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Infinite Geodesic Rays in the Space of Kähler Potentials

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Abstract. In this paper we prove the existence of solutions of a degenerate complex Monge-Ampére equation on a complex manifold. Applying our existence result to a special degeneration of complex structure, we show how to associate to a change of complex structure an infinite length geodetic ray in the space of Kähler potentials. We also prove an existence result for the initial value problem for geodesics. We end this paper with a discussion of a list of open problems indicating how to relate our reults to the existence problem for extremal metrics.

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1. – Introduction

It has been shown by Mabuchi ([18]), Semmes ([20]) and Donaldson ([11]) that the space of Kähler metrics cohomologous to a fixed one has a Riemannian structure of infinite dimensional symmetric space of negative curvature. Being its dimension not finite, the usual properties about existence, uniqueness and regularity for geodesics do not hold a priori. They appear crucial in a number of applications, which go from Hamiltonian dynamics, Monge-Ampére equations and existence and uniqueness of Kähler-Einstein (and, more generally, extremal) metrics. One of the most difficult problems is about the existence of solutions for this geodesic equation. The only examples known up to now are only trivial ones, i.e., those coming from one-parameter subgroups of automorphisms of the underlying manifold, some special examples on S^2 ([11]), and on toric varieties ([14]). In [5], X. X. Chen has proved the existence of geodesics in a weak sense. He used this weaker version of geodesics to deduce the uniqueness of extremal metrics when the first Chern class is non-positive. Despite all this, a general geometric construction is desired and one needs deeper understanding of how the geodesics are related to the geometry of the underlying manifold.

In [23], the K-stability was introduced through special degenerations to study the existence of Kähler-Einstein metrics with positive scalar curvature.

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This paper was partly motivated by exploring the connection between special degenerations and geodesics in the space of Kähler metrics. They can be both used to define certain stability of underlying manifolds. This will provide us a further understanding of the geometric invariant theory in terms of symplectic geometry of the space of Kähler metrics. The other motivation of this paper is to provide a new and general geometric construction of nontrivial solutions for the geodesic equation.

Our method is to relate the equation for the geodesics directly to special degenerations of complex structures of underlying Kähler manifolds. We will make use of the one-parameter group action of automorhisms on a special degeneration of complex manifolds, however, generic manifold in the special degeneration may have only finitely many automorhisms. We will show that if the degeneration is non trivial, then our general existence result produce geodesic of infinite length.

The same method used to produce these infinite geodesic rays gives also an existence result for the initial value problem for geodesic with analytical initial datum. The only known explicit examples of geodesics, given by Donaldson on S^2 ([11]), show that analyticity is not necessary. On the other hand, our result, combined with the openess result proven in [12], gives a great number of geodesics on any manifold with smooth initial datum. The problem of finding a necessary and sufficient condition on the initial data for the solution of the geodesic equation remains open.

To apply the study of geodesics to the problem of extremal metrics it is crucial to study the behaviour of the Mabuchi (or K-) energy along geodesics. In particular it is important to understand whether the K-energy blows up on a finite length geodesic going to the boundary. This and other geometrical important functionals along geodesics have been studied by Chen ([6]) and Calabi-Chen ([4]). It seems reasonable to expect that infinite legth geodesics correspond to some degeneration of the complex structure of the underlying manifold. This would fit well in the picture of the relationship between the existence of extremal metrics and some stability property of the underlying algebraic manifold. Also it seems very important to understand which property of these geodesic rays is related to the nontriviality of those generalized Futaki invariant ([9]). Note that generalized Futaki invariants were used to define the K-stability in [23].

We will end the paper with a discussion of a list of problems related to the geometry of geodesics in the space of Kähler metrics that we believe should be of great interest for future study.

2. – Brief summary of known results

In this section we recall the geometric setting for our analytic problem, and in doing so we try to follow as close as possible the exposition of Donaldson in [11]. We refer the reader also to the excellent survey paper [7].

To start with, we need a compact Kähler manifold M of complex dimension n, with a Kähler form ω_0. Then, the space of Kähler potentials of Kähler forms in the same cohomology class is given by

$$\mathcal{H} = \{\phi \in \mathcal{C}^\infty(M) \mid \omega_\phi = \omega_0 + i\partial\bar{\partial}\phi > 0\}.$$

Being $\mathcal{H}$ an open subset of the space of smooth functions on M, it is clear that its tangent space is $\mathcal{C}^\infty(M)$. Let us also denote by $dVol_\phi = \frac{1}{n!}\omega_\phi^n$ the volume form induced by each function. Given two smooth functions $\psi, \chi \in T_\phi\mathcal{H}$ for some point $\phi \in \mathcal{H}$, we can define their scalar product by

$$(\psi, \chi)_\phi = \int_M \psi\chi dVol_\phi.$$

Of course such a norm gives the possibility to calcultate the lenght of a path, and therefore we can study which curves in $\mathcal{H}$ are geodesics. A direct calculation gives the first appearance of the geodesic equation in its simplest form; a path $\phi(t) \in \mathcal{H}$ is a geodesic if and only if

$$\phi''(t) - \frac{1}{2} \| \nabla_t\phi'(t) \|^2_{\phi(t)} = 0, \tag{1}$$

where $'$ stands for differentiation in t and ∇_t denotes the covariant derivative for the metric $\omega_{\phi(t)}$.

The notion of geodesics $\phi(t)$ with a tangent vector field $\psi(t)$, allows us to define a connection on the tangent bundle to $\mathcal{H}$ by the following

$$D_t(\psi(t)) = \frac{\partial\psi(t)}{\partial t} - \frac{1}{2}(\nabla_t\psi(t), \nabla_t\phi'(t))_\phi \tag{2}$$

which is torsion-free and metric-compatible.

Of course real constants act on $\mathcal{H}$, so if we call $\mathcal{H}_0 = \mathcal{H}/\mathbb{R}$, we can say that, being our manifold always assumed to be compact, $\mathcal{H}_0$ is the space of Kähler metrics on cohomologous to ω_0. Moreover the natural splitting $\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$ is Riemannian for the structure just defined.

It has been a remarkable discovery due to Mabuchi, Semmes and Donaldson, how richly this geodesic equation is related to the geometry of M.

1. *Geodesics and complex Monge-Ampère equations*: suppose $\phi(t), t \in [0, 1]$ is indeed a geodesic in $\mathcal{H}$. It is convenient to think of this path as a function $\Phi: M \times [0, 1] \times S^1 \to \mathbb{R}$, given by $\Phi(x, t, s) = \phi_t(x)$, i.e. independent of the S^1 coordinate s. Let Ω_0 be the pull-back of ω_0 to $M \times [0, 1] \times S^1$ under the projection map, and put $\Omega_\Phi = \Omega_0 + \partial\bar{\partial}\Phi$ on $M \times [0, 1] \times S^1$ (note that we are taking on $[0, 1] \times S^1$ the standard complex structure). Then the result is that $\phi(t)$ is a geodesic iff

$$\Omega_\Phi^{n+1} = 0 \tag{3}$$

(see Proposition 3 in [11]).

2. *Geodesics and Wess-Zumino-Witten equations*: The result above induces one to study the complex Monge-Ampère which seems to have a simpler form than our original equation (1). An even more tempting approach is to take a Riemann surface R with boundary, and look at some special maps from R to $\mathcal{H}$.
Let us fix a a map $\rho: \mathcal{C}^\infty(\partial(V \times R))$. A map $f: R \to \mathcal{H}$ with ρ as boundary data, and whose associated map is $F: M \times R \to \mathbb{R}$, is said to satisfy the Wess-Zumino-Witten equations iff it is a critical point of the functional whose first variation is (it is a direct calculation to show that such a formula defines a functional)

$$\delta I_\rho(F) = \frac{1}{(n+1)!} \int_{M \times R} \delta F \, \Omega_F^{n+1} . \tag{4}$$

Therefore f satisfies the Weiss-Zumino-Witten equations if and only if $\Omega_F^{n+1} = 0$. (see Proposition 4 in [11]).
3. *Geodesics, foliations and holomorphic curves in the diffeomorphism group of M*: Let us go back to the solution Ω_Φ of the equation 3 given by a geodesic. Ω_Φ is a two-form on $M \times [0, 1] \times S^1$ which is closed, positive on the M-slices, and of type $(1, 1)$ (note that in this section we could take any Riemann surface instead of the cylinder $[0, 1] \times S^1$).
Being Ω_Φ a solution of (3) tells us that its null space at each point in nontrivial, and all the above conditions imply that such a space at every point has to be a complex line transverse to the M-slices.
A very significant variant of this interpretation is the following (see [20]): consider on the space of maps $\mathrm{Map}(M, M)$ the natural complex structure given by the complex structure on M. Then a foliation on $M \times [0, 1] \times S^1$ with leaves transverse to the Riemann surface, can be thought as a map (after fixing a base point on the Riemann surface) $F: [0, 1] \times S^1 \to \mathrm{Diff}(M)$ given by projection along the leaves, and the fact the leaves are complex tells us that F is in fact a holomorphic map. Being Ω_Φ of type $(1, 1)$, we know that the image of F is contained in the subspace of $\mathrm{Diff}(M)$, $\mathcal{Y}$, of diffeomorphisms which preserve the type of ω_0.
4. *Geodesics, K-energy and uniqueness of Extremal Kähler metrics*: One of the main reasons which raises interest in the problem of existence of geodesics is its connection with the problem of existence and uniqueness of constant scalar curvature Kähler metrics in a fixed cohomology class (and more generally of extremal metrics), a well known question asked by Calabi. The literature on this problem is too vast to give here any account of it. We point out [2], [3], [5], [16] and [23], and references therein. Here let us just mention that Mabuchi in [18] and [19] has defined a functional (now known as Mabuchi or K-Energy) whose critical points are the extremal Kähler metrics. It has been observed (Corollary 11 in [11]) that the existence of a *smooth* geodesic connecting two extremal metrics implies the existence of a holomorphic automorphism of M which pulls back one metric on

the other, therefore giving essential uniqueness for such a metric. Such a result can be interpreted by saying that *the K-energy is convex along geodesic*, a fact highly nontrivial and useful in the whole theory. It is important to note that in his argument regularity of the geodesic is crucial. X. Chen ([5]) made a very significant step towards the understanding of the regularity question for geodesic, proving that the solution to equation (3) is always $\mathcal{C}^{1,1}$. Geometrically this means that singularity might occur for example if null-directions for the symplectic form start arising, but not blow-ups. Even if this is not smoothness (which cannot be hoped), he has been able to prove that it is enough to conclude uniqueness for the extremal metric in the case of non-positive first Chern class of the manifold.

3. – Solutions to the complex Monge-Ampère

In this section we prove the existence of solutions of equation (3) with analytic boundary data in a sense described below. We will show that this give rise to a Cauchy problem for which the standard techniques about convergence of formal solutions of analytic non-characteristic equations can be applied. The solutions interesting in the problems about geodesics have some special symmetry as we will explain in the next section. This would reduce our problem to a genuine codimension one problem. Nevertheless we can give a general existence problem for which the solutions with S^1 symmetry can be easily extracted.

The first result gives a partial answer to the initial value problem for geodesics. We discuss plausible generalizations at the end of the paper.

THEOREM 3.1. *Let ω_M be a real analytic Kähler metric on M and let $\psi_0\colon M \to \mathbb{R}$ be a real analytic function. Then there exists $\epsilon > 0$ and a unique analytic family of functions $\phi_t(x) = \phi(x,t)$, $\mid t \mid < \epsilon$ s.t. $(\omega_M + \partial\bar{\partial}\phi)^{n+1} = 0$ on $M \times \Delta_\epsilon$ and $\frac{d}{dt}\phi_t(x) = \psi_0(x)$ and $\phi_0(x) = 0$, where $\Delta_\epsilon = \{S \in \mathbb{C} \mid \mid S \mid < \epsilon\}$.*

PROOF. In this situation one can apply directly Cauchy-Kowalevski theorem provided we show that (3) is an analytic non-characteristic equation.

This is easily done by writing locally (3) as

$$\det\begin{pmatrix} g_{\alpha\bar{\beta}} + \dfrac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta} & \dfrac{\partial^2\phi}{\partial S \partial z_\alpha} \\ \dfrac{\partial^2\phi}{\partial S \partial \bar{z}_\beta} & \dfrac{\partial^2\phi}{\partial S \partial \bar{S}} \end{pmatrix} = 0\,.$$

The only thing to observe is of course the positivity of the coefficient of $\frac{\partial^2\phi}{\partial S\partial\bar{S}}$, and that if ϕ depends only on $t = \mathrm{Re}(S)$, $\frac{\partial\phi}{\partial S} = \frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\bar{S}}$, $\frac{\partial^2\phi}{\partial S\partial\bar{S}} = \frac{\partial^2\phi}{\partial t^2}$. □

In fact it is interesting to write down the solution as a power series in t. To this aim let us write $\phi(x,t) = \sum_{k\geq 1}\theta_k(x)t^k$ so that $\theta_1 = \psi_0$. By plugging this into (3) we get a recursive formula to define θ_k inductively. To write simpler formulae let us just restrict ourselves to the case of $\dim_{\mathbb{C}} M = 1$ and z be a local coordinate on M. Then $(\omega_M + \partial\bar{\partial}\phi)^2 = 0$ becomes

$$\theta_2 = \frac{\partial\theta_1}{\partial z}\frac{\partial\theta_1}{\partial\bar{z}}\frac{dz\wedge d\bar{z}\wedge dS\wedge d\bar{S}}{\omega_M\wedge dS\wedge d\bar{S}},$$

and for $k > 2$

$$k(k-1)\theta_k = \sum_{\substack{j+m=k\\ j,m\geq 1}} \left\{ m(m-1)\theta_m \frac{\partial^2\theta_j}{\partial z\partial\bar{z}} + mj\left(\frac{\partial\theta_m}{\partial z}\frac{\partial\theta_j}{\partial\bar{z}} + \frac{\partial\theta_j}{\partial z}\frac{\partial\theta_m}{\partial\bar{z}}\right) + j(j-1)\theta_j\frac{\partial^2\theta_m}{\partial z\partial\bar{z}} \right\} \frac{dz\wedge d\bar{z}\wedge dS\wedge d\bar{S}}{\omega_M\wedge dS\wedge d\bar{S}}.$$

Without relying on general theorems one can also check that this formal solution converges locally.

This kind of solution of (3) give short time existence for geodesics starting at an analytic Kähler metric with analytic initial tangent vector.

The main existence theorem we prove is the following. It is a priori independent of our questions about geodesics. We will show later how to impose symmetry conditions on the solutions so to be able to use them to construct geodesic rays.

THEOREM 3.2. *Let D be a divisor of a Kähler manifold V of complex dimension n, and let ω_D be an analytic Kähler form defined on* $\mathrm{reg}(D) = D\setminus\mathrm{sing}(D)$. *Let further $\widetilde{\omega}_V$ be any closed analytic $(1,1)$ form which extends ω_D in a neighborhood V of* $\mathrm{reg}(D)$.

Then there exist a neighborhood $\widetilde{V}$ of $\mathrm{reg}(D)$ *in V and a function $\phi\colon \widetilde{V}\to\mathbb{R}$ s.t.*

1. $\omega = \widetilde{\omega}_V + \partial\bar{\partial}\phi$ *is smooth on* $\widetilde{V}\setminus\mathrm{sing}(D)$;
2. $\omega_{|\mathrm{reg}(D)} = \omega_D$;
3. $\omega^n = 0$.

PROOF. The method we want to use to construct the function ϕ is closely related to the one used in [8] and [24]. The idea is to write in a sufficiently small neighborhood of $\mathrm{reg}(D)$

$$\phi = \sum_{m>0}\sum_{i+j=m+1} S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\overline{\theta_{ij}} \tag{5}$$

where S is a defining section of the line bundle, L_D, associated to the divisor, so that $S^i\bar{S}^j$ is a section of $L_D^i\otimes\overline{L_D}^j$, and the θ_{ij}s are smooth sections of $L_D^{-i}\otimes\overline{L_D}^{-j}$. Clearly, since ϕ has to be real valued, $\theta_{ij} = \overline{\theta_{ji}}$.

Let us now fix an hermitian metric $\|\quad\|$ on the normal bundle to D in V and denote by $\widetilde{\omega}$ its curvature form (which is in general different from ω_D) and by D its covariant derivative (there should be no confusion with the divisor ...).

We now seek (some) θ_{ij}s for which the equation is satisfied. In order to do this the first step is to calculate

$$\begin{aligned}\partial\bar{\partial}\phi = \sum_{m>0}\sum_{i+j=m+1} \Bigg\{ & ij(S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\overline{\theta_{ij}})\frac{DS\wedge\overline{DS}}{|S|^2} \\ & + iS^i\bar{D}\theta_{ij}\wedge\bar{S}^j\frac{DS}{S} + jS^i\bar{S}^jD\theta_{ij}\wedge\frac{\overline{DS}}{\bar{S}} \\ & + iS^j\bar{S}^iD\overline{\theta_{ij}}\wedge\frac{\overline{DS}}{\bar{S}} + jS^j\bar{S}^i\bar{D}\overline{\theta_{ij}}\wedge\frac{DS}{S} + S^iD\bar{D}\bar{S}^j\theta_{ij} \\ & + S^jD\bar{D}\bar{S}^i\overline{\theta_{ij}} + S^i\bar{S}^jD\bar{D}\theta_{ij} + S^j\bar{S}^iD\bar{D}\overline{\theta_{ij}}\Bigg\}\,. \end{aligned} \tag{6}$$

Being S a holomorphic section we have

$$\begin{aligned}\bar{D}DS^j &= jS^j\widetilde{\omega} \\ \bar{D}D\theta_{ij} &= -D\bar{D}\theta_{ij} - (i-j)\theta_{ij}\widetilde{\omega}\,,\end{aligned} \tag{7}$$

which readily implies that

$$(\partial\bar{\partial}\phi)^l\wedge\widetilde{\omega}_V^{n-l} = \widetilde{\omega}^l\wedge\widetilde{\omega}_V^{n-l} + \mathcal{O}(2m+2)\,, \tag{8}$$

for $l \geq 2$.

Given a positive integer k, we now look at the equation $\omega^n = 0$ disregarding all the terms vanishing along D of order bigger than k (we will say "modulo S^k" from now on). For example by equation (6) and (8),

1. $\omega^n = 0$ modulo S reduces to

$$2\theta_{11}DS\wedge\overline{DS}\wedge\widetilde{\omega}_V^{n-1} + \widetilde{\omega}_V^n + \sum_{l=2}^{n}\binom{n}{l}\widetilde{\omega}^l\wedge\widetilde{\omega}_V^{n-l} = 0 \tag{9}$$

2. $\omega^n = 0$ modulo S^2 reduces to

$$\begin{aligned}0 = \{ & 2SDS\wedge\bar{D}\theta_{20} + 2\theta_{11}DS\wedge\overline{DS} + 2SD\theta_{11}\wedge\overline{DS} \\ & + 2\bar{S}\bar{D}\theta_{11}\wedge DS + 2\bar{S}D\theta_{02}\wedge\overline{DS} + 4(S\overline{\theta_{12}} + \bar{S}\theta_{12})DS\wedge\overline{DS}\}\wedge\widetilde{\omega}_V^{n-1} \\ & + \widetilde{\omega}_V^n + \sum_{l=2}^{n}\binom{n}{l}\widetilde{\omega}^l\wedge\widetilde{\omega}_V^{n-l}\end{aligned} \tag{10}$$

and so on.

We can use (9) to define θ_{11}, which is then smooth since $DS\wedge\overline{DS}\wedge\widetilde{\omega}_V^{n-1} > 0$ near D, so that the equation $\omega^n = 0$ holds modulo S. Let us then define $\theta_{20} = \theta_{02} = 0$. We can use equation (2) as a definition for θ_{21} and θ_{12} in terms of θ_{11}, its derivatives and other already fixed pieces. If we define also $\theta_{30} = \theta_{03} = 0$ we can write more easily the equation modulo S^3 when we see the terms coming from equation (8) coming in.

$\omega^n = 0$ modulo S^3 reduces to

$$
\begin{aligned}
(11)\qquad 0 = \text{all above terms} &+ [S^2(6\theta_{31}DS \wedge \bar{D}S + 2D\theta_{21} \wedge \bar{D}S) \\
&+ \bar{S}S(8\theta_{22}DS \wedge \bar{D}S + 4D\theta_{12} \wedge \bar{D}S + 4\bar{D}\theta_{21} \wedge DS \\
&+ 2\theta_{11}\widetilde{\omega} + 2D\bar{D}\theta_{11}) + \bar{S}^2(6\theta_{13}DS \wedge \bar{D}S + 2\bar{D}S \wedge DS)] \wedge \widetilde{\omega}_V^{n-1} \\
&+ \widetilde{\omega}_V^n + 4S\bar{S}(\theta_{11}^2\widetilde{\omega} + \theta_{11}D\bar{D}\theta_{11}) \wedge DS \wedge \bar{D}S \wedge \widetilde{\omega}_V^{n-2}
\end{aligned}
$$

$$
(12)\qquad + \sum_{l=2}^{n} \binom{n}{l} \widetilde{\omega}^l \wedge \widetilde{\omega}_V^{n-l} .
$$

It is clear that this phenomenon happens modulo S^k for any k, i.e. the equation modulo S^k can be used to define inductively all θ_{ij}s with $i + j = k + 1$ in terms of the θ_{ij}s with $i + j \leq k$ and their derivatives.

Defining θ_{ij} in this way we get a formal power series which solves our complex Monge-Ampère. We claim that this is in fact a convergent series. This can be seen directly by noticing that a fixed θ_{ij}, with $i + j = k + 1$, is defined by at most (since cancellations might occur)

1. 8 terms coming from choosing the highest term in $\partial\bar{\partial}\phi$;
2. for $i' < i$, $j' < j$ each term with coefficient $S^{i'}\bar{S}^{j'}$ in the expression on $\partial\bar{\partial}\phi$ gets multiplied by an appropriate terms in the expression of $\widetilde{\omega}_V^{n-1}(S, \bar{S})$. So how many of these terms do we get? For each $m = i' + j'$ the total number is $\frac{1}{2}\sum_{m<k} m(m + 1) < \frac{1}{2}k^2(k + 1)$.
3. the remaining terms coming form equation (8). Clearly the number of such terms is bounded by a polynomial in k too.

Adding up the two contributions, we see that the number of terms in the expression defining θ_{ij} grows polynomially as $i + j$ grows. It is now straightforward (though very lengthy) to adapt the classical proof of the convergence of the formal solution with any analytic initial data. In our case the initial data $\phi_{|D}$ and its first derivatives in S and $\bar{S}$ vanish. □

As we mentioned in the previous sections, to apply the above result to the question of the existence of geodesics we need to study the question of dependence of the solution on the S^1 parameter in the situation of algebraic degeneration. We first need to set up correctly the question we face: let V be in fact a holomorphic family over the unit disc $\Delta \subset \mathbb{C}$ and let $\pi\colon V \to \Delta$ the corresponding submersion. We want to use Theorem 3.2 when $D = \pi^{-1}(0)$. Clearly the S^1-action on Δ gives an S^1-action on the normal bundle of D in V. We therefore want to know whether we can choose the solution to the equation (3) equivariantly. The following theorem answers affirmatively:

THEOREM 3.3. *In the situation just described, we can choose ϕ to satisfy the extra property of being S^1-invariant.*

PROOF. Of course the S^1-action σ on the normal bundle preserves its fibers, and we can consider a defining section to be an eigenvector for σ in the sense that $\sigma^*(S) = \alpha S$, for some constant α. Now, we can certainly choose the first extension $\widetilde{\omega}_V$ of ω_D to be S^1-invariant. Moreover we can put on L_D an S^1-invariant hermitian metric so that both its covariant derivative and its curvature form have this property. The whole point of the proof is now to look at the recursive formulae for the θ_{ij}s. For example,

1. $\omega^n = 0$ modulo S reduces to

$$2\theta_{11} DS \wedge \overline{DS} \wedge \widetilde{\omega}_V^{n-1} + \widetilde{\omega}_V^n + \sum_{l=2}^{n} \binom{n}{l} \widetilde{\omega}^l \wedge \widetilde{\omega}_V^{n-l} = 0\,. \tag{13}$$

The invariance of $\widetilde{\omega}_V$ and of D implies that θ_{11} is invariant too.

2. $\omega^n = 0$ modulo S^2 reduces to

$$\begin{aligned} 0 = \{&2\theta_{11} DS \wedge \overline{DS} + \bar{D}\theta_{11} \wedge \bar{S} DS \\ &+ SD\theta_{11} \wedge \overline{DS} + SD\theta_{11} \wedge \overline{DS} + \bar{S}\bar{D}\theta_{11} \wedge DS \\ &+ 4(S\overline{\theta_{12}} + \bar{S}\theta_{12}) DS \wedge \overline{DS}\} \wedge \widetilde{\omega}_V^{n-1} + \widetilde{\omega}_V^n \\ &+ \sum_{l=2}^{n} \binom{n}{l} \widetilde{\omega}^l \wedge \widetilde{\omega}_V^{n-l}\,. \end{aligned} \tag{14}$$

This defines θ_{12} by

$$\begin{aligned} &\bar{S}(4\theta_{12} DS \wedge \overline{DS} + 2DS \wedge \bar{D}\theta_{11}) \wedge \widetilde{\omega}_D^{n-1} \\ &\qquad + \widetilde{\omega}_D^n + \sum_{l=2}^{n} \binom{n}{l} \widetilde{\omega}^l \wedge \widetilde{\omega}_V^{n-l} = 0 \end{aligned} \tag{15}$$

and θ_{21} by

$$\begin{aligned} &S(4\theta_{21} DS \wedge \overline{DS} + 2D\theta_{11} \wedge \overline{DS}) \wedge \widetilde{\omega}_D^{n-1} \\ &\qquad + \widetilde{\omega}_D^n + \sum_{l=2}^{n} \binom{n}{l} \widetilde{\omega}^l \wedge \widetilde{\omega}_V^{n-l} = 0\,. \end{aligned} \tag{16}$$

Composing the above expressions with σ we see that θ_{21} has the variance of $\bar{S}$ and θ_{12} has the variance of S; therefore $S^2\bar{S}\theta_{21} + S\bar{S}^2\theta_{12}$ (which is the term modulo S^3 in the expansion of ϕ) is indeed S^1-invariant.

Once again we leave to the reader the simple observation that this phenomenon occurs at every step of the definition of the θ_{ij}s, and therefore the proof is complete. □

One can also try to find directly S^1-invariant solutions by relying on the classical Cauchy-Kowalevski theorem. This is again a non-characteristic probelm as in Theorem 3.1. Unfortunately if one writes the local form of the Monge-Ampère in polar coordinates one gets an anlytic equation away from the divisor (which has become a real hypersurface for the problem with symmetry). So one has to show that these fake singularities of the coefficients do not interfere with the smoothness of the solution.

REMARK 3.1. We wish to stress the fact that the solution to the complex Monge-Ampère given in the two theorems above is not unique (the choice of $\widetilde{\omega}_V$ and of the terms of the form $\theta_{i,0}$ and $\theta_{0,j}$ leaves for example great free dom ...). In fact we have the freedom left by a complex gauge (which appears in the choice of the section S) and the action of the symplectomorphisms. The choice of $\theta_{i,0}$ and $\theta_{0,j}$ to vanish seems to play the analogue role of the choice of Bochner coordinates (i.e. those coordinates for which higher order mixed derivatives of the Kähler potential vanish) when studying the problem of approximating a polarized Kähler metric by Bergmann metrics ([22]).

4. – Infinite rays

The aim of this section is to prove that under some geometric assumption on V, the geodesic rays constructed in the previous section have infinite length.

Even though probably not the most general situation in which these geodesic will have infinite length, we will prove this directly in the setup more natural for prooceding to find the link of theory developed in this paper and the one of obstructions to the existence of KählerEinstein metrics as studied in [23].

Let us then recall that an almost Fano variety is an irreducible, normal variety, such that for some m, the pluri-anticanonical bundle $K_{Y_{\mathrm{reg}}}^{-m}$ extends to an ample line bundle, L, over Y. In this situation a choice of a basis of $H^0(Y, L^k)$ defines an embedding of Y into some $\mathbb{C}P^N$. Indeed, this can be reversed in the following sense: if Y is an irreducible, normal subvariety in some projective space, which is the limit of a sequence of smooth compact Kähler manifolds of positive first Chern class in the same projective space, then Y is an almost Fano variety.

Therefore almost Fano varieties arise naturally as degenerations of Fano, but we want to look at a particular class of degenerations, so called *special*. First recall that a vector field v on Y is called *admissible*, if it generates a family of automorphisms $\tau_v(t)$ of Y s.t. $\tau_v(t)^*L = L$. A degeneration $\pi: V \to \Delta$ is then called *special* if there exists a holomorphic vector field v on V such that $\pi_* v = -t\frac{\partial}{\partial t}$, and which therefore generate the one-parameter subgroup $z \to e^{-t}z$ on Δ.

Special degenerations have the property that $\pi^{-1}(t)$ is biholomorphic to $\pi^{-1}(s)$ whenever s and t are different from zero. On the other hand $\pi^{-1}(0)$

might not be biholomorphic to the general fibre. We then call M a Fano smooth manifold isomorphic to $M_t = \pi^{-1}(t)$, and $Y = \pi^{-1}(0)$.

Being V special, there exists an embedding of V into some product $\mathbb{C}P^N \times \Delta$, in such a way that v induces a one-parameter subgroup $G = \{\sigma(z)\}_{z\in\mathbb{C}^*} \subset SL(N+1,\mathbb{C})$, which then satisfies $\sigma(z)(M) = M_{e^{-z}}$. We denote by ω_{FS} the standard Fubini-Study metric on $\mathbb{C}P^N$.

Theorem 3.3 applied to this situation implies that there exists on some neighborhood V of reg(Y) a function $\phi\colon V \to \mathbb{R}$ s.t. $\widetilde{\omega}_V + \partial\bar{\partial}\phi$ satisfies the Monge-Ampère and $\phi = \phi(p, \mid S \mid^2)$, where S is a defining section of the normal bundle to Y. By fixing ϵ_0 sufficiently small we can find a biholomorphism $\Sigma\colon M_{\epsilon_0} \times [x_0,\infty) \times S^1 \to \pi^{-1}(\overline{\Delta_{\epsilon_0}} \setminus \{0\}) \subset V$.

Σ is really nothing but "gluing" the $\sigma(z)$ together (so one can check that $x_0 = -log\epsilon_0$ even this won't be necessary in the sequel).

Clearly $\Sigma^*(\widetilde{\omega}_V + \partial\bar{\partial}\phi)$ still satisfies the Monge-Ampère on $M_{\epsilon_0} \times [x_0,\infty) \times S^1$. Moreover $\phi = \phi(p,|S|^2)$, and $\widetilde{\omega}_V$ is S^1-invariant as well, and $\phi(\Sigma(p,x,y)) = e^{-x-iy}$. Therefore $\Sigma^*(\widetilde{\omega}_V + \partial\bar{\partial}\phi) = \Omega_0 + \partial\bar{\partial}\Phi$, where $\Phi = \Phi(x)$, $x \in [x_0,\infty)$, does not depend on y, and hence it defines a genuine geodesic on M_{ϵ_0}.

We claim that if the complex structure jumps in the degeneration, i.e. if Y is not biholomorphic to M, then this geodesic has infinite length.

We note that since the geodesic associated to a solution of the complex Monge-Ampère equation is automatically parametrized by a multiple of arc length, and being our curve parametrized on an infinite segment, the only possibility to contradict the infiniteness of our ray is that the curve is indeed constant. This can be seen explicitely in the following way: the length of the curve of potentials between $x_0 = -log\epsilon$ and x_1 as x_1 goes to infinity is given by

$$L = \sqrt{\int_{x_0}^{x_1}\int_M \mid \Phi'_x \mid^2 dVol_\Omega dx}$$
$$= \sqrt{(x_1 - x_0)\int_M (\Phi')_0^2 dVol_{\Omega_{x_0}}}\,.$$

Therefore the only way to have a finite length curve is that the family of potential does not depend on s. We now claim that this would contradict the jump in complex structure.

We now assume that the degeneration is non trivial and that each component of Y has multiplicity one. Here Y is allowed to be non smooth and reducible, even though we do not have a general existence theorem for the solution of the Monge-Ampére equation in the non smooth case.

Indeed, $\|\Phi_x\|_{C^0}$, diverges as x goes to $+\infty$. We can prove this by contradiction: if $\|\Phi_x\|_{C^0}$ were uniformly bounded, then $\sigma(x)\colon M \to M_{e^{-x}}$ would converge to a holomorphic map $\tau\colon M \to Y$. Let us look at $\tau^{-1}(p)$ for any $p \in Y$. If $\tau^{-1}(p)$ is of complex dimension greater than zero, then for any small neighborhood U of p, $\sigma(x)(\tau^{-1}(p))$ is contained in U, for any sufficiently small t. This is clearly contradicts maximum principle for holomorphic maps.

Similarly, since each component of Y has multiplicity one, $\tau^{-1}(p)$ has at most one component. Therefore τ is a biholomorphism and therefore the degeneration is trivial, contradicting our assumption.

We summarize the above discussion in the following:

THEOREM 4.1. *If $\pi: V \to \Delta$ is a non trivial special degeneration, whose central fibre is smooth, then the geodesic produced by solving the degenerate complex Monge-Ampère in Theorem* 3.2 *has infinite length.*

A number of natural questions arise naturally from our results. We wish to point out explicitely some of them.

QUESTION 1. It should be possible to generalize our results to special degenerations such that the central fibre may have mild singularities. Moreover it would be also interesting to see what are the milder assumptions on the initial data for the Monge-Ampére equation to get the convergence of the formal solution. In the similar setting of constructing Kähler metrics on Grauert tubes ([1], [15]), analyticity is essential and cannot be dropped (we wish to thank Prof. D. Burns for extremely useful discussions on this problem). In the situation studied in this paper more flexibility is allowed and we know of many examples of geodesics which do not satisfy these assumptions. It is likely that the right assumption is a compatibility condition between the intial data on ϕ and on the given Kähler form.

QUESTION 2. It is tempting to draw an analogy between the problem of approximating Kähler metrics by projectively induced metrics, and the one of approximating geodesics segments by paths induced by subgroups of linear groups of a fixed embedding into a projective space. Tian ([22]) proved that any polarized Kähler metric is a limit of Bergmann metrics as the degree of the projective embedding goes to infinity. It would be very interesting to know whether the analogue "dynamical" result holds for geodesic segments.

QUESTION 3. If the previous problem can be solved, one may ask if there is an expansion of the approximation in term of geometrical data of the Kähler manifold. For a fixed Kähler metric this is indeed possible and gives rise to many interesting results ([12], [17], [25]).

QUESTION 4. It is interesting to associate to each geodesic of infinite lenght a generalized Futaki invariant. It was done for special degenerations by Tian and Ding-Tian ([23], [9]). This will allow us to construct analytic obstructions to the existence of extremal and Kähler- Einstein metrics in terms of geodesics.

QUESTION 5. In order to understand the relationship between the geodesics studied here and the existence problem for extremal metrics, we observe that on the geodesics constructed with our method the time derivative of the K-energy has a limit as time goes to infinity (this follows from the fact that the sequence of Kähler metrics converges to a metric). We believe this property should characterize infinite-length geodesics corresponding to degenerations of the complex structure. If this turns out to be true, they should therefore be the relevant ones to study the stability of the underlying algebraic manifold, which is the crucial issue for the existence problem for extremal metrics.

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