{-1} . Next we show that zero-class
 161 admissibility guarantees that x even lies in $C(0, \infty; X)$.

162 **PROPOSITION 2.5.** *If $\Sigma(A, B)$ is Z -zero-class admissible, then for every $x_0 \in X$*
 163 *and every $u \in Z$ the mild solution of (2.4), given by (2.5), satisfies $x \in C([0, \infty); X)$.*

164 *Proof.* Since x is given by (2.5), it suffices to consider the case $x_0 = 0$. Let $u \in Z$.
 165 We have to show that $t \mapsto \Phi_t u := \int_0^t T_{-1}(s)Bu(s) ds$ is continuous. The proof is
 166 divided into two steps.

167 First, note that $t \mapsto \Phi_t u$ is right-continuous on $[0, \infty)$. In fact, by

$$168 \quad \Phi_{t+h}u - \Phi_t u = T(t) \int_0^h T_{-1}(s)Bu(s+t) ds,$$

169 $h > 0$, and Z -zero-class admissibility, it follows that

$$171 \quad \|\Phi_{t+h}u - \Phi_t u\| \leq c(h)\|T(t)\| \|u(\cdot + t)\|_{Z(0,h;U)} \rightarrow 0$$

172 for $h \searrow 0$ (where we used properties (d), (e) of Z).

173 Second, we show that $t \mapsto \Phi_t$ is left-continuous on $(0, \infty)$. Since $(\Phi_t - \Phi_{t-h})u =$
 174 $(\Phi_t - \Phi_{t-h})u|_{(0,t)}$, we can assume that $u \in Z(0, t; U)$. Clearly,

$$175 \quad (\Phi_t - \Phi_{t-h})u = T(t-h) \int_0^h T_{-1}(s)Bu(s+t-h) ds.$$

176 It follows that

$$177 \quad \left\| \int_0^h T_{-1}(s)Bu(s+t-h) ds \right\| \leq c(h)\|u(\cdot + t-h)\|_{Z(0,h;U)} \\ 178 \quad \leq c(h)\|u(\cdot + t-h)\|_{Z(0,t;U)} \\ 179 \quad \leq c(h)\|u\|_{Z(0,t;U)} \xrightarrow{h \searrow 0} 0,$$

181 where the last two inequalities hold by properties (e) and (d) of Z . Since $(T(t))_{t \geq 0}$
 182 is uniformly bounded on compact intervals, we conclude that $\|\Phi_{t+h}u - \Phi_t u\| \rightarrow 0$ as
 183 $h \rightarrow 0$. \square

184 *Remark 2.6.* If $\Sigma(A, B)$ is admissible with respect to L^p , $1 \leq p < \infty$, then,
 185 by Hölder's inequality, $\Sigma(A, B)$ is L^q -zero-class admissible for any $q > p$. Thus,
 186 Proposition 2.5 implies that the mild solution of (2.4) lies in $C(0, \infty; X)$ for all $u \in L^q$.
 187 Moreover, this continuity even holds for $u \in L^p$, which was already shown by G. Weiss
 188 in his seminal paper [31, Prop. 2.3] on admissible control operators. However, there,
 189 a direct, but similar proof is used without using the notion of zero-class admissibility.
 190 As stated in [31, Problem 2.4], it is an interesting open problem whether the continuity
 191 of x is implied by L^∞ -admissibility. By Proposition 2.5, the answer is 'yes' in the case
 192 of L^∞ -zero-class admissibility. See also Section 6.

193 To introduce input-to-state stability, we will need the following well-known func-
 194 tion classes from Lyapunov theory. Here, \mathbb{R}_0^+ denotes the set of nonnegative real
 195 numbers.

$$\begin{aligned} 196 \quad \mathcal{K} &= \{\mu: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \mu(0) = 0, \mu \text{ continuous, strictly increasing}\}, \\ 197 \quad \mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x \rightarrow \infty} \theta(x) = \infty\}, \\ 198 \quad \mathcal{L} &= \{\gamma: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \gamma \text{ continuous, strictly decreasing, } \lim_{t \rightarrow \infty} \gamma(t) = 0\}, \\ 199 \quad \mathcal{KL} &= \{\beta: (\mathbb{R}_0^+)^2 \rightarrow \mathbb{R}_0^+ \mid \beta(\cdot, t) \in \mathcal{K} \forall t \geq 0 \text{ and } \beta(s, \cdot) \in \mathcal{L} \forall s > 0\}. \end{aligned}$$

201 **DEFINITION 2.7.** *The system $\Sigma(A, B)$ is called input-to-state stable with respect*
 202 *to Z (or Z -ISS), if there exist functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that for every*
 203 *$t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$*

- 204 (i) $x(t)$ lies in X and
- 205 (ii) $\|x(t)\| \leq \beta(\|x_0\|, t) + \mu(\|u\|_{Z(0, t; U)})$.

206 *The system $\Sigma(A, B)$ is called integral input-to-state stable with respect to Z (or*
 207 *Z -iISS), if there exist functions $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for every $t \geq 0$,*
 208 *$x_0 \in X$ and $u \in Z(0, t; U)$*

- 209 (i) $x(t)$ lies in X and
- 210 (ii) $\|x(t)\| \leq \beta(\|x_0\|, t) + \theta \left(\int_0^t \mu(\|u(s)\|_U) ds \right)$.

211 *The system $\Sigma(A, B)$ is called uniformly bounded energy bounded state with re-*
 212 *spect to Z (or Z -UBEBS), if there exist functions $\gamma, \theta \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and a constant*
 213 *$c > 0$ such that for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$*

- 214 (i) $x(t)$ lies in X and
- 215 (ii) $\|x(t)\| \leq \gamma(\|x_0\|) + \theta \left(\int_0^t \mu(\|u(s)\|_U) ds \right) + c$.

216 **Remark 2.8.** 1. By the inclusion of L^p spaces on bounded intervals we ob-
 217 tain that L^p -ISS (L^p -iISS, L^p -UBEBS) implies L^q -ISS (L^q -iISS, L^q -UBEBS)
 218 for all $1 \leq p < q \leq \infty$. Further the inclusions $L^\infty \subseteq E_\Phi \subseteq L_\Phi \subseteq L^1$ and
 219 $Z \subseteq L_{loc}^1$ yield a corresponding chain of implications of ISS, iISS and UBEBS.

220 2. Note that in general the integral $\int_0^t \mu(\|u(s)\|_U) ds$ in the inequalities defining
 221 Z -iISS and Z -UBEBS may be infinite. In that case, the inequalities hold
 222 trivially. This indicates that the major interest in iISS and UBEBS lies in
 223 the case $Z = L^\infty$, in which the integral is always finite.

224 **2.2. Relations between the stability notions.** Recall that the semigroup
 225 $(T(t))_{t \geq 0}$ is called exponentially stable, if there exist constants $M, \omega > 0$ such that

$$226 \quad (2.8) \quad \|T(t)\| \leq M e^{-\omega t}, \quad t \geq 0.$$

227

228 LEMMA 2.9. Let $(T(t))_{t \geq 0}$ be exponentially stable and $\Sigma(A, B)$ be Z -admissible.
 229 Then the following holds.

- 230 (i) $\Sigma(A, B)$ is infinite-time Z -admissible.
 231 (ii) $\Sigma(A, B)$ is Z -iISS if and only if there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for
 232 every $u \in Z(0, 1; U)$,

$$233 \quad (2.9) \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq \theta \left(\int_0^1 \mu(\|u(s)\|_U) ds \right).$$

234 Moreover, if (2.9) holds, then $\Sigma(A, B)$ is Z -iISS with the same choice of μ .

235 *Proof.* By the representation of the solution (2.5) for $x_0 = 0$, it follows that the
 236 condition in (ii) is necessary for Z -iISS. For the sufficiency it is enough to consider
 237 $x_0 = 0$ by exponential stability. Therefore, both (i) and (ii) hold if we can show
 238 that there exists $C > 0$ such that for any $t > 0$ and $u \in Z(0, t; U)$, there exists
 239 $\tilde{u} \in Z(0, 1; U)$ such that the following three inequalities hold:

$$240 \quad \left\| \int_0^t T_{-1}(s)Bu(s) ds \right\| \leq C \left\| \int_0^1 T_{-1}(s)B\tilde{u}(s) ds \right\|,$$

$$241 \quad \|\tilde{u}\|_{Z(0,1;U)} \leq \|u\|_{Z(0,t;U)},$$

$$242 \quad \int_0^1 \mu(\|\tilde{u}(s)\|_U) ds \leq \int_0^t \mu(\|u(s)\|_U) ds \quad \forall \mu \in \mathcal{K}.$$

244 Without loss of generality, we assume that $t \in \mathbb{N}$, otherwise extend u suitably by
 245 the zero-function. By splitting the integral, substitution and the fact that $\Sigma(A, B)$ is
 246 Z -admissible, we get for $u \in Z(0, t; U)$,

$$247 \quad \left\| \int_0^t T_{-1}(s)Bu(s) ds \right\| = \left\| \sum_{k=0}^{t-1} \int_k^{k+1} T_{-1}(s)Bu(s) ds \right\|$$

$$248 \quad = \left\| \sum_{k=0}^{t-1} T(k) \int_0^1 T_{-1}(s)Bu(s+k) ds \right\|$$

$$249 \quad \leq \sum_{k=0}^{t-1} \|T(k)\| \max_{k=0, \dots, t-1} \left\| \int_0^1 T_{-1}(s)Bu(s+k) ds \right\|$$

$$250 \quad \leq C \cdot \max_{k=0, \dots, t-1} \left\| \int_0^1 T_{-1}(s)Bu(s+k) ds \right\|,$$

252 where $C < \infty$ only depends on the exponentially stable semigroup $(T(t))_{t \geq 0}$. Choose
 253 $\tilde{u} = u(\cdot + k_0)|_{(0,1)}$, where k_0 is the argument such that the above maximum is at-
 254 tained. Clearly, $\int_0^1 \mu(\|\tilde{u}(s)\|_U) ds \leq \int_0^t \mu(\|u(s)\|_U) ds$. We now use the properties of
 255 Z described in Assumption 2.1. By (d), $u(\cdot + k_0) \in Z(0, t; U)$ and $\|u(\cdot + k_0)\|_{Z(0,t;U)} \leq$
 256 $\|u\|_{Z(0,t;U)}$. Therefore, Property (e) implies that $\tilde{u} \in Z(0, 1; U)$ with $\|\tilde{u}\|_{Z(0,1;U)} \leq$
 257 $\|u(\cdot + k_0)\|_{Z(0,t;U)} \leq \|u\|_{Z(0,t;U)}$. \square

258 Note that (i) in Lemma 2.9 for the case $Z = L^p$ is well-known and can e.g. be found
 259 in [30] for $p = 2$.

260 PROPOSITION 2.10. Let $Z \subseteq L^1_{loc}(0, \infty; U)$ be a function space. Then we have:

- 261 (i) The following statements are equivalent
 262 (a) $\Sigma(A, B)$ is Z -ISS,

- 263 (b) $\Sigma(A, B)$ is Z -admissible and $(T(t))_{t \geq 0}$ is exponentially stable,
 264 (c) $\Sigma(A, B)$ is Z -infinite-time admissible and $(T(t))_{t \geq 0}$ is exponentially stable.
 265
 266 (ii) If $\Sigma(A, B)$ is Z -iISS, then the system is Z -admissible and $(T(t))_{t \geq 0}$ is exponentially stable,
 267
 268 (iii) If $\Sigma(A, B)$ is Z -UBEBS, then the system is Z -admissible and $(T(t))_{t \geq 0}$ is bounded, that is, (2.8) holds for $\omega = 0$.
 269

270 *Proof.* Clearly, Z -ISS, Z -iISS and Z -UBEBS imply Z -admissibility (consider $x_0 =$
 271 0 in (2.5) and observe that $x(t) \in X$ for all $t > 0$). Further, Z -admissibility and
 272 exponential stability of $(T(t))_{t \geq 0}$ show Z -ISS, see Remark 2.4. If, $\Sigma(A, B)$ is Z -
 273 ISS or Z -iISS, by setting $u = 0$, it follows that $\|T(t)\| < 1$ for sufficiently large t ,
 274 which shows that $(T(t))_{t \geq 0}$ is exponentially stable. It is easy to see that Z -UBEBS
 275 implies boundedness of $(T(t))_{t \geq 0}$. Finally, by Remark 2.4 items (b) and (c) in (i) are
 276 equivalent. \square

277 **PROPOSITION 2.11.** *If $1 \leq p < \infty$, then the following are equivalent*

- 278 (i) $\Sigma(A, B)$ is L^p -ISS,
 279 (ii) $\Sigma(A, B)$ is L^p -iISS,
 280 (iii) $\Sigma(A, B)$ is L^p -UBEBS and $(T(t))_{t \geq 0}$ is exponentially stable.

281 *Proof.* Clearly, by the definition of iISS and UBEBS, (ii) \Rightarrow (iii). By Proposition
 282 2.10, (iii) \Rightarrow (i). Thus in view of Proposition 2.10 it remains to show that L^p -infinite-
 283 time admissibility and exponential stability imply L^p -iISS. Indeed, L^p -infinite-time
 284 admissibility and exponential stability show for $x_0 \in X$ and $u \in L^p(0, t; U)$ that

$$285 \quad \|x(t)\| \leq M e^{-\omega t} \|x_0\| + c_\infty \|u\|_{L^p(0, t; U)}$$

$$286 \quad = M e^{-\omega t} \|x_0\| + c_\infty \left(\int_0^t \|u(s)\|_U^p ds \right)^{1/p},$$
 287

288 which shows L^p -iISS. \square

289 *Remark 2.12.* Let $1 \leq p < \infty$. If the system $\Sigma(A, B)$ is L^p -admissible and
 290 $(T(t))_{t \geq 0}$ is exponentially stable, then the system $\Sigma(A, B)$ is L^p -ISS with the fol-
 291 lowing choices for the functions β and μ :

$$292 \quad \beta(s, t) := M e^{-\omega t} s \quad \text{and} \quad \mu(s) := c_\infty s.$$

293 Here the constants M and ω are given by (2.8) and $c_\infty = \sup_{t \geq 0} c(t)$.

294 **PROPOSITION 2.13.** *If $\Sigma(A, B)$ is L^∞ -iISS, then $\Sigma(A, B)$ is L^∞ -zero-class admis-*
 295 *sible.*

296 *Proof.* If $\Sigma(A, B)$ is L^∞ -iISS, then there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for
 297 all $t > 0$, $u \in L^\infty(0, t; U)$, $u \neq 0$

$$298 \quad (2.10) \quad \frac{1}{\|u\|_\infty} \left\| \int_0^t T_{-1}(s) B u(s) ds \right\| \leq \theta \left(\int_0^t \mu \left(\frac{\|u(s)\|_U}{\|u\|_\infty} \right) ds \right).$$

299 Since the function μ is monotonically increasing and $\|u(s)\|_U \leq \|u\|_\infty$ a.e., the right-
 300 hand side of (2.10) is bounded above by $\theta(t\mu(1))$ which converges to zero as $t \searrow 0$. \square

301 We illustrate the relations of the different stability notions with respect to L^∞
 302 discussed above in the diagram depicted in Figure 2.1.

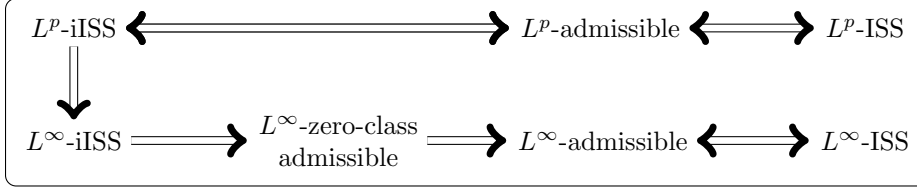


FIG. 2.1. Relations between the different stability notions with respect to L^p , $p < \infty$, and L^∞ for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable.

303 PROPOSITION 2.14. Suppose that B is a bounded operator from U to X and $Z \subseteq$
 304 $L^1_{loc}(0, \infty; U)$ is a function space as in Section 2.1. Then the following statements are
 305 equivalent.

- 306 (i) $(T(t))_{t \geq 0}$ is exponentially stable,
- 307 (ii) $\Sigma(A, B)$ is Z -admissible and $(T(t))_{t \geq 0}$ is exponentially stable,
- 308 (iii) $\Sigma(A, B)$ is Z -infinite-time admissible and $(T(t))_{t \geq 0}$ is exponentially stable,
- 309 (iv) $\Sigma(A, B)$ is Z -ISS,
- 310 (v) $\Sigma(A, B)$ is Z -iISS,
- 311 (vi) $\Sigma(A, B)$ is Z -UBEBS and $(T(t))_{t \geq 0}$ is exponentially stable,
- 312 (vii) $\Sigma(A, B)$ is L^1_{loc} -admissible and $(T(t))_{t \geq 0}$ is exponentially stable.

313 If Z satisfies Assumption (B), then the above assertions are equivalent to

- 314 (viii) $\Sigma(A, B)$ is Z -zero-class admissible and $(T(t))_{t \geq 0}$ is exponentially stable.

315 *Proof.* By Proposition 2.10 we have (v) \Rightarrow (vi) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i), and
 316 Proposition 2.11 and Remark 2.8 prove (vii) \Rightarrow (v). The implication (i) \Rightarrow (vii)
 317 follows from the fact that by the boundedness of B we have $x(t) \in X$ for all $t \geq 0$ and
 318 all $u \in L^1(0, t; U)$. Clearly, (viii) \Rightarrow (ii). Thus it remains to show that if Z satisfies
 319 Assumption (B), then (i) \Rightarrow (viii). Let $(T(t))_{t \geq 0}$ be exponentially stable, that is, there
 320 exist constants $M, \omega > 0$ such that (2.8) holds. Therefore, for any $u \in L^1(0, t; U)$,

$$\begin{aligned}
 321 \quad \|x(t)\| &\leq M e^{-\omega t} \|x_0\| + M \|B\| \int_0^t e^{-\omega(t-s)} \|u(s)\|_U ds \\
 322 \quad (2.11) \quad &\leq M e^{-\omega t} \|x_0\| + M \|B\| \int_0^t \|u(s)\|_U ds. \\
 323
 \end{aligned}$$

324 Using that $Z(0, t; U)$ is continuously embedded in $L^1(0, t; U)$, we conclude that

$$325 \quad (2.12) \quad \|x(t)\| \leq M e^{-\omega t} \|x_0\| + M \|B\| \kappa(t) \|u\|_{Z(0, t; U)}$$

326 for all $t \geq 0$. If Assumption (B) holds, then the embedding constants $\kappa(t)$ tend to 0
 327 as $t \searrow 0$. Hence, (2.12) shows that (i) implies (viii). \square

328 For the special case $Z = L^p(0, \infty; U)$, parts of the equivalences in Proposition 2.14
 329 can already be found in [19].

330 *Remark 2.15.* Note that in Proposition 2.14, the assertions are independent of
 331 Z as the assertions only rest on exponential stability. In particular, if one of the
 332 equivalent conditions hold, then the system $\Sigma(A, B)$ is L^p -ISS with the following
 333 choices for the functions β and μ

$$334 \quad \beta(s, t) := M e^{-\omega t} s \quad \text{and} \quad \mu(s) := \frac{M}{\omega q} \|B\| s,$$

335 where q is the Hölder conjugate of p , and L^p -iISS with

$$336 \quad \beta(s, t) := Me^{-\omega t}s, \quad \mu(s) := s, \quad \text{and} \quad \theta(s) := sM\|B\|.$$

337 Here the constants M and ω are given by (2.8). Although, in this case a system is
 338 L^p -ISS or L^p -iISS for all p if this holds for some p , the choices for the functions μ ,
 339 however, do depend on p . Note that if B is unbounded, then the question whether a
 340 system is L^p -ISS or L^p -iISS crucially depends on p .

341 Furthermore, note that in the trivial case $X = U = \mathbb{C}$ and $A = -1$, $B = 1$, we have
 342 that the system $\Sigma(A, B)$ is not L^1 -zero-class admissible.

343 **3. ISS from the viewpoint of Orlicz spaces.** In this section we relate L^∞ -
 344 ISS and L^1 -ISS to ISS with respect to Orlicz spaces E_Φ corresponding to a Young
 345 function Φ . The use of Orlicz spaces is motivated by the idea of understanding the
 346 integral appearing in the definition of iISS, (1.2), as some type of norm. For the
 347 definition and fundamental properties of Orlicz spaces and Young functions, we refer
 348 to the Appendix. The main results of this section are summarized in the following
 349 three theorems.

350 **THEOREM 3.16.** *The following statements are equivalent.*

- 351 (i) *There is a Young function Φ such that the system $\Sigma(A, B)$ is E_Φ -ISS,*
- 352 (ii) *$\Sigma(A, B)$ is L^∞ -iISS,*
- 353 (iii) *$(T(t))_{t \geq 0}$ is exponentially stable and there is a Young function Φ such that*
 354 *the system $\Sigma(A, B)$ is E_Φ -UBEBS.*

355 If Φ satisfies the Δ_2 -condition (see Definition A.42) more can be said.

356 **THEOREM 3.17.** *If Φ is a Young function that satisfies the Δ_2 -condition, then the*
 357 *following are equivalent.*

- 358 (i) *$\Sigma(A, B)$ is E_Φ -ISS,*
- 359 (ii) *$\Sigma(A, B)$ is E_Φ -iISS,*
- 360 (iii) *$\Sigma(A, B)$ is E_Φ -UBEBS and $(T(t))_{t \geq 0}$ is exponentially stable.*

361 **Remark 3.18.** Since L^p -spaces are examples of Orlicz spaces where the Δ_2 -condition
 362 is satisfied, Theorem 3.17 can be seen as a generalization of Proposition 2.11.

363 **THEOREM 3.19.** *The following statements are equivalent.*

- 364 (i) *$\Sigma(A, B)$ is L^1 -ISS,*
- 365 (ii) *$\Sigma(A, B)$ is L^1 -iISS,*
- 366 (iii) *$\Sigma(A, B)$ is E_Φ -ISS for every Young function Φ .*

367 The proofs of Theorems 3.16, 3.17 and 3.19 are given at the end of this section.

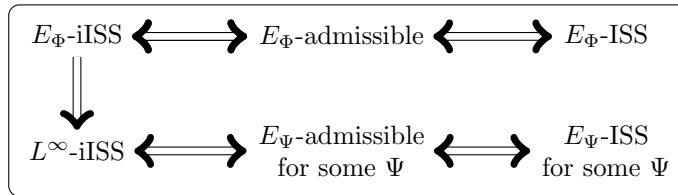


FIG. 3.2. Relations between the different stability notions with respect to Orlicz spaces for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable and that Φ satisfies the Δ_2 -condition.

368 LEMMA 3.20. Let $\Sigma(A, B)$ be L^∞ -iISS. Then there exist $\tilde{\theta}, \Phi \in \mathcal{K}_\infty$ such that Φ
 369 is a Young function which is continuously differentiable on $(0, \infty)$ and

$$370 \quad (3.13) \quad \left\| \int_0^t T_{-1}(s)Bu(s) ds \right\| \leq \tilde{\theta} \left(\int_0^t \Phi(\|u(s)\|_U) ds \right)$$

371 for all $t > 0$ and $u \in L^\infty(0, t; U)$.

372 *Proof.* By assumption, $(T(t))_{t \geq 0}$ is exponentially stable and there exist $\theta \in \mathcal{K}_\infty$
 373 and $\mu \in \mathcal{K}$ such that (2.9) holds for $Z = L^\infty$. Without loss of generality we can assume
 374 that μ belongs to \mathcal{K}_∞ . By Lemma 14 in [23] there exist a convex function $\mu_v \in \mathcal{K}_\infty$ and
 375 a concave function $\mu_c \in \mathcal{K}_\infty$ such that both are continuously differentiable on $(0, \infty)$
 376 and $\mu \leq \mu_c \circ \mu_v$ holds on $[0, \infty)$. Now for any Young function $\Psi: [0, \infty) \rightarrow [0, \infty)$ it
 377 is straightforward to check that $\mu_c \circ \Psi^{-1}$ is a concave function and hence we have by
 378 Jensen's inequality

$$379 \quad \theta \left(\int_0^1 \mu(\|u(s)\|_U) ds \right) \leq \theta \left(\int_0^1 \mu_c \circ \mu_v(\|u(s)\|_U) ds \right) \\ \leq (\theta \circ \mu_c \circ \Psi^{-1}) \left(\int_0^1 (\Psi \circ \mu_v)(\|u(s)\|_U) ds \right).$$

380 Using Remark 3.2.7 in [15] it is easy to see that $\Phi := \Psi \circ \mu_v$ is a Young function.
 381 Taking $\tilde{\theta} := \theta \circ \mu_c \circ \Psi^{-1}$ we obtain the desired estimate for $t = 1$. By Lemma 2.9, the
 382 assertion follows. \square

383 *Proof of Theorem 3.16. (i) \Rightarrow (ii):* Since $\Lambda(s) = s^2$ defines a Young function with
 384 $\Lambda(1) = 1$, it can be easily seen that

$$385 \quad \Phi_1(s) = \begin{cases} \Phi(s), & s < 1, \\ \Phi(\Lambda(s)), & s \geq 1, \end{cases}$$

386 defines another Young function such that $\Phi \leq \Phi_1$. Furthermore, Φ_1 increases essen-
 387 tially more rapidly than Φ (see Def. A.43), since the composition $\Phi \circ \Lambda$ of two Young
 388 functions Φ, Λ is known to be increasing essentially more rapidly than Φ (see page
 389 114 of [14]). We define $\theta: [0, \infty) \rightarrow [0, \infty)$ by

$$390 \quad \theta(\alpha) = \sup \left\{ \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \mid u \in L^\infty(0, 1; U), \int_0^1 \Phi_1(\|u(s)\|_U) ds \leq \alpha \right\},$$

391 for $\alpha > 0$ and $\theta(0) = 0$. Clearly, θ is non-decreasing. Admissibility with respect to
 392 E_Φ and Remark A.40.4 yield that for $u \in L^\infty(0, 1; U)$,

$$393 \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq c(1)\|u\|_{E_\Phi(0,1;U)} \leq c(1) \left(1 + \int_0^1 \Phi_1(\|u(s)\|_U) ds \right).$$

394 Hence, $\theta(\alpha) < \infty$ for all $\alpha \geq 0$.

395 If we can show that $\lim_{t \searrow 0} \theta(t) = 0$, then, by Lemma 2.5 in [3], there exists $\tilde{\theta} \in \mathcal{K}_\infty$
 396 such that $\theta \leq \tilde{\theta}$ pointwise. Therefore, let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive real
 397 numbers converging to 0. By the definition of θ , for any $n \in \mathbb{N}$ there exists $u_n \in$
 398 $L^\infty(0, 1; U)$ such that

$$399 \quad \int_0^1 \Phi_1(\|u_n(s)\|_U) ds \leq \alpha_n$$

400 and

$$401 \quad (3.14) \quad \left\| \theta(\alpha_n) - \left\| \int_0^1 T_{-1}(s) B u_n(s) ds \right\| \right\| < \frac{1}{n}.$$

402 Hence the sequence $(\|u_n(\cdot)\|_U)_{n \in \mathbb{N}}$ is Φ_1 -mean convergent to zero (see Def. A.41). By
403 Theorem A.44, the sequence even converges to zero with respect to the norm of the
404 space $L_\Phi(0, 1)$, and thus also in $E_\Phi(0, 1)$. Hence

$$405 \quad \lim_{n \rightarrow \infty} \|u_n\|_{E_\Phi(0,1;U)} = \lim_{n \rightarrow \infty} \|\|u_n(\cdot)\|_U\|_{E_\Phi(0,1)} = 0,$$

406 where we used Remark A.40.2. Hence, by admissibility,

$$407 \quad \left\| \int_0^1 T_{-1}(s) B u_n(s) ds \right\| \leq c(1) \|u_n\|_{E_\Phi(0,1;U)} \rightarrow 0,$$

408 as $n \rightarrow \infty$. Altogether we obtain that

$$409 \quad \begin{aligned} \theta(\alpha_n) &\leq \left| \theta(\alpha_n) - \left\| \int_0^1 T_{-1}(s) B u_n(s) ds \right\| \right| + \left\| \int_0^1 T_{-1}(s) B u_n(s) ds \right\| \\ &\leq \frac{1}{n} + c(1) \|u_n\|_{E_\Phi(0,1;U)}, \end{aligned}$$

410 and thus, $\lim_{n \rightarrow \infty} \theta(\alpha_n) = 0$.

411 Therefore, there exists $\tilde{\theta} \in \mathcal{K}_\infty$ such that $\theta \leq \tilde{\theta}$ pointwise. Furthermore, Φ_1 is a
412 Young function, in particular we have $\Phi_1 \in \mathcal{K}_\infty$. The definition of θ yields that

$$413 \quad \left\| \int_0^1 T_{-1}(s) B u(s) ds \right\| \leq \theta \left(\int_0^1 \Phi_1(\|u(s)\|_U) ds \right) \leq \tilde{\theta} \left(\int_0^1 \Phi_1(\|u(s)\|_U) ds \right)$$

414 for all $u \in L^\infty(0, 1; U)$. By Lemma 2.9, we conclude that $\Sigma(A, B)$ is L^∞ -iISS.

415

416 (ii) \Rightarrow (i): Now assume that $\Sigma(A, B)$ is L^∞ -iISS. We need to show that for some
417 Young function Φ the system $\Sigma(A, B)$ is E_Φ -ISS. By Proposition 2.10(i) it suffices
418 to show that there is a Young function Φ such that $\int_0^t T_{-1}(s) B u(s) ds \in X$ for all
419 $u \in E_\Phi(0, t)$. Note that since $E_\Phi(0, t; U) \subset L^1(0, t; U)$ for any Young function Φ ,
420 the integral always exists in X_{-1} . By assumption, $\int_0^t T_{-1}(s) B u(s) ds \in X$ for all
421 $u \in L^\infty(0, t)$. By Lemma 3.20, there exist $\tilde{\theta} \in \mathcal{K}_\infty$ and a Young function Φ such that
422 (3.13) holds. Let $u \in E_\Phi$. By definition, there is a sequence $(u_n)_{n \in \mathbb{N}} \subset L^\infty(0, t; U)$
423 such that $\lim_{n \rightarrow \infty} \|u_n - u\|_{E_\Phi(0,t;U)} = 0$. Since $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in
424 $E_\Phi(0, t; U)$, we can assume without loss of generality that $\|u_n - u_m\|_{E_\Phi(0,t;U)} \leq 1$ for
425 all $m, n \in \mathbb{N}$. By [15, Lemma 3.8.4 (i)] this implies that for all $n, m \in \mathbb{N}$,

$$426 \quad \int_0^t \Phi(\|u_n(s) - u_m(s)\|_U) ds \leq \|u_n - u_m\|_{E_\Phi(0,t;U)}.$$

427 Together with (3.13) and the monotonicity of $\tilde{\theta}$, this yields

$$428 \quad \begin{aligned} \left\| \int_0^t T_{-1}(s) B (u_n(s) - u_m(s)) ds \right\| &\leq \tilde{\theta} \left(\int_0^t \Phi(\|u_n(s) - u_m(s)\|_U) ds \right) \\ &\leq \tilde{\theta} (\|u_n - u_m\|_{E_\Phi(0,t;U)}). \end{aligned}$$

429 Hence $(\int_0^t T_{-1}(s)Bu_n(s) ds)_{n \in \mathbb{N}}$ is a Cauchy sequence in X and thus converges. Let y
 430 denote its limit. Since $E_\Phi(0, t; U)$ is continuously embedded in $L^1(0, t; U)$, see Remark
 431 A.40.3, it follows that

$$432 \quad \lim_{n \rightarrow \infty} \int_0^t T_{-1}(s)Bu_n(s) ds = \int_0^t T_{-1}(s)Bu(s) ds$$

433 in X_{-1} . Since X is continuously embedded in X_{-1} , we conclude that

$$434 \quad y = \int_0^t T_{-1}(s)Bu(s) ds.$$

435 Thus, we have shown that $\int_0^t T_{-1}(s)Bu(s) ds \in X$ for all $u \in E_\Phi$ and hence $\Sigma(A, B)$
 436 is admissible with respect to E_Φ .

437

438 (i) \Rightarrow (iii): This follows since for all $u \in E_\Phi(0, t; U)$ it holds that $u \in \tilde{L}_\Phi(0, t; U)$
 439 and

$$440 \quad \|u\|_{E_\Phi} \leq 1 + \int_0^t \Phi(\|u(s)\|_U) ds,$$

441 see Remark A.40.4.

442 (iii) \Rightarrow (i): This follows by (iii) and (i) of Proposition 2.10. \square

443 *Proof of Theorem 3.17.* The implications (ii) \Rightarrow (iii) \Rightarrow (i) follow, analogously as
 444 for the L^p -case, by Proposition 2.10.

445 (i) \Rightarrow (ii): Similarly to the proof of Theorem 3.16, we can define a non-decreasing
 446 function θ by

$$447 \quad \theta(\alpha) = \sup \left\{ \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \mid u \in E_\Phi(0, 1; U), \int_0^1 \Phi(\|u(s)\|_U) ds \leq \alpha \right\},$$

448 for $\alpha > 0$ and $\theta(0) := 0$. By E_Φ -admissibility and Remark A.40.4, we have that

$$449 \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq c(1)\|u\|_{E_\Phi(0, 1; U)} \leq c(1) \left(1 + \int_0^1 \Phi(\|u(s)\|_U) ds \right),$$

450 for $u \in E_\Phi(0, 1; U) \subset \tilde{L}_\Phi(0, t; U)$. Hence, θ is well-defined. In analogy to the proof
 451 of Theorem 3.16, it remains to show that θ is right-continuous at 0. This follows
 452 because Φ satisfies the Δ_2 -condition. In fact, if the latter is true, it is known that a
 453 sequence $(u_n)_{n \in \mathbb{N}}$ in E_Φ converges to 0 if and only if the sequence is Φ -mean convergent
 454 to zero (see Def. A.41). Therefore, $\alpha_n \searrow 0$ implies that there exists a sequence
 455 $u_n \in E_\Phi(0, 1; U)$ that converges to 0 in E_Φ and such that

$$456 \quad \left| \theta(\alpha_n) - \left\| \int_0^1 T_{-1}Bu_n(s) ds \right\| \right| \leq \frac{1}{n}, \quad n \in \mathbb{N}.$$

457 By E_Φ -admissibility, we conclude that $\theta(\alpha_n) \rightarrow 0$ as $n \rightarrow \infty$.

458 Hence, by Lemma 2.4 in [3], we find $\tilde{\theta} \in \mathcal{K}_\infty$ such that $\theta \leq \tilde{\theta}$ pointwise. By definition
 459 of θ , this implies

$$460 \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq \tilde{\theta} \left(\int_0^1 \Phi(\|u(s)\|_U) ds \right)$$

461 for all $u \in E_\Phi(0, 1; U)$. Finally, Lemma 2.9 yields that $\Sigma(A, B)$ is E_Φ -iISS. \square

462 *Proof of Theorem 3.19.* By Propositions 2.10 and 2.11, we only need to show the
 463 equivalence of (i) and (iii). That (i) implies (iii) follows immediately since E_Φ is
 464 continuously embedded in L^1 .

465 Conversely, let $\Sigma(A, B)$ be E_Φ -admissible for every Young function Φ . According to
 466 Proposition 2.10 (a), we have to show that $\Sigma(A, B)$ is L^1 -admissible. Let $t > 0$ and
 467 $u \in L^1(0, t; U)$. It remains to prove that $\int_0^t T_{-1}(s)Bu(s) ds \in X$. By [14, p. 61], there
 468 exists a Young function Φ satisfying the Δ_2 -condition such that $\|u(\cdot)\|_U \in L_\Phi^1$. The
 469 Δ_2 -condition implies that $E_\Phi = L_\Phi$ and $E_\Phi(0, t; U) = L_\Phi(0, t; U)$, see [24, p. 303] or
 470 [26, Thm. 5.2]. Thus $\int_0^t T_{-1}(s)Bu(s) ds \in X$ by assumption. \square

471 **PROPOSITION 3.21.** *Let $\Sigma(A, B)$ be L^∞ -ISS. If there exist a nonnegative function
 472 $f \in L^1(0, 1)$, $\theta \in \mathcal{K}$, a constant $c > 0$ and a Young function μ such that for every
 473 $u \in L^1(0, 1; U)$ with $\int_0^1 f(s)\mu(\|u(s)\|_U) ds < \infty$ one has*

$$474 \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq c + \theta \left(\int_0^1 f(s)\mu(\|u(s)\|_U) ds \right),$$

475 *then $\Sigma(A, B)$ is L^∞ -iISS.*

476 *Proof.* By Theorem 3.16 and Proposition 2.10 it is sufficient to show that there
 477 is a Young function Φ such that the system $\Sigma(A, B)$ is E_Φ -admissible. Theorem A.33
 478 implies that there exists a Young function Ψ such that $f \in \tilde{L}_\Psi(0, 1)$. Let $\tilde{\Phi}$ be the
 479 complementary Young function to Ψ . We define the Young function Φ by $\Phi := \tilde{\Phi} \circ \mu$.
 480 Using Remark A.36 for $u \in E_\Phi(0, 1; U)$ we obtain

$$481 \quad \left\| \int_0^1 T_{-1}(s)Bu(s) ds \right\| \leq c + \theta \left(\int_0^1 f(s)\mu(\|u(s)\|_U) ds \right) \\
 482 \quad \leq c + \theta \left(\int_0^1 \Psi(f(s)) ds + \int_0^1 \tilde{\Phi}(\mu(\|u(s)\|_U)) ds \right). \\
 483$$

484 This shows that for all $u \in E_\Phi(0, 1; U)$ we have

$$485 \quad \int_0^1 T_{-1}(s)Bu(s) ds \in X,$$

486 that is, $\Sigma(A, B)$ is E_Φ -admissible. \square

487 **4. Stability of parabolic diagonal systems.** In the previous section we have
 488 proved that for infinite-dimensional systems L^∞ -iISS implies L^∞ -ISS. It is an open
 489 question whether the converse implication holds. Here, we give a positive answer for
 490 parabolic diagonal systems, which are a well-studied class of systems in the literature,
 491 see e.g. [30].

492 Throughout this section we assume that $U = \mathbb{C}$, $1 \leq q < \infty$ and that the operator A

¹In [14, p. 61] it is actually shown that for given $f \in L^1(0, t)$, there exists a Young function Q
 such that $f \in L_{Q \circ Q}(0, t)$ and such that Q satisfies the Δ' -condition, i.e.,

$$\exists c, u_0 > 0 \forall u, v \geq u_0 : Q(uv) \leq cQ(u)Q(v).$$

In fact, it is easy to see that this property implies that $Q \circ Q$ satisfies

$$\forall u \geq u_0 : (Q \circ Q)(\ell u) \leq k(\ell)(Q \circ Q)(u),$$

for some $\ell > 1$ and $k(\ell) > 0$, which is known to be equivalent to $Q \circ Q$ satisfying the Δ_2 -condition,
 see [14, p. 23].

493 possesses a q -Riesz basis of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a
 494 sector in the open left half-plane \mathbb{C}_- . More precisely, $(e_n)_{n \in \mathbb{N}}$ is a q -Riesz basis of X ,
 495 if $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis and for some constants $c_1, c_2 > 0$ we have

$$496 \quad c_1 \sum_k |a_k|^q \leq \left\| \sum_k a_k e_k \right\|^q \leq c_2 \sum_k |a_k|^q$$

497 for all sequences $(a_k)_{k \in \mathbb{N}}$ in $\ell^q = \ell^q(\mathbb{N})$. Thus without loss of generality we can
 498 assume that $X = \ell^q$ and that $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of ℓ^q . We further assume
 499 that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ lies in \mathbb{C} with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and that there exists a
 500 constant $k > 0$ such that $|\operatorname{Im} \lambda_n| \leq k |\operatorname{Re} \lambda_n|$, $n \in \mathbb{N}$, i.e., $(\lambda_n)_n \subset \mathbb{C} \setminus S_{\pi/2+\theta}$ for some
 501 $\theta \in (0, \pi/2)$, where

$$502 \quad S_{\pi/2+\theta} = \{z \in \mathbb{C} \mid |z| > 0, |\arg z| < \pi/2 + \theta\}.$$

503 Then the linear operator $A: D(A) \subset \ell^q \rightarrow \ell^q$, given by

$$504 \quad Ae_n = \lambda_n e_n, \quad n \in \mathbb{N},$$

505 and $D(A) = \{(x_n) \in \ell^q \mid \sum_n |x_n \lambda_n|^q < \infty\}$, generates an analytic exponentially
 506 stable C_0 -semigroup $(T(t))_{t \geq 0}$ on ℓ^q , which is given by $T(t)e_n = e^{t\lambda_n} e_n$. An easy
 507 calculation shows that the extrapolation space $(\ell^q)_{-1}$ is given by

$$508 \quad (\ell^q)_{-1} = \left\{ x = (x_n)_{n \in \mathbb{N}} \mid \sum_n \frac{|x_n|^q}{|\lambda_n|^q} < \infty \right\},$$

$$509 \quad \|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^q}.$$

511 Thus the linear bounded operator B from \mathbb{C} to $(\ell^q)_{-1}$ can be identified with a sequence
 512 $(b_n)_{n \in \mathbb{N}}$ in \mathbb{C} satisfying

$$513 \quad \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\lambda_n|^q} < \infty.$$

514 Thanks to the sectoriality condition for $(\lambda_n)_{n \in \mathbb{N}}$ this equivalent to

$$515 \quad \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^q} < \infty.$$

516 The following result shows that, under the above assumptions, the system $\Sigma(A, B)$
 517 is L^∞ -iISS. Thus for this class of systems L^∞ -iISS is equivalent to L^∞ -ISS, and both
 518 notions are implied by $B \in (\ell^q)_{-1}$, that is, $\sum_n \frac{|b_n|^q}{|\lambda_n|^q} < \infty$. The following theorem
 519 generalizes the main result in [7], where the case $q = 2$ is studied.

520 **THEOREM 4.22.** *Let $U = \mathbb{C}$, and suppose that the operator A possesses a q -Riesz*
 521 *basis of X that consists of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a*
 522 *sector in the open left half-plane \mathbb{C}_- with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and $B \in \mathcal{L}(\mathbb{C}, X_{-1})$. Then*
 523 *the system $\Sigma(A, B)$ is L^∞ -iISS, and hence also L^∞ -ISS and L^∞ -zero-class admissible.*

524 *Remark 4.23.* In the situation of Theorem 4.22, $\Sigma(A, B)$ is L^∞ -iISS if and only
 525 if $\Sigma(A, B)$ is L^∞ -ISS.

526 *Proof of Theorem 4.22.* Without loss of generality we may assume $X = \ell^q$ and
 527 that $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of ℓ^q . Let $f: (0, \infty) \rightarrow [0, \infty)$ be defined by

$$528 \quad f(s) = \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^{q-1}} e^{\operatorname{Re} \lambda_n s}.$$

529 Then it is easy to see that f belongs to $L^1(0, \infty)$. Now for $u \in L^1(0, 1)$ with
 530 $\int_0^1 f(s)|u(s)|^q ds < \infty$ we obtain (denoting by q' the Hölder conjugate of q)

$$\begin{aligned} 531 \quad & \left\| \int_0^1 T_{-1}(s) B u(s) ds \right\|_{\ell^q}^q = \sum_{n \in \mathbb{N}} |b_n|^q \left| \int_0^1 e^{\lambda_n s} u(s) ds \right|^q \\ 532 \quad & \leq \sum_{n \in \mathbb{N}} |b_n|^q \left(\int_0^1 e^{\operatorname{Re} \lambda_n s} |u(s)| ds \right)^q \\ 533 \quad & = \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{(\operatorname{Re} \lambda_n)^q} \left(\int_0^1 |\operatorname{Re} \lambda_n| e^{\operatorname{Re} \lambda_n s} |u(s)| ds \right)^q \\ 534 \quad & \leq \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{(\operatorname{Re} \lambda_n)^q} \left(\int_0^1 |\operatorname{Re} \lambda_n| e^{\operatorname{Re} \lambda_n s} |u(s)|^q ds \right) \left(\int_0^1 |\operatorname{Re} \lambda_n| e^{\operatorname{Re} \lambda_n s} ds \right)^{q/q'} \\ 535 \quad & \leq \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^q} \left(\int_0^1 |\operatorname{Re} \lambda_n| e^{\operatorname{Re} \lambda_n s} |u(s)|^q ds \right) \\ 536 \quad & = \int_0^1 \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^{q-1}} e^{\operatorname{Re} \lambda_n s} |u(s)|^q ds \\ 537 \quad & = \int_0^1 f(s) |u(s)|^q ds \\ 538 \quad & < \infty. \end{aligned}$$

540 This shows that the system $\Sigma(A, B)$ is L^∞ -ISS and the claim now follows from
 541 Proposition 3.21. \square

542 *Remark 4.24.* Theorem 4.22 states that L^∞ -admissibility implies E_Φ -admissibility
 543 for some Young function Φ in the case of parabolic diagonal systems. A natural ques-
 544 tion is whether Φ can always be chosen such that the Δ_2 -condition is satisfied. Looking
 545 at the proof and having in mind that L^1 equals the union of all spaces E_Ψ where Ψ
 546 satisfies the Δ_2 -condition, this could be expected. However, the answer is negative,
 547 which can be seen as follows. For a Young function Φ satisfying the Δ_2 -condition
 548 there exist constants $x_0 > 0$ and $p \in \mathbb{N} \setminus \{1\}$ such that

$$549 \quad \Phi(x) \leq x^p, \quad x > x_0,$$

550 see [14, p. 25]. This implies that $E_\Phi \supset L^p$, see e.g. [15, Sec. 3.17]. However, there exists
 551 Young functions that do not satisfy the latter estimate, e.g., $\Phi(x) = e^{x-1} - x - e^{-1}$.
 552 In Example 5.29, $\Sigma(A, B)$ is not L^p -admissible for any $p < \infty$, which, with the above
 553 reasoning, implies that the system cannot be E_Φ -admissible for any Φ satisfying the
 554 Δ_2 -condition.

555 **LEMMA 4.25.** *Let μ be a positive regular Borel measure supported on a sector S_ϕ*
 556 *with $\phi \in (0, \frac{\pi}{2})$, and let $1 \leq q < \infty$. Then the following are equivalent:*

- 557 (i) The Laplace transform $\mathcal{L}: L^\infty(0, \infty) \rightarrow L^q(\mathbb{C}_+, \mu)$ is bounded,
 558 (ii) The function $s \mapsto 1/s$ lies in $L^q(\mathbb{C}_+, \mu)$.

559 *Proof.* (i) \Rightarrow (ii): Taking $f(t) = 1$ for $t \geq 0$ we have that $\mathcal{L}f(s) = 1/s$ and the
 560 result follows.

561 (ii) \Rightarrow (i): For $f \in L^\infty(0, \infty)$ and $s \in \mathbb{C}_+$ we have

$$562 \quad \left| \int_0^\infty f(t)e^{-st} dt \right| \leq \|f\|_\infty \int_0^\infty |e^{-st}| dt \leq \|f\|_\infty / (\operatorname{Re} s) \leq M \|f\|_\infty / |s|,$$

563 where M is a constant depending only on ϕ . Now Condition (ii) implies that \mathcal{L} is
 564 bounded. \square

565 **THEOREM 4.26.** *Suppose that A possesses a q -Riesz basis of X consisting of eigen-*
 566 *vectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a sector in the open left half-plane*
 567 *\mathbb{C}_- and $B \in X_{-1}$. Then the following assertions are equivalent.*

- 568 (i) $\Sigma(A, B)$ is infinite-time L^∞ -admissible,
 569 (ii) $\sup_{\lambda \in \mathbb{C}_+} \|(\lambda - A)^{-1}B\| < \infty$,
 570 (iii) The function $s \mapsto 1/s$ lies in $L^q(\mathbb{C}_+, \mu)$, where μ is the measure $\sum |b_k|^2 \delta_{-\lambda_k}$.

571 *Proof.* By [9, Thm 2.1], admissibility is equivalent to the boundedness of the
 572 Laplace transform $\mathcal{L}: L^\infty(0, \infty) \rightarrow L^q(\mathbb{C}_+, \mu)$, and hence (i) and (iii) are equivalent
 573 by Lemma 4.25. Note that

$$574 \quad \|(\lambda - A)^{-1}B\|^q = \sum_k \frac{|b_k|^q}{|\lambda - \lambda_k|^q}.$$

575 Now if (ii) holds, then (iii) also holds, letting $\lambda \rightarrow 0$. Conversely, if (iii) holds, then
 576 by sectoriality we have that

$$577 \quad \sum_k \frac{|b_k|^q}{|\operatorname{Re} \lambda_k|^q} < \infty,$$

578 and hence $\sum_k |b_k|^q / |\lambda - \lambda_k|^q$ is bounded independently of $\lambda \in \mathbb{C}_+$, that is, (ii) holds. \square

579 **Remark 4.27.** Let $\mathfrak{b}_p(X)$ denote the set of L^p -admissible control operators from
 580 \mathbb{C} to X for a given A . By Theorem 4.22, we have that $\mathfrak{b}_\infty(X) = X_{-1}$ for exponentially
 581 stable, parabolic diagonal systems. Using [32, Theorem 6.9], and the inclusion of the
 582 L^p -spaces, we obtain the following chain of inclusions for $X = \ell^q$ with $q > 1^2$

$$583 \quad (4.15) \quad X = \mathfrak{b}_1(X) \subset \mathfrak{b}_p(X) \subset \mathfrak{b}_\infty(X) = X_{-1}.$$

584 It is not so hard to show that the equality $\mathfrak{b}_\infty(X) = X_{-1}$ does not hold in general if
 585 the exponential stability is dropped. In fact, a counterexample on $X = \ell^2$ with the
 586 standard basis is given by $\lambda_n = 2^n$, $n \in \mathbb{Z}$, $b_n = 2^n/n$ for $n > 0$ and $b_n = 2^n$ for
 587 $n < 0$.

588 The relations of the different stability notions with respect to L^∞ for parabolic
 589 diagonal systems are summarized in the diagram shown in Figure 4.3.

590 5. Some Examples.

591 *Example 5.28.* Let us consider the following boundary control system given by the
 592 one-dimensional heat equation on the spatial domain $[0, 1]$ with Dirichlet boundary

²here, $q = 1$ is also allowed if $(T^*(t))_{t \geq 0}$ is strongly continuous.

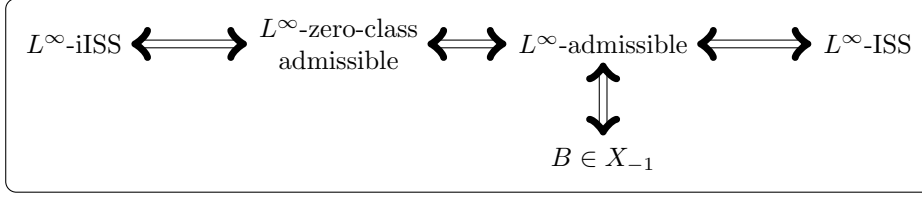


FIG. 4.3. Relations between the different stability notions for parabolic diagonal system (assuming that the semigroup is exponentially stable).

593 control at the point 1,

$$\begin{aligned}
 594 \quad & x_t(\xi, t) = ax_{\xi\xi}(\xi, t), \quad \xi \in (0, 1), \quad t > 0, \\
 595 \quad & x(0, t) = 0, \quad x(1, t) = u(t), \quad t > 0, \\
 596 \quad & x(\xi, 0) = x_0(\xi),
 \end{aligned}$$

598 where $a > 0$. It can be shown that this system can be written in the form $\Sigma(A, B)$ in
 599 (2.4). Here $X = L^2(0, 1)$ and

$$\begin{aligned}
 600 \quad & Af = f'', \quad f \in D(A), \\
 601 \quad & D(A) = \{f \in H^2(0, 1) \mid f(0) = f(1) = 0\}.
 \end{aligned}$$

603 Moreover, with $\lambda_n = -a\pi^2 n^2$,

$$604 \quad Ae_n = \lambda_n e_n, \quad n \in \mathbb{N},$$

605 where the functions $e_n = \sqrt{2}\sin(n\pi\cdot)$, $n \geq 1$, form an orthonormal basis of X .
 606 With respect to this basis, the operator $B = a\delta'_1$ can be identified with $(b_n)_{n \in \mathbb{N}}$
 607 for $b_n = (-1)^n \sqrt{2}an\pi$, $n \in \mathbb{N}$. Therefore,

$$608 \quad \sum_{n \in \mathbb{N}} \frac{|b_n|^2}{|\lambda_n|^2} = \frac{1}{3} < \infty,$$

609 which shows that $B \in X_{-1}$. By Theorem 4.22, we conclude that the system is L^∞ -
 610 iISS. Moreover, we obtain the following L^∞ -ISS and L^∞ -iISS estimates:

$$\begin{aligned}
 611 \quad & \|x(t)\|_{L^2(0,1)} \leq e^{-a\pi^2 t} \|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}} \|u\|_{L^\infty(0,t)}, \\
 612 \quad & \|x(t)\|_{L^2(0,1)} \leq e^{-a\pi^2 t} \|x_0\|_{L^2(0,1)} + c \left(\int_0^t |u(s)|^p ds \right)^{1/p}, \\
 613
 \end{aligned}$$

614 for $p > 2$ and some constant $c = c(p) > 0$. For the second inequality, we used the
 615 fact that $\Sigma(A, B)$ is even L^p -admissible for $p > 2$, as it can be shown by applying
 616 Theorem 3.5 in [9]. We note that a slightly weaker L^∞ -ISS estimate for this system
 617 can also be found in [12].

618 *Example 5.29.* As remarked, Example 5.28 provides a system $\Sigma(A, B)$ which is
 619 even L^p -admissible for $p > 2$. In the following we present a system which is L^∞ -
 620 admissible, but not L^p -admissible for any $p < \infty$. In order to find such an example,
 621 we use the characterization of L^p -admissibility from [9, Thm. 3.5].
 622 Let $X = \ell^2$ and let $(\lambda_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ define a parabolic diagonal system $\Sigma(A, B)$ as in

623 Section 4. Furthermore, let $p \in (2, \infty)$. Then $\Sigma(A, B)$ is infinite-time L^p -admissible
 624 if and only if

$$625 \quad \left(2^{-\frac{2n(p-1)}{p}} \mu(Q_n) \right)_{n \in \mathbb{Z}} \in \ell^{\frac{p}{p-2}}(\mathbb{Z}),$$

626 where $\mu = \sum_{n \in \mathbb{Z}} |b_n|^q \delta_{\lambda_n}$ and $Q_n = \{z \in \mathbb{C} \mid \operatorname{Re} z \in (2^{n-1}, 2^n]\}$, $n \in \mathbb{Z}$.

627 We choose $\lambda_n = -2^n$ and $b_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$. Clearly, $B = (b_n) \in X_{-1}$. Then we
 628 have that

$$629 \quad 2^{-\frac{2n(p-1)}{p}} \mu(Q_n) = 2^{-\frac{2n(p-1)}{p}} \frac{2^{2n}}{n^2} = \frac{2^{\frac{2n}{p}}}{n^2},$$

630 and thus for $p > 2$,

$$631 \quad \left(\left(2^{-\frac{2n(p-1)}{p}} \mu(Q_n) \right)^{\frac{p}{p-2}} \right)_{n \in \mathbb{Z}} = \left(\frac{2^{\frac{2n}{p-2}}}{n^{\frac{2p}{p-2}}} \right)_{n \in \mathbb{Z}} \notin \ell^1.$$

632 Hence, $\Sigma(A, B)$ is not L^p -admissible for any $p > 2$, and therefore also not for any
 633 $p \geq 1$. However, since $\sum_{n \in \mathbb{N}} |b_n|^2 / |\operatorname{Re} \lambda_n|^2 = \sum_{n \in \mathbb{N}} 1/n^2 < \infty$, Theorem 4.22 shows
 634 that $\Sigma(A, B)$ is L^∞ -iISS and, in particular infinite-time L^∞ -admissible.

635 We observe that by Theorem 3.16, there exists a Young function Φ such that $\Sigma(A, B)$
 636 is E_Φ -admissible. However, as the system is not L^p -admissible, such Φ cannot satisfy
 637 the Δ_2 -condition, see Remark 4.24.

638 **6. Conclusions and Outlook.** In this paper, we have studied the relation be-
 639 tween input-to-state stability and integral input-to-state stability for linear infinite-
 640 dimensional systems with a (possibly) unbounded control operator and inputs in gen-
 641 eral function spaces. In this situation, ISS is equivalent to admissibility together with
 642 exponential stability of the semigroup. We have related the notions of iISS with re-
 643 spect to L^1 and L^∞ to ISS with respect to Orlicz spaces. The known result that ISS
 644 and iISS are equivalent for L^p -inputs with $p < \infty$, was generalized to Orlicz spaces
 645 that satisfy the Δ_2 -condition. Moreover, we have shown that for parabolic diagonal
 646 systems and scalar input, the notions of L^∞ -iISS and L^∞ -ISS coincide.

647 Among possible directions for future research are the investigation of the non-
 648 analytic diagonal case, general analytic systems and the relation of zero-class admissi-
 649 bility and input-to-state stability. Recently, the results on parabolic diagonal systems
 650 have been adapted to more general situations of analytic semigroups – the crucial tool
 651 being the holomorphic functional calculus for such semigroups [10]. Furthermore, ver-
 652 sions ISS and iISS for strongly stable semigroups rather than exponentially stable can
 653 be studied, see [22].

654 Finally, we mention that the existence of a counterexample for one of the unknown
 655 (converse) implications in Figure 2.1 can be related to the following open question
 656 posed by G. Weiss in [31, Problem 2.4].

657 **Question A:** *Does the mild solution x belong to $C([0, \infty), X)$ for any $x_0 \in X$ and*
 658 *$u \in Z = L^\infty(0, \infty; U)$ provided that $\Sigma(A, B)$ is L^∞ -admissible?*

659 Although we do not provide an answer to this question, we relate it to

660 PROPOSITION 6.30. *At least one of the following assertions is true.*

- 661 1. *The answer to Question A is positive for every system $\Sigma(A, B)$.*
- 662 2. *There exists a system $\Sigma(A_0, B_0)$, with A_0 generating an exponentially stable*
 663 *semigroup and $\Sigma(A_0, B_0)$ is L^∞ -admissible, but not L^∞ -zero-class admissible.*

664 *Proof.* This follows directly from Proposition 2.5. □

665 **Appendix A. Orlicz Spaces.** In this section we recall some basic definitions
 666 and facts about Orlicz spaces. More details can be found in [14, 15, 1, 35]. For the
 667 generalization to vector-valued functions see [24, VII, Sec. 7.5]. In the following $I \subset \mathbb{R}$
 668 is an open bounded interval, U is a Banach space and $\Phi: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a function.

669 **DEFINITION A.31.** *The Orlicz class $\tilde{L}_\Phi(I; U)$ is the set of all equivalence classes
 670 (w.r.t. equality almost everywhere) of Bochner-measurable functions $u: I \rightarrow U$ such
 671 that*

$$672 \quad \rho(u; \Phi) := \int_I \Phi(\|u(x)\|_U) dx < \infty.$$

673 In general, $\tilde{L}_\Phi(I; U)$ is not a vector space. Of particular interest are Orlicz classes
 674 generated by Young functions.

675 **DEFINITION A.32.** *A function $\Phi: [0, \infty) \rightarrow \mathbb{R}$ is called a Young function (or
 676 Young function generated by φ) if*

$$677 \quad \Phi(t) = \int_0^t \varphi(s) ds, \quad t \geq 0,$$

678 where the function $\varphi: [0, \infty) \rightarrow \mathbb{R}$ has the following properties: φ is right-continuous
 679 and nondecreasing, $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$.

680 **THEOREM A.33** ([15, Thm. 3.2.3 and Thm. 3.2.5]). *Let Φ be a Young function.
 681 Then $\tilde{L}_\Phi(I; U)$ is a convex set and $\tilde{L}_\Phi(I; U) \subset L^1(I; U)$. Conversely, for $u \in L^1(I; U)$
 682 there is a Young function Φ such that $u \in \tilde{L}_\Phi(I; U)$.*

683 **DEFINITION A.34.** *Let Φ be the Young function generated by φ . Then Ψ defined
 684 by*

$$685 \quad \Psi(t) = \int_0^t \psi(s) ds \quad \text{with} \quad \psi(t) = \sup_{\varphi(s) \leq t} s, \quad t \geq 0,$$

686 is called the complementary function to Φ .

687 The complementary function of a Young function is again a Young function. If
 688 φ is continuous and strictly increasing in $[0, \infty)$, i.e. belongs to \mathcal{K}_∞ , then ψ is the
 689 inverse function φ^{-1} and vice versa. We call Φ and Ψ a pair of complementary Young
 690 functions.

691 **THEOREM A.35** (Young's inequality, [35, Thm. I, p. 77]). *Let Φ, Ψ be a pair of
 692 complementary Young functions and φ, ψ their generating functions. Then*

$$693 \quad uv \leq \Phi(u) + \Psi(v), \quad \forall u, v \in [0, \infty).$$

694 Equality holds if and only if $v = \varphi(u)$ or $u = \psi(v)$.

695 **Remark A.36.** Let Φ, Ψ be a pair of complementary Young functions, $u \in \tilde{L}_\Phi(I)$
 696 and $v \in \tilde{L}_\Psi(I)$. By integrating Young's inequality we get

$$697 \quad \int_I |u(x)v(x)| dx \leq \rho(u; \Phi) + \rho(v; \Psi)$$

698 We are now in the position to define the Orlicz spaces for which several equivalent
 699 definitions exist. Here we use the so-called *Luxemburg norm*.

700 DEFINITION A.37. For a Young function Φ , the set $L_\Phi(I; U)$ of all equivalence
 701 classes (w.r.t. equality almost everywhere) of Bochner-measurable functions $u: I \rightarrow U$
 702 for which there is a $k > 0$ such that

$$703 \quad \int_I \Phi(k^{-1} \|u(x)\|_U) dx < \infty$$

704 is called the Orlicz space. The Luxemburg norm of $u \in L_\Phi(I; U)$ is defined as

$$705 \quad \|u\|_\Phi := \|u\|_{L_\Phi(I; U)} := \inf \left\{ k > 0 \mid \int_I \Phi(k^{-1} \|u(x)\|) dx \leq 1 \right\}.$$

706 For the choice $\Phi(t) := t^p$, $1 < p < \infty$, the Orlicz space $L_\Phi(I; U)$ equals the vector-
 707 valued L^p -spaces with equivalent norms.

708 THEOREM A.38 ([15, Thm. 3.9.1]). $(L_\Phi(I; U), \|\cdot\|_\Phi)$ is a Banach space.

709 Clearly, $L^\infty(I, U)$ is a linear subspace of $L_\Phi(I, U)$.

710 DEFINITION A.39. The space $E_\Phi(I, U)$ is defined as

$$711 \quad E_\Phi(I, U) = \overline{L^\infty(I, U)}^{\|\cdot\|_{L_\Phi(I; U)}}.$$

712 The norm $\|\cdot\|_{E_\Phi(I; U)}$ refers to $\|\cdot\|_{L_\Phi(I; U)}$.

713 If $U = \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then we write $L_\Phi(I) := L_\Phi(I; \mathbb{K})$ and $E_\Phi(I) := E_\Phi(I; \mathbb{K})$
 714 for short.

715 Remark A.40. The Banach spaces $E_\Phi(I; U)$ and $L_\Phi(I; U)$ have the following
 716 properties:

- 717 1. $E_\Phi(I; U)$ is separable, see e.g. [26, Thm. 6.3].
- 718 2. For a measurable $u: I \rightarrow U$, $u \in L_\Phi(I; U)$ if and only if $f = \|u(\cdot)\|_U \in L_\Phi(I)$.
 719 This follows from the fact that $\|u\|_\Phi = \|f\|_\Phi$. Thus, $(u_n)_{n \in \mathbb{N}} \subset L_\Phi(I; U)$
 720 converges to 0 if and only if $(\|u_n(\cdot)\|_U)_{n \in \mathbb{N}}$ converges to 0 in $L_\Phi(I)$.
- 721 3. Let Φ, Ψ be a pair of complementary Young functions. The extension of
 Hölder's inequality to Orlicz spaces reads: for any $u \in L_\Phi(I)$ and $v \in L_\Psi(I)$,
 it holds that $uv \in L^1(I)$ and

$$\int_I |u(s)v(s)| ds \leq 2\|u\|_{L_\Phi(I)} \|v\|_{L_\Psi(I)},$$

721 see [15, Thm. 3.7.5 and Rem. 3.8.6]. This implies that for $u \in L_\Phi(I; U)$,

$$722 \quad \|u\|_{L^1(0, t; U)} = \int_0^t \|u(s)\|_U ds \leq 2\|\chi_{(0, t)}\|_\Psi \|u\|_\Phi,$$

723 i.e., $L_\Phi(I; U)$ is continuously embedded in $L^1(I; U)$. Moreover, $\|\chi_{(0, t)}\|_\Psi \rightarrow 0$
 724 as $t \searrow 0$, where $\chi_{(0, t)}$ denotes the characteristic function of the interval $(0, t)$.

- 725 4. $E_\Phi(I; U) \subset \tilde{L}_\Phi(I; U) \subset L_\Phi(I; U)$, see e.g. [26, Thm. 5.1]. For $u \in \tilde{L}_\Phi(I; U)$,

$$726 \quad \|u\|_\Phi \leq \rho(\|u(\cdot)\|_U; \Phi) + 1 < \infty.$$

727 DEFINITION A.41 (Φ -mean convergence). A sequence $(u_n)_{n \in \mathbb{N}}$ in $L_\Phi(I)$ is said
 728 to converge in Φ -mean to $u \in L_\Phi(I)$ if

$$729 \quad \lim_{n \rightarrow \infty} \rho(u_n - u; \Phi) = \lim_{n \rightarrow \infty} \int_I \Phi(|u_n(x) - u(x)|) dx = 0.$$

730 DEFINITION A.42. We say that a Young function Φ satisfies the Δ_2 -condition if

$$731 \quad \exists k > 0, u_0 \geq 0 \forall u \geq u_0 : \quad \Phi(2u) \leq k\Phi(u).$$

732 It holds that $E_\Phi(I; U) = \tilde{L}_\Phi(I; U) = L_\Phi(I; U)$ if Φ satisfies the Δ_2 -condition.

733 DEFINITION A.43. Let Φ and Φ_1 be two Young functions. We say that the func-
734 tion Φ_1 increases essentially more rapidly than the function Φ if, for arbitrary $s > 0$,

$$735 \quad \lim_{t \rightarrow \infty} \frac{\Phi(st)}{\Phi_1(t)} = 0.$$

736 THEOREM A.44 ([14, Thm. 13.4]). Let Φ, Φ_1 be Young functions such that Φ_1
737 increases essentially more rapidly than Φ . If $(u_n)_{n \in \mathbb{N}} \subset L_{\Phi_1}(I)$ converges to 0 in
738 Φ_1 -mean, then it also converges in the norm $\|\cdot\|_\Phi$.

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