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Infinite Hierarchical Potential Games

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Abstract. Hierarchical potential games with infinite strategy sets are considered. For these games, pessimistic Stackelberg equilibria are characterized as minimum points of the potential function; properties are studied and illustrated with examples.

Key Words. Potential games, hierarchical decision making, multilevel optimization problems.

1. Introduction

Let us consider a strategic form game

$$G = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle,$$

where the player set is $N = \{1, \ldots, n\}$, for each $i \in N$ the strategy set of play ers i is X_i, and $u_i: \prod_{i \in N} X_i \rightarrow \mathscr{R}$ is the payoff function of player i. We suppose that $X_1, \ldots, X_n$ are compact subsets of Euclidean spaces and define

$$X = \prod_{i \in N} X_i, \qquad x = (x_1, \ldots, x_n) \in X,$$

and for $i \in N$,

$$X_{-i} = \prod_{j \in N - \{i\}} X_j, \qquad x_{-i} = \prod_{j \in N - \{i\}} x_j.$$

The players, placed in the order $1, 2, \ldots, n$, want to minimize their own payoff and each player $i \in N$ chooses a strategy $x_i \in X_i$ after observing the

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moves of all the preceding players in $\{1, \ldots, i-1\}$. Players have no information about the choices of the following players. When player $i \in N$ acts, we say that players $1, \ldots, i-1$ are leaders or predecessors of player i and players $i+1, \ldots, n$ are followers of player i. In the case of finite strategy sets, this game is the hierarchical game considered in Ref. 1, when only one player operates at each stage. In this paper, we are concerned with the case of infinite strategy sets.

For a hierarchical game with two players, an equilibrium concept was introduced by Von Stackelberg in 1934 (Ref. 2) for the static case in the context of economic competitions and was expanded to the dynamic case by Chen and Cruz (Ref. 3). Several papers can be found dealing with various aspects of the Stackelberg problem, such as existence results, optimality conditions, and numerical methods; see the bibliographical references in Refs. 4–5. The Stackelberg concept was generalized to weak or pessimistic Stackelberg equilibrium and to strong or optimistic one by Leitmann in 1978 (Ref. 6). There are difficulties to obtain existence results for weak Stackelberg equilibria (they may not exist even for nice payoff functions, see Ref. 4) and stability results (convergence of solutions together with the corresponding values with respect to perturbations of the data) for weak and strong equilibria. Consequently, for both types of equilibria, different notions of approximate equilibrium were introduced (Refs. 7–10).

In this paper, with reference to infinite strategy spaces, we consider a special case of hierarchical games, i.e., the hierarchical potential games as introduced in Ref. 4 for finite strategies spaces and give an existence result for weak Stackelberg equilibrium.

In potential games, introduced by Monderer and Shapley in 1996 (Ref. 11), the change in payoff for a unilaterally deviating player is measured by a potential function; for these games, the argmin set of the potential function refines the Nash equilibrium set. Many propeties of potential games have been studied (Ref. 12–15). For the infinite hierarchical potential games considered in this paper, the argmin set of the potential is characterized as the set of weak Stackelberg equilibria.

Let us recall the definition of a hierarchical potential game.

Definition 1.1. Let G be a hierarchical game; G is called a hierarchical potential game if there exist functions $P: X \mapsto \mathscr{R}$ and $d_i: \prod_{k=1}^{i-1} X_k \mapsto \mathscr{R}$, with $i > 1$ and $d_1 \in \mathscr{R}$, such that

$$u_1(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) + d_1,$$

$$u_i(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) + d_i(x_1, \ldots, x_{i-1}), \qquad i > 1.$$

The function P is called a potential for G.

Examples of economic applications of hierarchical potential games are the sequential production situations (Ref. 1).

The paper is organized as follows. In Section 2, several definitions are given and illustrated by examples. In Section 3, an existence result is given together with some properties of hierarchical potential games. A discussion follows in Section 4.

2. Preliminaries

For $n = 2$, the classical definition of equilibrium for hierarchical games was given by Von Stackelberg in 1934 (Ref. 2).

Definition 2.1. Let $\bar{x}_1 \in X_1$ be such that

$$u_1(\bar{x}_1, \tilde{x}_2(\bar{x}_1)) = \inf_{x_1 \in X_1} u_1(x_1, \tilde{x}_2(x_1)),$$

where $\tilde{x}_2(x_1) \in X_2$ is the unique solution $\forall x_1 \in X_1$ to the lower-level problem

$$\inf_{x_2 \in X_2} u_2(x_1, x_2).$$

If $\bar{x}_2 = \tilde{x}_2(\bar{x}_1)$, then $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ is called a Stackelberg equilibrium.

Example 2.1. Let $X_1 = X_2 = [-2, 2]$ and

$$u_1(x_1, x_2) = x_1^2 + 3x_1 x_2 + x_2^2,$$

$$u_2(x_1, x_2) = -2x_1^2 + 3x_1 x_2 + x_2^2.$$

Then, for any $x_1 \in X_1$, we have

$$\tilde{x}_2(x_1) = \begin{cases} 2, & \text{if } x_1 \in [-2, -4/3[, \\ -(3/2)x_1, & \text{if } x_1 \in [-4/3, 4/3[, \\ -2, & \text{if } x_1 \in [4/3, 2], \end{cases}$$

and the function $u_1(x_1, \tilde{x}_2(x_1))$ has two minimum points $x_1 = 2$ and $x_1 = -2$. Then, we have two Stackelberg equilibria: $(2, -2)$ and $(-2, 2)$.

The uniqueness assumption in Definition 2.1 can be removed. If we assume that the follower (player 2) does the worst to the leader (player 1), the previous definition is modified as follows (see Refs. 6 and 16).

Definition 2.2. For all $x_1 \in X_1$, let

$$S_2(x_1) := \arg\min_{x_2 \in X_2} u_2(x_1, x_2),$$

$$S_1 := \arg\min_{x_1 \in X_1} \phi_1(x_1),$$

where

$$\phi_1(x_1) = \sup_{x_2 \in S_2(x_1)} u_1(x_1, x_2).$$

Any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ such that $\bar{x}_1 \in S_1$ and $\bar{x}_2 \in S_2(\bar{x}_1)$ is called a weak or pessimistic Stackelberg equilibrium.

Example 2.2. Let $X_1 = X_2 = [0, 1]$ and

$$u_1(x_1, x_2) = -x_1 - 2x_2,$$

$$u_2(x_1, x_2) = \max\{0, (x_2 - x_1)x_2\}.$$

In this case, we have $S_2(x_1) = [0, x_1]$ and $\phi_1(x_1) = -x_1$, so any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ such that $\bar{x}_1 = 1$ and $\bar{x}_2 \in [0, 1]$ is a weak (pessimistic) Stackelberg equilibrium.

For n players, Definition 2.2 is modified as follows (see Ref. 4).

Definition 2.3. We define recursively

$$S_n(x_1, \ldots, x_{n-1}) := \arg\min_{x_n \in X_n} u_n(x_1, \ldots, x_n), \qquad \forall (x_1, \ldots, x_{n-1}) \in X_{-n},$$

$$S_k(x_1, \ldots, x_{k-1}) := \arg\min_{x_k \in X_k} \phi_k(x_1, \ldots, x_k), \forall (x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} X_i,$$

$$\forall k \in \{n-1, \ldots, 2\},$$

$$S_1 := \arg\min_{x_1 \in X_1} \phi_1(x_1),$$

where

$$\phi_k(x_1, \ldots,, x_k)$$

$$:= \sup_{x_{k+1} \in S_{k+1}(x_1,\ldots,x_k)} \ \sup_{x_{k+2} \in S_{k+2}(x_1,\ldots,x_{k+1})} \cdots \sup_{x_n \in S_n(x_1,\ldots,x_{n-1})} u_k(x_1, \ldots, x_n),$$

$$\text{for all } k \in \{1, \ldots, n-1\} \quad \text{and} \quad (x_1, \ldots, x_k) \in \prod_{i=1}^{k} X_i.$$

Any $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ such that $\bar{x}_1 \in S_1$ and $\bar{x}_k \in S_k(\bar{x}_1, \ldots, \bar{x}_{k-1})$ for $k > 2$ is called a weak (pessimistic) Stackelberg equilibrium.

In the following, we will consider special hierarchical games. More precisely, we will investigate the existence and some properties of weak Stackelberg equilibria in the case in which G is a hierarchical potential game as in Definition 1.1.

The game in Example 2.1, with

$$P(x_1, x_2) = u_1(x_1, x_2), \qquad d_1 = 0, \qquad d_2(x_1) = -3x_1^2,$$

is a hierarchical potential game.

3. Results

For a hierarchical potential game G with potential P, let G^P be the game obtained from G by replacing all payoff functions u_i by the potential P; i.e., G^P is the game in which all players have the same payoff function P. Moreover, for each player $i \in N$, denote the collections S_i in G and G^P respectively by S_i^G and S_i^P.

Proposition 3.1. Let $G = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N}\rangle$ be a hierarchical potential game with potential P. Then, for each player $i \in N$, $S_i^G = S_i^P$.

Proof. Fix $(x_1, \ldots, x_n) \in X$. Then,

$$\begin{aligned} S_n^G(x_1, \ldots, x_{n-1}) &= \arg\min_{x_n \in X_n} u_n(x_1, \ldots, x_n) \\ &= \arg\min_{x_n \in X_n} [P(x_1, \ldots, x_n) + d_n(x_1, \ldots, x_{n-1})] \\ &= \arg\min_{x_n \in X_n} P(x_1, \ldots, x_n) \\ &= S_n^P(x_1, \ldots, x_{n-1}), \end{aligned}$$

so

$$S_n^G = S_n^P.$$

Assume that the equality $S_i^G = S_i^P$ has been established for all players with index i larger that $k \in \{2, \ldots, n-1\}$. We show also that

$$S_k^G = S_k^P.$$

The proof for $k = 1$ is identical, up to minor modifications in the notation.

Fix

$$(x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} X_i.$$

Then,

$$
\begin{aligned}
&S_k^G(x_1, \ldots, x_{k-1}) \\
&= \arg \min_{x_k \in X_k} \sup_{x_{k+1} \in S_{k+1}^G(x_1,\ldots,x_k)} \cdots \sup_{x_n \in S_n^G(x_1,\ldots,x_{n-1})} u_k(x_1, \ldots, x_n) \\
&= \arg \min_{x_k \in X_k} \sup_{x_{k+1} \in S_{k+1}^G(x_1,\ldots,x_k)} \cdots \sup_{x_n \in S_n^G(x_1,\ldots,x_{n-1})} [P(x_1, \ldots, x_n) + d_k(x_1, \ldots, x_{k-1})] \\
&= \arg \min_{x_k \in X_k} \sup_{x_{k+1} \in S_{k+1}^G(x_1,\ldots,x_k)} \cdots \sup_{x_n \in S_n^G(x_1,\ldots,x_{n-1})} P(x_1, \ldots, x_n) \\
&= \arg \min_{x_k \in X_k} \sup_{x_{k+1} \in S_{k+1}^P(x_1,\ldots,x_k)} \cdots \sup_{x_n \in S_n^P(x_1,\ldots,x_{n-1})} P(x_1, \ldots, x_n) \\
&= S_k^P(x_1, \ldots, x_{k-1}),
\end{aligned}
$$

and so,

$$S_k^G = S_k^P. \qquad \square$$

Proposition 3.2. Let $G = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a hierarchical potential game with potential P. The following claims are equivalent:

(a) $\bar{x} \in X$ is a weak Stackelberg equilibrium of G;
(b) $\bar{x} \in X$ is a weak Stackelberg equilibrium of G^P;
(c) $\bar{x} \in \arg\min_{x \in X} P(x)$.

Proof. The equivalence between (a) and (b) follows from Proposition 3.1. The equivalence between (b) and (c) follows from the following claim. For each $k \in N$ and each $(x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} X_i$, the following two claims are equivalent:

$$x_m \in S_m^P(x_1, \ldots, x_{m-1}), \qquad \text{for each } m \in N, m \geq k,$$

$$(x_m, \ldots, x_n) \in \arg \min_{(z_m,\ldots,z_n)} P(x_1, \ldots, x_{m-1}, z_m, \ldots, z_n).$$

This equivalence holds by the definition of S_n^P if $k = n$. Assume that the equivalence holds for all players with index larger than $k \in \{2, \ldots, n-1\}$. We show also that it holds for player k. The proof for $k = 1$ is identical, up to minor modifications in the notation. Fix

$$(x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} X_i.$$

Since

$$x_m \in S_m^P(x_1, \ldots, x_{m-1}), \qquad m \in N, m > k,$$

we know that

$$(x_{k+1}, \ldots, x_n) \in \arg \min_{(z_{k+1}, \ldots, z_n)} P(x_1, \ldots, x_k, z_{k+1}, \ldots, z_n).$$

Then,

$$\begin{aligned} x_k &\in S_k^P(x_1, \ldots, x_{k-1}) \\ &= \arg \min_{x_k \in X_k} \sup_{x_{k+1} \in S_{k+1}^P(x_1, \ldots, x_k)} \cdots \sup_{x_n \in S_n^P(x_1, \ldots, x_{n-1})} P(x_1, \ldots, x_n) \\ &= \arg \min_{x_k \in X_k} \min_{x_{k+1} \in X_{k+1}} \cdots \min_{x_n \in X_n} P(x_1, \ldots, x_n). \end{aligned}$$ □

In Example 2.1, it is easy to prove that the global minimum points of the function

$$P(x_1, x_2) = u_1(x_1, x_2) = x_1^2 + 3x_1x_2 + x_2^2$$

on the square $[-2, 2]^2$ are $(2, -2)$ and $(-2, 2)$.

A consequence of this result is the following existence theorem.

Theorem 3.1. Let $G = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a hierarchical potential game with potential P. If P is a lower semicontinuous function on X, then G has at least one weak Stackelberg equilibrium.

Example 3.1. Let $X_1 = X_2 = [0, 1]$, and let

$$u_1(x_1, x_2) = P(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2 = 0, \\ 1 - x_1 - x_2, & \text{otherwise,} \end{cases}$$

$$u_2(x_1, x_2) = P(x_1, x_2) + 2x_1 = \begin{cases} 0, & \text{if } x_1 = x_2 = 0, \\ 1 + x_1 - x_2, & \text{otherwise.} \end{cases}$$

In this case, the potential function P is lower semicontinuous, not continuous, and the minimum on the square $[0, 1]^2$ is reached in $(1, 1)$, which is also the weak Stackelberg equilibrium.

Remark 3.1. A hierarchical potential game G is a particular case of potential games (exact potential games in Ref. 11) according to the following characterization given in Ref. 12: a game $\Gamma = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is a potential game iff

$$u_i = c_i + d_i, \qquad \forall i \in N,$$

where $c_i = w_i P$ for $w_i \in \mathscr{R}$ and $P: X \mapsto \mathscr{R}$, $d_i: X \mapsto \mathscr{R}$ not depending on x_i.

In Ref. 11, it is observed that the set of all strategy profiles that minimize P is a subset of the Nash equilibria, that is, any $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ such that

$$u_i(\bar{x}_1, \ldots, \bar{x}_n) = \min_{y \in X_i} u_i(y, \bar{x}_{-i}), \qquad \forall i = 1, \ldots, n.$$

From Proposition 3.2, the set of the weak Stackelberg equilibria is a subset of the Nash equilibria for a hierarchical potential game. In Example 2.1, the Nash equilibria of G are $\{(2, -2); (0, 0); (-2, 2\}$.

Remark 3.2. We can obtain a result analogous to Proposition 3.2 for strong Stackelberg equilibrium. Let us recall the following definition (see Ref. 4).

Definition 3.1. We define recursively

$$T_n(x_1, \ldots, x_{n-1}) := \arg \min_{x_n \in X_n} u_n(x_1, \ldots, x_n), \qquad \forall (x_1, \ldots, x_{n-1}) \in X_{-n},$$

$$T_k(x_1, \ldots, x_{k-1}) := \arg \min_{x_k \in X_k} \psi_k(x_1, \ldots, x_k), \qquad \forall (x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} X_i,$$

$$\forall k \in \{n-1, \ldots, 2\},$$

$$T_1 := \arg \min_{x_1 \in X_1} \psi_1(x_1),$$

where

$$\psi_k(x_1, \ldots, x_k) := \inf_{x_{k+1} \in T_{k+1}(x_1, \ldots, x_k)} \; \inf_{x_{k+2} \in T_{k+2}(x_1, \ldots, x_{k+1})} \cdots \inf_{x_n \in T_n(x_1, \ldots, x_{n-1})} u_k(x_1, \ldots, x_n),$$

$$\text{for all } k \in \{1, \ldots, n-1\} \text{ and } (x_1, \ldots, x_k) \in \prod_{i=1}^{k} X_i.$$

Any $\bar{x}_1, \ldots, \bar{x}_n) \in X$ such that $\bar{x}_1 \in T_1$ and $\bar{x}_k \in T_k(\bar{x}_1, \ldots, \bar{x}_{k-1})$ for all $k > 2$ is called a strong (optimistic) Stackelberg equilibrium.

Analogous to the derivation above, one can prove the following result.

Proposition 3.3. Let $G = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a hierarchical potential game with potential P. The following claims are equivalent:

(a) $\bar{x} \in X$ is a strong Stackelberg equilibrium of G;
(b) $\bar{x} \in X$ is a strong Stackelberg equilibrium of G^P;
(c) $\bar{x} \in \arg \min_{x \in X} P(x)$.

Consequently, the set of weak and strong Stackelberg equilibria of hierarchical potential games coincide with the set of minimizers of a potential function.

We consider now continuously differentiable games; it is possible to give an integrability condition in order to recognize a potential function P:

Proposition 3.4. Let G be a hierarchical game; let X_i be an interval of real numbers, and let $u_i \in C^1(X)$, $\forall i = 1, \dots, n$. G is a hierarchical potential game with potential function $P: X \mapsto \mathscr{R}$ iff $P \in C^1(X)$ and, $\forall i = 1, \dots, n$,

$$\partial u_i / \partial x_j = \partial P / \partial x_j, \qquad \forall j \geq i.$$

Proof. In fact, for $i \geq 2$ we have

$$\begin{aligned} \partial u_i / \partial x_j &= \partial P / \partial x + \partial d_i / \partial x_j \\ &= \begin{cases} \partial P / \partial x_j, & \text{if } i \leq j, \\ \partial P / \partial x_j + \partial d_i / \partial x_j, & \text{if } 1 \leq j < i, \end{cases} \end{aligned}$$

Moreover,

$$(\partial / \partial x_j)(u_j - u_{j-1}) = \partial P / \partial x_j - \partial P / \partial x_j = 0.$$

Then, $u_j - u_{j-1}$ does not depend on x_j for $j = 2, \dots, n$. Let $k > j$ for $j = 2, \dots, n-1$; we have

$$(\partial / \partial x_k)(u_j - u_{j-1}) = 0.$$

Then, $u_j - u_{j-1}$ does not depend on x_k for $k > j$. So,

$$u_j - u_{j-1} = d_j(x_1, \dots, x_{j-1}).$$ □

4. Discussion

In this paper, we presented the concept of hierarchical potential game as a special case of hierarchical games, which are games with n levels of hierarchy in the decision-making process. For hierarchical potential games, we considered pessimistic Stackelberg strategies and optimistic Stackelberg strategies.

We proved that the set of pessimistic and optimistic Stackelberg equilibria of a hierarchical potential game coincide with the set of minimizers of a potential function. Consequently, sufficient conditions for the existence of pessimistic and optimistic Stackelberg strategies are obtained.

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