Infinite Particle Systems and Multi-Dimensional Renewal Theory

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1. Introduction and statement of results. The obvious extension of the main theorem of one-dimensional renewal theory to the multi-dimensional case is degenerate in that all limits are zero or infinity. In this paper we will consider multi-dimensional renewal theory in the context of an infinite particle system. By this device we will be able to obtain a non-degenerate theory which closely parallels one-dimensional renewal theory. The methods used below depend on a closely related work of the author [13] and on the main theorem of one-dimensional renewal theory, proven in its final form by Feller and Orey [6].

Let φ denote a probability measure on d-dimensional space R^d, Let $\varphi^{(n)}$ denote the n-fold convolution of φ with itself and set

$$\mu = \sum_{n=0}^{\infty} \varphi^{(n)}.$$

Then μ is the *renewal measure*.

The behavior of μ depends on the smallest closed subgroup X of R^d containing the support of φ. Without loss of generality we can assume that

$$X = \{x = (x^1, \cdots, x^d) \mid x^{d_1+1}, \cdots, x^d \quad \text{are integers}\}.$$

If $d_1 = d$, then $X = R^d$ and if $d_1 = 0$, then $X = Z^d$, where Z^d denotes the set of points in R^d all of whose coordinates are integers.

Let $| \ |$ denote Haar measure on X defined as the product of Lebesgue measure on the first d_1 coordinates of X and counting measure on the last $d - d_1$ coordinates. Let $\mathcal{B}$ denote the collection of relatively compact Borel subsets of X. Let $\mathcal{A}$ denote the subcollection of $A \in \mathcal{B}$ such that $|\partial A| = 0$.

For some results it will be necessary to assume that φ satisfies

Condition 1. *Some* $\varphi^{(n)}$ *is non-singular with respect to Haar measure on* X.

Let $\mathcal{A}^* = \mathcal{B}$ if φ satisfies Condition 1 and let $\mathcal{A}^* = \mathcal{A}$ otherwise.

Let $\cdot$ denote the usual dot product on R^d and let v denote a vector in R^d of unit length. Let the non-negative constant $\kappa = \kappa_v$ be defined as follows: if

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$$M_v = \int (x \cdot v) \varphi\, (dx)$$

is finite and strictly positive, set $\kappa = M_v^{-1}$; if

$$\mu(\{x \mid |x \cdot v| \leqq 1\}) = \infty,$$

set $\kappa = \infty$; otherwise set $\kappa = 0$. In the expression "$x \cdot v \to \infty$" let it be understood that x ranges over X. With this notation we can state the

One-dimensional renewal theorem. *Let $d = 1$.*

(i) If $\kappa < \infty$, then

$$\lim_{x \cdot v \to \infty} \mu(x + A) = \kappa\, |A|, \qquad A \in \mathfrak{A}^*. \tag{1.1}$$

(ii) If $\kappa = \infty$, then

$$\mu(x + A) = \infty, x \in X \quad \textit{and } A \in \mathfrak{A}^* \quad \textit{with} \quad |A| > 0. \tag{1.2}$$

When restricted to $A \in \mathfrak{A}$ this important theorem was obtained in its final form by Feller and Orey [6] (see Feller [5]). The extension to $\mathfrak{B}$ under Condition 1 was given by the author [9].

One is tempted to extend this result to the multi-dimensional case $d \geqq 2$. What is *essentially* known in this case can be stated as the

Multi-dimensional renewal theorem. *Let $d \geqq 2$. Then either*

$$\mu(x + A) = \infty, \quad x \in X \quad \textit{and} \quad A \in \mathfrak{A}^* \quad \textit{with} \quad |A| > 0, \tag{1.3}$$

or

$$\lim_{|x| \to \infty} \mu(x + A) = 0, \qquad A \in \mathfrak{A}^*. \tag{1.4}$$

Results in this direction were first obtained by Chung [2]. Extensions can be found in Feller [5] and Spitzer [8]. Although the theorem has never been published in the full generality stated above, most likely the methods of [5] and [8] could be used to yield the result. At any rate the theorem is stated for motivation only, and will not be used in the sequel.

As a consequence of the above theorem we have the

Corollary. *Let $d \geqq 2$ and suppose that*

$$\lim_{x \cdot v \to \infty} \mu(x + A) = L, \qquad A \in \mathfrak{A}. \tag{1.5}$$

Then either $L = 0$ or $L = \infty$.

(See Problem 12 of Feller [5, p. 372] for an equivalent result.) In other words finite non-zero limits are impossible.

These results suggest that in order to obtain finite non-zero limits, the multi-dimensional case should be reformulated. One line of research in this direction has been taken by Bickel and Yahav [1] and Farrell [3] and [4].

In this paper we will reformulate the problem in terms of infinite particle systems along the lines suggested by the author [13]. In particular we will consider measures ν such that if $\kappa < \infty$, then

$$\lim_{x \cdot v \to \infty} (\nu * \mu)(x + A) = \kappa\lambda\, |A|, \qquad A \in \mathcal{A}^*. \tag{1.6}$$

To this end we recall some notation from [13]. Let $v_1 = v, v_2, \cdots, v_d$ be an orthonormal basis of R^d. Set

$$\Gamma_m = \{x \in R^d \mid 0 \leqq x \cdot v_k < m \text{ for } 2 \leqq k \leqq d\}.$$

Let $\mathfrak{J}_v$ denote the collection of all sets $T \subseteq R^d$ of the form

$$T = T_{a_1, a_2} = \{x \in R^d \mid a_1 \leqq x \cdot v \leqq a_2\},$$

where a_1 and a_2 are finite.

Let $\mathfrak{N}_v$ denote the collection of measures ν supported by $X \cap T$ for some $T \in \mathfrak{J}_v$. For $0 < \lambda < \infty$ let $\mathfrak{N}_{v,\lambda}$ denote the subcollection of measures $\nu \in \mathfrak{N}_v$ such that

$$\lim_{m \to \infty} \nu(x + \Gamma_m)/m^{d-1} = \lambda \tag{1.7}$$

uniformly for $x \in R^d$.

From now on we assume that $d \geqq 2$. Our candidate for the multi-dimensional renewal theorem is

Theorem 1. *Let* $0 < \lambda < \infty$ *and* $\nu \in \mathfrak{N}_v$.

(*i*) *If* $0 \leqq \kappa < \infty$ *and* $\nu \in \mathfrak{N}_{v,\lambda}$, *then* (1.6) *holds.*

(*ii*) *If* $0 < \kappa < \infty$ *and* (1.6) *holds, then* $\nu \in \mathfrak{N}_{v,\lambda}$.

(*iii*) *If* $\kappa = \infty$ *and* $\nu \in \mathfrak{N}_{v,\lambda}$, *then*

$$(\nu * \mu)(x + A) = \infty, \quad x \in X \quad \textit{and} \quad A \in \mathcal{A}^* \quad \textit{with } |A| > 0. \tag{1.8}$$

Next we will consider random measures N such that $EN \in \mathfrak{N}_v$ and, for some $0 < \lambda < \infty$,

$$\lim_{m \to \infty} E\, |N(x + \Gamma_m)/m^{d-1} - \lambda| = 0 \tag{1.9}$$

uniformly for $x \in R^d$. Then $EN \in \mathfrak{N}_{v,\lambda}$.

Theorem 2. *Let* $0 < \lambda < \infty$ *and* $EN \in \mathfrak{N}_v$.

(*i*) *If* $0 \leqq \kappa < \infty$ *and* (1.9) *holds, then*

$$\lim_{x \cdot v \to \infty} E\, \big|(N * \mu)(x + A) - \kappa\lambda\, |A|\big| = 0, \qquad A \in \mathcal{A}^*. \tag{1.10}$$

(*ii*) *If* $0 < \kappa < \infty$ *and* (1.10) *holds, then* (1.9) *holds.*

(*iii*) *If* $\kappa = \infty$ *and* (1.9) *holds, then*

$$P\{(N * \mu)(x + A) = \infty, \quad x \in X\} = 1, \quad A \in \mathcal{A}^* \quad \textit{with} \quad |A| > 0. \tag{1.11}$$

These results have interesting probabilistic consequences, but to state them we need more notation.

Let X_0, X_1, X_2, denote a random walk on X with transition distribution φ. Then $X_0, X_1 - X_0, X_2 - X_1, \cdots$ are independent, $X_1 - X_0, X_2 - X_1, \cdots$ have distribution φ, and the distribution of X_0 is supported by X. Let P_x and E_x denote probabilities and expectations when $X_0 = x$.

Let $Y = Y(X_0, X_1, X_2, \cdots)$ denote a random variable defined on the process. For $x \varepsilon X$, let Y_x be defined by

$$Y_x(X_0, X_1, \cdots) = Y(X_0 - x, X_1 - x, \cdots).$$

Then

$$E_x Y_x = E_0 Y. \tag{1.12}$$

Suppose N is a random counting measure. Let the particles in N undergo independent random walks starting from their respective positions in N and with transition distribution φ. Corresponding to each particle and its random walk, we obtain a value of Y. Let V_Y be the random variable denoting the sum of these values of Y over all the particles. Then

$$EV_y = \int \nu(dy) E_y Y \tag{1.13}$$

and

$$E[V_Y \mid N] = \int N(dy) E_y Y. \tag{1.14}$$

Examples. Let B denote a Borel subset of X and set

$$\sigma_B = \min [n \geqq 0 \mid X_n \varepsilon B]$$

($\sigma_B = \infty$ if $X_n \notin B$ for all n). Let A denote a Borel subset of X and 1_A the indicator function of A. Set

$$Y = N_B(A) = \sum_{n=0}^{\sigma_B} 1_A(X_n).$$

Let $V_B(A) = V_Y$, where $Y = N_B(A)$. Then $N_B(A)$ denotes the total occupation time of A by the time when B is hit for a single process, and $V_B(A)$ denotes the corresponding sum over all the particles. In particular $V_B(B)$ denotes the total number of particles hitting B at least once, and $V_\phi(B)$ denotes the total occupation time of B by all the particles.

As another example let $N_B^r(A)$ be defined by $N_B^r(A) = 1$ if $N_B(A) = r$ and $N_B^r(A) = 0$ if $N_B(A) \neq r$ (here r denotes a strictly positive integer). Let $V_B^r(A) = V_Y$, where $Y = N_B^r(A)$. Then $V_B^r(A)$ denotes the total number of particles which have visited A exactly r times by the time of their first visit to B. Note that if $A \subseteq B$, then $N_B^1(A) = N_B(A)$ and $V_B^1(A) = V_B(A)$.

Naturally in order to obtain results about Y or V_Y we will need to impose further conditions on Y. Let $\mathfrak{D}_v$ denote the collection of Borel subsets D of X such that

$$\inf [x \cdot v \mid x \in D] > -\infty$$

and either $|\partial D| = 0$ or Condition 1 holds. Let $\mathfrak{D}_Y$ denote the subcollection of $D \in \mathfrak{D}_v$ such that if $X_j \notin D$ for $0 \leqq j < i$, then

$$Y(X_0, X_1, \cdots) = Y(X_i, X_{i+1}, \cdots).$$

Note that $\mathfrak{D}_{Y_x} = x + \mathfrak{D}_Y$.

Let $\mathcal{Y}$ denote the collection of Y's such that

(i) $\mathfrak{D}_Y$ is non-empty,
(ii) either Condition 1 holds, or $P_x(y \in \cdot)$ are continuous everywhere in x in the sense of weak convergence distributions, and
(iii) for some compact set $A \subseteq X$ and $a < \infty$

$$|Y| \leqq a \sum_{n=0}^{\infty} 1_A(X_n).$$

Let $\mathcal{Y}_1$ denote subcollection of $Y \in \mathcal{Y}$ which take on only the values 0 and 1.

In our example suppose $A \in \mathfrak{A}^*$ and $B \in \mathfrak{D}_v$. Then $N_B(A) \in \mathcal{Y}$ and $N_B^r(A) \in \mathcal{Y}_1$.

Let $\tilde{\varphi}$ denote the measure dual to φ defined by $\tilde{\varphi}(dx) = \varphi(-dx)$. Let $\tilde{E}_x$ and $\tilde{P}_d$, $x \in X$, denote probabilities and expectations corresponding to the dual random walk on X with transition distribution $\tilde{\varphi}$. Set

$$\tilde{L}_B(x) = \tilde{P}_x(X_n \notin B \quad \text{for all} \quad n \geqq 1).$$

Theorem 3. *Let* $0 \leqq \kappa < \infty$, $0 \leqq \lambda < \infty$, $\nu \in \mathfrak{N}_{v,\lambda}$, $Y \in \mathcal{Y}$, *and* $D \in \mathfrak{D}_Y$. *Then*

$$\lim_{x \cdot v \to \infty} \int \nu(dy) E_y Y_x = \lambda\kappa \int_D \tilde{L}_D(y) E_y Y. \tag{1.15}$$

Theorem 4. *Let* $0 \leqq \kappa < \infty$, $0 \leqq \lambda < \infty$, $EN \in \mathfrak{N}_v$ *and* (1.9) *hold,* $Y \in \mathcal{Y}$ *and* $D \in \mathfrak{D}_Y$. *Then*

$$\lim_{x \cdot v \to \infty} E \left| \int N(dy) E_y Y_x - \lambda\kappa \int_D \tilde{L}_D(y) E_y Y \right| = 0. \tag{1.16}$$

Theorems 3 and 4 do not require N to be a *counting* measure. If N is a counting measure however, these results can be reinterpreted by use of (1.13) and (1.14).

Corollary 1. *Under the conditions of Theorem* 3

$$\lim_{x \cdot v \to \infty} E V_{Y_x} = \lambda\kappa \int_D \tilde{L}_D(y) E_y Y. \tag{1.17}$$

Corollary 2. *Under the conditions of Theorem* 4

$$\lim_{x \cdot v \to \infty} E \left| E[V_{Y_x} \mid N] - \lambda\kappa \int_D \tilde{L}_D(y) E_y Y \right| = 0. \tag{1.18}$$

The next result concerns the asymptotic distribution of V_{Y_x}.

Theorem 5. *Let* $0 \leqq \kappa < \infty$, $0 \leqq \lambda < \infty$, $EN \in \mathfrak{N}_v$ *and* (1.9) *hold*, $Y \in \mathcal{Y}$, *and* $D \in \mathfrak{D}_Y$.

(*i*) *As* $x \cdot v \to \infty$, V_{Y_x} *has asymptotic distribution which is infinitely divisible with mean*

$$\lim_{x \cdot v \to \infty} EV_{Y_x} = \lambda\kappa \int_D \tilde{L}_D(y) E_y Y$$

and log characteristic function of the form

$$\int_{-\infty}^{\infty} (e^{i\theta z} - 1) M(dz),$$

where

$$M(A) = \lambda\kappa \int_D \tilde{L}_D(y) P_y(Y \in A)\, dy.$$

(*ii*) *If* $Y_1, \cdots, Y_k$ *are in* $\mathcal{Y}$ *and the product of any two of them is identically zero, then* $V_{(Y_1)_x}, \cdots, V_{(Y_1)_x}$ *are asymptotically independent as* $x \cdot v \to \infty$.

The final result summarizes the application of these theorems to the examples of Y defined above.

Theorem 6. *Let* $0 \leqq \kappa < \infty$, $0 \leqq \lambda < \infty$, $EN \in \mathfrak{N}_v$ *and* (1.9) *hold*, $A \in \mathfrak{A}^*$ *and* $B \in \mathfrak{D}_v$.

(*i*) $V^r_{x+B}(x + A)$ *has, as* $x \cdot v \to \infty$, *an asymptotic Poisson distribution with mean*

$$\lambda\kappa \int_A \tilde{L}_{A \cup B}(y) E_y N^r_B(A).$$

(*ii*) *For different values of* r, $V^r_{x+B}(x + A)$ *are asymptotically independent as* $x \cdot v \to \infty$.

(*iii*) $V_{x+B}(x + A)$ *has, as* $x \cdot v \to \infty$, *an asymptotic distribution which is infinitely divisible with mean*

$$\lambda\kappa \int_A \tilde{L}_B(y)\, dy$$

and log characteristic function

$$\lambda\kappa \sum_{r=1}^{\infty} (e^{i\theta r} - 1) \int_A \tilde{L}_{A \cup B}(y) E_y N^r_B(A).$$

(*iv*) *If* $A \subseteq B$ (*in particular if* $A = B$), *then* $V_{x+B}(x + A)$ *has, as* $x \cdot v \to \infty$, *an asymptotic distribution which is Poisson with mean*

$$\lambda\kappa \int_A \tilde{L}_B(y)\, dy.$$

In all of these cases the mean of the limiting distribution is the limit of the means.

2. Preliminaries. Some of the arguments used in proving Theorem 1 are valid in a more general setting. In order to break up the rather long proof of Theorem 1, we will present some of these results in this section.

Let X denote a locally compact Abelian group, let $|\ |$ denote Haar measure on X, and let $\mathfrak{A}$ denote the collection of relatively compact Borel subsets of X such that $|\partial A| = 0$. Let G denote a closed subgroup of X, let $|\ |_G$ denote Haar measure on G, and let $\mathfrak{A}_G$ denote the collection of relatively compact Borel subsets A of G such that $|\partial A|_G = 0$. We can think of $|\ |_G$ as a measure on X with support G.

Let $H = X/G$ and let M be the natural homomorphism from X onto H. Then M is a continuous open map, $M(x + y) = M(x) + M(y)$, x and y in X, and $M(x) = 0$ if and only if $x \in G$. Let $|\ |_H$ denote Haar measure on H, normalized so that if H is compact, then $|H|_H = 1$.

Proposition 1. *H is compact if and only if there is a compact subset C of X such that M maps C onto H or equivalently such that $(x + G) \cap C \neq \phi$, $x \in X$.*

Proof. Suppose H is compact. Then there exist finitely many compact subsets $N_1, \cdots, N_n$ of X such that

$$M(N_1) \cup \cdots \cup M(N_n) = H.$$

Then $C = N_1 \cup \cdots \cup N_n$ is a compact subset of X such that $M(C) = H$.

Let C be a compact subset of X such that $(x + G) \cap C \neq \phi$, $x \in X$. Choose $x \in X$. Choose $y \in (x + G) \cap C$ and set $g = y - x \in G$. Then $M(y) = M(g + x) = M(x)$. Thus M maps C onto H.

Let C be a compact subset of H such that M maps C onto H. Let A_α, $\alpha \in B$, be an open covering of H. Then $M^{-1}(A_\alpha)$, $\alpha \in B$, is an open covering of X and hence of C. Thus there is a finite subset B' of B such that $M^{-1}(A_\alpha)$, $\alpha \in B'$, is an open covering of C. Consequently A_α, $\alpha \in B'$, is an open covering of H. This shows that H is compact. Choose $x \in X$. There is a $y \in C$ such that $M(y) = M(x)$ and hence $M(x - y) = 0$. Thus $x - y = g \in G$, so that $y \in (x + G) \cap C$ and consequently $(x + G) \cap C \neq \phi$.

Proposition 2. *If H is compact there is a compact subset C of X and a $c > 0$ such that $|x + C|_G \geqq c$, $x \in X$.*

Proof. Let C_1 be a compact subset of X such that M maps C_1 onto H. Let C_2 be a compact subset of X such that $|C_2|_G = c > 0$. Set $C = C_2 - C_1$. Then C is compact. Choose $x \in X$. There is a $y \in C_1$ such that $x - y = g \in G$. Now

$$|x + C|_G = |x - g + C|_G = |y + C_2 - C_1|_G \geqq |C_2|_G \geqq c.$$

Proposition 3. *Let H be compact and suppose that $|G| > 0$. Then H is finite.*

Proof. Let A be a non-empty open relatively compact subset of X. Let C be a compact subset of G such that $|A| > 0$. Then $|A + C| < \infty$. Let $h_1, \cdots, h_n$ be distinct elements of $M(A)$ with $h_m = M(x_m)$ and $x_m \in A$, $1 \leqq m \leqq n$. Then $x_m + C$, $1 \leqq m \leqq n$, are disjoint subsets of $|A + C|$ and $n \leqq |A + C|/|C|$. Thus

$M(A)$ is a finite non-empty open set. Consequently H is descrete. Since it is compact, it must be finite.

Let $\mathfrak{A}^G$ denote the collection of relatively compact Borel subsets A of X having the property that for all $\epsilon > 0$ there is a symmetric open subset P of G containing the origin and such that for $x \in X$.

$$|x + A + P|_G - \epsilon \leqq |x + A|_G \leqq |\{y \mid y + P \subseteq x + A\}|_G + \epsilon.$$

For $\alpha \in X$, let $|\ |_{G+\alpha}$ denote the measure on X defined by $|A|_{G+\alpha} = |A - \alpha|_G$.

The next proposition extends somewhat Corollary 2.1 of [12].

Proposition 4. *Let $p_n^\beta(x)$, $n \geqq 1$, $x \in X$, and $\beta \in B$ (B an index set) be non-negative and bounded uniformly in n, x, and β. Let $\alpha_n \in X$, $n \geqq 1$, and let μ_n^β, $n \geqq 1$ and $\beta \in B$, be measures on X having support in $\alpha_n + G$. Let K_m, $m \geqq 1$, be probability measures in X having support in G and weakly convergent as $m \to \infty$ to the probability measure concentrated at the origin. Suppose that for each $m \geqq 1$ and relatively compact Borel subset A of X*

$$\lim_{n\to\infty} \left((K_m * \mu_n^\beta)(x + A) - \int_{x+A} p_n^\beta(y)\, |dy|_{G+\alpha_n} \right) = 0$$

uniformly for $x \in X$ and $\beta \in B$. Then for each $A \in \mathfrak{A}^G$

$$\lim_{n\to\infty} \left(\mu_n^\beta(x + A) - \int_{(x+A)\cap G} p_n^\beta(y)\, |dy|_{G+\alpha_n} \right) = 0$$

uniformly for $x \in X$ and $\beta \in B$.

Proof. The proof of this proposition is similar to that of Theorem 2.1 of [12] and, except for one detail, will be omitted. The detail is that $|x + A|_G$, $x \in X$, has a finite upper bound. To see this, observe that if $|x + A|_G > 0$, then $x + a = g$ for some $a \in A$ and $g \in G$. In this case

$$|x + A|_G = |x - g + A|_G = |A - a|_G \leqq |A - A|_G < \infty.$$

The final result of this section is an application of the point-wise ergodic theorem.

Proposition 5. *Let H be compact. Suppose $x_0 \in X$ is such that*

$$\bigcup_{n=-\infty}^{\infty} (nx_0 + G)$$

is dense in X. Then there is a finite positive constant c such that for $A \in \mathfrak{A}$

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} |nx_0 + x + A|_G = c\, |A| \tag{2.1}$$

uniformly for $x \in X$.

Proof. Set $h_0 = M(x_0)$. *If $c(h)$, $h \in H$, is a continuous character on H such that $c(h_0) = 1$, then $c(h) = 1$, $h \in H$.*

To see this note that for $-\infty < n < \infty$ and $g \in G$

$$c(M(nx_0 + g)) = c(nM(x_0)) = c(nh_0) = 1,$$

and hence $c(M(x)) = 1$, $x \in X$. Since M is an onto map, $c(h) = 1$, $h \in H$.

Let $T : H \to H$ be given by $Th = h_0 + h$. Then *T is an ergodic transformation on H.* To prove this we need only show that if $I(h)$, $h \in H$, is a bounded real-valued function and if I is invariant with respect to T, *i.e.*, if

$$I(h_0 + h) = I(h), \qquad \text{a.e.} \quad h \in H,$$

then $I(h)$ is almost everywhere equal to a constant.

Let $\{c_k(h),\ h \in H\}$ be a complete orthonormal basis of continuous characters in $\mathfrak{L}_2(H)$. Then there are constants a_k such that in $\mathfrak{L}_2(H)$

$$\begin{aligned}\sum_k a_k c_k(h) &= I(h) = I(h_0 + h)\\ &= \sum_k a_k c_k(h_0 + h)\\ &= \sum_k a_k c_k(h_0) c_k(h).\end{aligned}$$

Consequently $a_k(c_k(h_0) - 1) = 0$ for all k. Therefore, either $a_k = 0$ or $c_k(h) = 1$, $h \in H$. This shows that $I(h)$ is almost surely equal to a constant.

It now follows from the pointwise ergodic theorem that *if $f(h)$, $h \in H$, is in $\mathfrak{L}_1(H)$, then*

$$\lim_{n\to\infty} N^{-1} \sum_{n=1}^{N} f(nh_0 + h) = \int_H f(h)\, dh, \quad \text{a.e.} \quad h \in H. \tag{2.2}$$

Let $C_c(X)$ denote the collection of real-valued functions on X which are continuous and have compact support. For $f \in C_c(X)$ set

$$J(f) = \int f(x)\, dx.$$

Also let I_f be the function on X defined by

$$I_f(x) = \int f(y - x)\, |dy|_G .$$

It is easily seen that I_f is bounded, uniformly continuous, and invariant under translation by elements of G. Thus we can define a function K_f on H by

$$K_f(h) = I_f(x) \quad \text{if} \quad h = M(x).$$

Then K_f is bounded and uniformly continuous on H and, in particular, is a Borel function.

Suppose from now on that H is compact. Then *there is a finite positive constant c such that for $f \in C_c(X)$*

$$\int_H K_f(h)\, dh = cJ(f). \tag{2.3}$$

To see this note that if g is the translate of f by an element of G, then

$$\int_H K_f(h)\,dh = \int_H K_g(h)\,dh$$

and (2.3) now follows from the uniqueness of Haar measure on X.

By (2.2) and (2.3) we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} K_f(nh_0 + h) = cJ(f) \quad \text{a.e.} \quad h \in H.$$

It follows from the continuity of K_f that this limit holds for all $h \in H$ and, in fact, uniformly for $h \in H$ (since H is compact). In other words for $f \in C_c(X)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f(y - nx_0 - x)\,|dy|_G = cJ(f) \tag{2.4}$$

uniformly for $x \in X$.

Suppose $A \in \mathcal{A}$ and $\epsilon > 0$. Then by Tietze's extension theorem and the regularity of Haar measure we can find non-negative functions $f_1 \in C_c(X)$ and $f_2 \in C_c(X)$ such that $f_1 \leqq 1_A \leqq f_2$ and $J(f_2) - \epsilon \leqq |A| \leqq J(f_1) + \epsilon$. It now follows by applying (2.4) to f_1 and f_2 and letting $\epsilon \to 0$ that (2.1) holds, as desired.

3. Proof of Theorem 1. We will now apply the results of Section 2 and go on to complete the proof of Theorem 1.

Let S denote the support of φ. Then S is a closed subset of R^d, $\varphi(S) = 1$, and X is the smallest closed subset of R^d containing S. Let G be the smallest closed subgroup of R^d containing $(S - S)$ (or, equivalently, containing some translate of S). Then G is a closed subgroup of X. Choose $s \in S$. Then $X \supseteq s + G \supseteq S$. Thus X is the smallest closed subgroup of R^d containing $s + G$. It follows that X is the closure of

$$\bigcup_{n=-\infty}^{\infty} (ns + G).$$

Clearly G is d-dimensional or $(d - 1)$-dimensional. If G is $(d - 1)$-dimensional, then

$$X = \bigcup_n (ns + G).$$

In this case $d_1 \leqq d - 1$ and, without loss of generality, we can assume that

$$G = \{x \in X \mid x^d = 0\}.$$

If G is d-dimensional, then it follows easily from Proposition 1 that X/G is compact and hence Propositions 2, 3, and 5 are applicable. Suppose additionally that Condition 1 holds. Then $|G| > 0$ and it now follows from Proposition 3 that X/G is finite. Thus there is a positive integer N_0 such that $N_0 h = 0$ for $h \in H$. We then have that for $s \in S$ and $A \in B$

$$N_0^{-1} \sum_{n=1}^{N_0} |ns + x + A|_G = c\,|A|, \qquad x \in X. \tag{3.1}$$

We summarize some of these results in

Proposition 6. *Suppose G is d-dimensional. Then there is a finite positive constant c such that for all $s \in S$ and $A \in \mathcal{A}^*$*

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} |ns + x + A|_G = c\,|A| \tag{3.2}$$

uniformly for $x \in X$. If also Condition 1 holds, there is a positive integer N_0 such that, for $s \in S$ and $A \in \mathcal{B}$, (3.1) holds.

In order to apply Proposition 4, we need to find sets in $\mathcal{A}^G$. For $0 \leqq a < \infty$ and $x = (x^1, \cdots, x^d) \in X$, set

$$a \odot x = (ax^1, \cdots, ax^{d_1}, x^{d_1+1}, \cdots, x^d).$$

Set $\Delta = \{x \in X \mid 0 \leqq x^i < 1, 1 \leqq i \leqq d\}$. Then we clearly have

Proposition 7. *For $0 \leqq a < \infty$, $a \odot \Delta \in \mathcal{A}^G$.*

Using Proposition 4, we can extend some results of Stone [10] to a form more useful here. We first recall some notation from [10].

Let χ denote a probability measure on R^d which satisfies Cramér's condition: for some finite positive constant c

$$\int_{R^d} e^{c|x|}\chi(dx) < \infty.$$

Let B be a compact subset of $\{t \in R^d \mid |t| < c\}$ containing 0 as an interior point. Let g denote the moment generating function of χ, defined by

$$g(t) = \int_{R^d} e^{t\cdot x}\chi(dx), \qquad t \in R^d.$$

Then g is finite and continuously differentiable any number of times on B and in particular

$$g'(0) = \int_{R^d} x\chi(dx) = m_0 .$$

Let χ_t , $t \in B$, denote the probability measure on R^d defined by

$$\chi_t(A) = \int_A (g(t))^{-1} e^{t\cdot x}\chi(dx).$$

Let m_t and Σ_t denote the mean and covariance matrix of χ_t and let $|\Sigma_t|$ denote the determinant of Σ_t . Let $\chi^{(n)}(\chi_t^{(n)})$ denote the n-fold convolution of $\chi(\chi_t)$ with itself.

Assume χ is such that the support of χ is S and let G and X be as before.

Assume also that G is d-dimensional. Then Σ_t is non-singular. Let p_{Σ_t} denote the density of a multivariate normal distribution with mean 0 and covariance Σ_t.

Proposition 8. (G *d-dimensional*)

(*i*) *Under the above definitions and conditions there is a constant* $0 < c < \infty$ *such that for* $A \in \mathfrak{A}^G$ *and* $s \in S$

$$\lim_{n\to\infty} [n^{d/2}\chi_t^{(n)}(x + A) - cp_{\Sigma_t}((x - m_t n)/\sqrt{n})\, |x + A|_{ns+G}] = 0 \tag{3.3}$$

uniformly for $x \in X$ *and* $t \in B$.

(*ii*) *If, additionally,* χ *satisfies Condition 1, then* $\mathfrak{A}^G$ *can be replaced by* $\mathfrak{B}$.

Proof. The proof of the first statement uses Proposition 4 but is otherwise similar to the proof of Theorem 3.2 of Stone [10]. The arguments used to prove the second statement are similar to those used in the proof of Theorem 3 of Stone [11].

We can express $\chi^{(n)}$ in terms of $\chi_t^{(n)}$ according to the formula

$$\chi^{(n)}(A) = \int_A (g(t))^n e^{-t\cdot x}\chi_t(dx).$$

Under the above assumptions (which guarantee that Σ_t is non-singular), m_t has a continuous inverse $t(m)(t(m_0) = 0)$ for t sufficiently small. Then $m_{t(x)} = x$ for x in some neighborhood of m_0. Using Proposition 8 we get

Proposition 9. Let G *be d-dimensional and* χ *satisfy Cramér's condition.* (*i*) *There exist finite positive constants* c_1 *and* c_2 *such that for* $s \in S$, $x \in X$, *and* $A \in \mathfrak{A}^G$

$$\chi^{(n)}(x + A) = c_1[g(t(x/n))]^n e^{-x\cdot t(x/n)}\cdot n^{-d/2} \cdot\left[\int_{x+A} e^{-(y-x)\cdot t(x/n)}\, |dy|_{ns+G} + o_n(1)\right], \tag{3.4}$$

where $o_n(1) \to 0$ *as* $n \to \infty$ *uniformly for* $|x - m_0 n| \leqq c_2 n$. (*ii*) *If, additionally,* χ *satisfies Condition 1, then* $\mathfrak{A}^G$ *can be replaced by* $\mathfrak{B}$.

Proposition 9 is used together with the well known easily proven result which we state as

Proposition 10. Let χ *have mean* m_0 *and satisfy Cramér's condition. Then for every finite positive constant* c *there is a* $\delta > 0$ *such that*

$$\lim_{n\to\infty} e^{\delta n}\chi(\{x \mid |x - m_0 n| \geqq cn\}) = 0. \tag{3.5}$$

Only slight modifications are needed in Proposition 9 if G is $(d - 1)$-dimensional.

Proposition 11. Let G *be* $(d - 1)$*-dimensional and* χ *satisfy Cramér's condition.* (i) *There exist finite positive constants* c_1 *and* c_2 *such that for* $s \in S$ *and* $A \in \mathfrak{A}$ *with* $A \subseteq G$ *and* $x \in nx + g$, (3.6) $$\chi^{(n)}(x + A) = c_1[g(t(x/n))]^n e^{-x \cdot t(x/n)} \cdot n^{-(d-1)/2} \cdot \left[\int_{x+A} e^{-(y-x)\cdot t(x/n)}\, dy + o_n(1) \right],$$

where $o_n(1) \to 0$ *as* $n \to \infty$ *uniformly for* $|x - m_0 n| \leqq c_2 n$. (ii) *If, additionally* χ *satisfies Condition 1, then* $\mathfrak{A}$ *can be replaced by* $\mathfrak{B}$.

The preceding results are of general interest. We now apply them to yield more specialized results necessary for the proof of Theorem 1.

Proposition 12. *Let* G *be d-dimensional and let* χ *have mean* m_0 *and satisfy Cramér's condition. Let* A_1 *and* A_2 *be relatively compact Borel subsets of* X *such that* $|A_1| = |A_2|$ *and either* χ *satisfies Condition* 1 *or* A_1 *and* A_2 *are in* $\mathfrak{A}^G \cap \mathfrak{A}$. *Let* B *be in* $\mathfrak{A}^G \cap \mathfrak{A}$ *and such that* $\inf_x |x + B|_G > 0$. *Let* C *be a compact subset of* X. *Then for any* $\epsilon > 0$ *there exist* $\delta > 0$ *and positive integers* i_0 *and* j_0 *such that for* $j \geqq j_0$, $y \in C$, *and* $x \in X$ *with* $|x - j| \leqq \delta_j$

(3.7) $$\left| \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x + y + A_1) - \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x + A_2) \right| \leqq \epsilon \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x + B).$$

Proof. In the notation of Proposition 9 set

$$c(n, x) = c_1 n^{-d/2}[g(t(x/n))]^n e^{-x.t(x/n)}.$$

We can assume that $0 \in C$. Choose $0 < \epsilon < 1$ and positive integer i_0. By Proposition 9 there is a $\delta > 0$ and a positive integer j_0 such that for $s \in S$, $j \geqq j_0$, $0 \leqq i \leqq i_0 - 1$, $y \in C$, and $x \in X$ with $|x - m_{0j}| \leqq \delta j$

(3.8) $$\chi^{(j+i)}(x + B) \geqq \tfrac{1}{2} c(j, x)\, |x + B|_{(j+i)s+G},$$

(3.9) $$\chi^{(j+i)}(x + y + A_1) \leqq (1 + \epsilon^2) c(j, x) \cdot \left[|x + y + A_1|_{(j+i)s+G} + \frac{\epsilon}{2} |x + B|_{(j+i)s+G} \right],$$

(3.10) $$\chi^{(j+i)}(x + y + A_1) \geqq (1 - \epsilon^2) c(j, x) \cdot \left[|x + y + A_1|_{(j+i)s+G} - \frac{\epsilon}{2} |x + B|_{(j+i)s+G} \right],$$

and (3.9) and (3.10) also hold with A_1 replaced by A_2.

Choose $\beta > 0$ such that

(3.11) $$|x + B|_G \geqq \beta, \qquad x \in X.$$

We can assume that ϵ is small enough so that for $n \geqq 0$

$$\epsilon\, |x + A_1|_{ns+G} \leqq \beta, \qquad x \in X,$$

and

$$\epsilon\, |x + A_2|_{ns+G} \leqq \beta, \qquad x \in X.$$

Applying these inequalities we get that for s, j, i, y and x as in (3.8)–(3.10)

$$(3.12)\quad |\chi^{(j+i)}(x+y+A_1) - \chi^{(j+i)}(x+A_2) - c(j,x) \cdot [|s+y+A_1|_{(j+i)s+G} - |x+A_2|_{(j+i)s+G}]| \leqq 4\epsilon c(j,x)\,|x+B|_{(j+i)s+G} \leqq 8\epsilon\chi^{(j+i)}(x+B).$$

It follows from Proposition 5 that we can assume i_0 is such that

$$\left| i_0^{-1} \sum_{i=0}^{i_0-1} \left| is + x + A_1 \right|_G - c\,|A_1| \right| \leqq \epsilon\beta/2, \qquad x \in X$$

and

$$\left| i_0^{-1} \sum_{i=0}^{i_0-1} \left| is + x + A_2 \right|_G - c\,|A_2| \right| \leqq \epsilon\beta/2, \qquad x \in X.$$

Since $|A_1| = |A_2|$, we get that for s, j, y, and x as above

$$(3.13)\quad \left| \sum_{i=0}^{i_0-1} \right| x+y+A_1|_{(j+i)s+G} - \sum_{i=0}^{i_0-1} |x+A_2|_{(j+i)s+G}| \leqq \epsilon\beta i_0 .$$

By (3.8) and (3.11)

$$\beta c(j,x) \leqq 2\chi^{(j+i)}(x+B)$$

and hence

$$(3.14)\quad \epsilon\beta i_0 c(j,x) \leqq 2\epsilon \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x+B).$$

From (3.12)–(3.14) we get that for $j \geqq j_0$, $y \in C$ and $x \in X$ with $|x - m_0 j| \leqq \delta j$

$$(3.15)\quad \left| \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x+y+A_1) - \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x+A_2) \right| \leqq 10\epsilon \sum_{i=0}^{i_0-1} \chi^{(j+i)}(x+B).$$

Replacing ϵ by $\epsilon/10$, we have (3.7) as desired.

Proposition 13. *Let G be $(d-1)$-dimensional and let χ have mean m_0 and satisfy Cramér's condition. Let A_1 and A_2 be relatively compact Borel subsets of G and such that $|A_1| = |A_2|$ and either χ satisfies Condition 1 or A_1 and A_2 are in $\mathfrak{A}$. Let $B \subseteq G$ be in $\mathfrak{A}$ and such that $|B|_G > 0$. Let $s_0 \in S$ and let C be a compact subset of G. Then for any $\epsilon > 0$ and integer i there exist $\delta > 0$ and positive integer j_0 such that for $j \geqq j_0$, $y \in C$, $x \in X$ with $|x - m_{0j}| \leqq \delta_j$*

$$(3.16)\quad |\chi^{(j+i)}(x+y+is_0+A_1) - \chi^{(j)}(x+A_2)| \leqq \epsilon\chi^{(j)}(x+B).$$

Proof. Proposition 13 follows easily from Proposition 10.

We now make a decomposition of φ similar to that used in Stone [10] and [11]. Note that

$$\int_{R^d} e^{-\alpha(|x|+1)}\varphi(dx)$$

decreases continuously from 1 to 0 as α increases from 0 to ∞. Thus we can find an α_0 such that

$$\int_{R^d} e^{-\alpha_0(|x|+1)}\varphi(dx) = \tfrac{1}{2}.$$

We now write $\varphi = (\psi + \chi)/2$, where

$$\chi(dx) = e^{-\alpha_0(|x|+1)}\varphi(dx)$$

and

$$\psi(dx) = (1 - e^{-\alpha_0(|x|+1)})\varphi(dx).$$

Then ψ and χ are both probability measures on R^d having support S. Also χ satisfies Cramér's condition. Let m_0 denote the mean of χ.

We have that

$$\varphi^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{2^k} \psi^{(k-j)} * \chi^{(j)}$$

and

$$\mu = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{2^k} \psi^{(k-j)} * \chi^{(j)}. \tag{3.17}$$

We will show next that parts of the right side of (3.17) are negligible in the limit and can safely be neglected.

For $\delta > 0$ let ${}_1\varphi_\delta^{(k)}$ be the measure defined by

$${}_1\varphi_\delta^{(k)}(A) = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{2^k} \int_{|y-jm_0|\geqq \delta j} \psi^{(k-j)}(A-y)\chi^{(j)}(dy).$$

Set

$${}_1\mu_\delta = \sum_{k=0}^{\infty} {}_1\varphi_\delta^{(k)}.$$

Lemma 1. *For* $\delta > 0$, ${}_1\mu_\delta(R^d) < \infty$.

Proof. By Proposition 10 we can choose $\epsilon > 0$ such that

$$\chi^{(j)}(\{y \mid |y - jm_0|\}) \geqq 2e^{-\epsilon j}.$$

Then

$$\begin{aligned} {}_1\varphi_\delta^{(k)}(R^d) &\leqq \sum_{j=0}^{k} \binom{k}{j} \frac{1}{2^k} 2e^{-\epsilon j} \\ &= 2\left(\frac{1+e^{-\epsilon}}{2}\right)^k \end{aligned}$$

and hence

$$ {}_1\mu_\delta(R^d) \leqq 2 \sum_{k=0}^{\infty} \left(\frac{1+e^{-\epsilon}}{2}\right)^k = \frac{4}{1-e^{-\epsilon}} < \infty . $$

Proof of Theorem 1. (*G d-dimensional*). We now complete the proof of Theorem 1 in the case that G is d-dimensional.

We can rearrange slightly the terms in (3.17), obtaining, for fixed $i_0 > 0$,

$$ \mu = \sum_{k=0}^{\infty} \psi^{(k)} * \sum_{j=0}^{\infty} \sum_{i=0}^{i_0-1} \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} \cdot 2^{-(k+i_0 j+i)} \chi^{(i_0 j+i)} . \tag{3.18} $$

It is easily seen that for fixed k

$$ \sum_{j=0}^{\infty} \sum_{i=0}^{i_0-1} \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} 2^{-(k+i_0 j+i)} < \infty $$

and that for any $\delta > 0$

$$ \sum_{k=0}^{\infty} \sum_{|k-i_0 j|>\delta k} \sum_{i=0}^{i_0-1} \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} 2^{-(k+i_0 j+i)} < \infty . $$

Set

$$ \begin{aligned} \mu_{\delta,k_0} = {} & \sum_{k=0}^{k_0-1} \psi^{(k)} * \sum_{j=0}^{\infty} \sum_{i=0}^{i_0-1} \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} 2^{-(k+i_0 j+i)} \chi^{(i_0 j+i)} \\ & + \sum_{k=k_0}^{\infty} \psi^{(k)} * \sum_{|k-i_0 j|>\delta k} \sum_{i=0}^{i_0-1} \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} 2^{-(k+i_0 j+i)} \chi^{(i_0 j+i)} . \end{aligned} $$

Then we obtain from the above observations

Lemma 2. *For* $0 \leqq k_0 < \infty$ *and* $\delta > 0$, $\mu_{\delta,k_0}(R^d) < \infty$.

Choose $0 < \epsilon < \frac{1}{2}$. Then there exist $k_0 > 0$ and $0 < \delta_0 < 1$ such that for $k \geqq k_0$, $|k - i_0 j| \leqq \delta_0 k$, and $0 \leqq i \leqq i_0 - 1$

$$ \begin{aligned} (1-\epsilon) \begin{bmatrix} k + i_0 j \\ i_0 j \end{bmatrix} 2^{-(k+i_0 j)} & \leqq \begin{bmatrix} k + i_0 j + i \\ i_0 j + i \end{bmatrix} 2^{-(k+i_0 j+i)} \\ & \leqq (1+\epsilon) \begin{bmatrix} k + i_0 j \\ i_0 j \end{bmatrix} 2^{-(k+i_0 j)} . \end{aligned} \tag{3.19} $$

Let A_1, A_2, B, and C be as in Proposition 12. Let δ, i_0, and j_0 be such that (3.7) holds, and $\delta \leqq \delta_0$ of the above paragraph. Let k_0 be such that $k_0(1 - \delta) \geqq j_0$.

Set

$$\mu_1(x, A) = \sum_{k=k_0}^{\infty} \sum_{|k-i_0j|\leqq\delta k} 2^{-(k+i_0j)} \binom{k+i_0j}{i_0j} \cdot \sum_{i=0}^{i_0-1} \int_{|x-z-i_0jm_0|\leqq\delta i_0j} \psi^{(k)}(dz)\chi^{(i_0j+i)}(x-z+A).$$

Then there is a measure μ_2 such that $\mu_2(R^d) < \infty$ and, for $x \in X$ and $y \in C$,

$$(3.20) \qquad (1-\epsilon)\mu(x, y+A_1) \leqq \mu(x+y+A_1) \leqq (1+\epsilon)\mu_1(x, y+A_1) + \mu_2(x+y+A_1),$$

$$(3.21) \qquad (1-\epsilon)\mu_1(x, A_2) \leqq \mu(x+A_2) \leqq (1+\epsilon)\mu_1(x, A_2) + \mu_2(x+A_2),$$

and

$$(3.22) \qquad (1-\epsilon)\mu_1(x, B) \leqq \mu(x+B) \leqq (1+\epsilon)\mu_1(x, B) + \mu_2(x+B).$$

From Proposition 12 we have for $x \in X$ and $y \in C$,

$$(3.23) \qquad |\mu_1(x, y+A_1) - \mu_1(x, A_2)| \leqq \epsilon\mu_1(x, B).$$

It follows from (3.20)–(3.23) that for $x \in X$ and $y \in C$,

$$(3.24) \qquad |\mu(x+y+A_1) - \mu(x+A_2)| \leqq \mu_2(x+y+A_1) + \mu_2(x+A_2) + 2\epsilon(\mu(x+y+A_1) + \mu(x+A_2) + \mu(x+B)).$$

In other words there is a compact subset D of X such that, for $x \in X$

$$(3.25) \qquad \sup_{y \in C} |\mu(x+y+A_1) - \mu(x+A_2)| \leqq 2\mu_2(x+D) + 6\epsilon\mu(x+D).$$

We summarize what we have obtained for far in

Proposition 14. *Let G be d-dimensional. Let A_1 and A_2 be relatively compact Borel subsets of X such that $|A_1| = |A_2|$ and either Condition 1 holds or A_1 and A_2 are in $\mathfrak{A}^G \cap \mathfrak{A}$. Let C be a compact subset of X. Then there is a compact subset D of X having the property that for any $\epsilon > 0$ there is a finite measure μ_2 on X such that for $x \in X$*

$$(3.26) \qquad \sup_{y \in C} |\mu(x+y+A_1) - \mu(x+A_2)| \leqq \mu_2(x+D) + \epsilon\mu(x+D).$$

It now follows from the one-dimensional renewal theorem, Theorem 1 of Stone [13], and Proposition 14 that Theorem 1 is true if G is d-dimensional and $\kappa < \infty$.

If $\kappa = \infty$ and the random walk corresponding to φ is recurrent then it follows that $\mu(A) = \infty$ for all $A \in \mathcal{A}^*$ with $|A| > 0$ (see Stone [9] for related results). Theorem 1 now follows immediately.

Suppose now that $\kappa = \infty$, but that $\mu(A) < \infty$ for all compact subsets A. Let $|A_1| > 0$. Then A_2 can be taken to have a non-empty interior. From (3.20)–(3.23) there is a finite measure μ_2 on X such that, for $x \in X$ and $y \in C$

$$\mu(x + y + A_1) \geqq \mu(x + A_2)/2 - \epsilon\mu(x + B) - \mu_2(x + A_2). \tag{3.27}$$

We can find a finite number of disjoint A_2's $\in \mathcal{A}^G \cap \mathcal{A}$ with $|A_2| = |A_1|$ whose union contains B and also contains Δ. Thus we get

Proposition 15. *Let G be d-dimensional. Let A be a relatively compact Borel subset of X such that $|A| > 0$ and either Condition 1 holds, or $A \in \mathcal{A}^G \cap \mathcal{A}$. Let C be a compact subset of X. Then there is a constant $c > 0$, a finite measure μ_2 on X and a compact subset D of X such that*

$$\inf_{y \in C} \mu(x + y + A) \geqq c\mu(x + \Delta) - \mu_2(x + D), \qquad x \in X. \tag{3.28}$$

Suppose again that $\kappa = \infty$. Then for $T \in \mathfrak{Z}_v$ and sufficiently large

$$\sum_{n \in T \cap Z^d} \mu(x + \Delta) = \infty. \tag{3.29}$$

From (3.27) and (3.28) we get

$$\sum_{n \in T \cap Z^d} \inf_{y \in C} \mu(n + y + A) = \infty.$$

Theorem 1 (iii) now follows from Theorem 3 (ii) of Stone [13].

Proof of Theorem 1. $G(d - 1)$-dimensional. We complete the proof of Theorem 1 by showing that the theorem is valid if G is $(d - 1)$-dimensional.

For positive integer k_0 and $\delta > 0$ let ${}_1\mu_{\delta, k_0}$ denote the measure defined by

$${}_1\mu_{\delta, k_0} = \sum_{k=0}^{k_0 - 1} \sum_{j=0}^{k} \binom{k}{j} 2^{-k} \psi^{(k-j)} * \chi^{(j)} + \sum_{k=k_0}^{\infty} \sum_{|k-2j| > \delta j} \binom{k}{j} \psi^{(k-j)} * \chi^{(j)}.$$

Then by arguments similar to those of Lemma 2, we get

Lemma 3. *For* $0 < k_0 < \infty$ $\delta > 0$, ${}_1\mu_{\delta, k_0}(R^d) < \infty$.

Let i be a fixed integer. Choose $0 < \epsilon < \frac{1}{2}$. Then there exist positive integers k_0 and $0 < \delta_0 < 1$ such that for $k \geqq k_0$ and $|k - 2j| \leqq \delta_0 k$

$$(1 + \epsilon)\binom{k}{j} 2^{-k} \leqq \binom{k + i}{j + i} 2^{-(k+i)} \leqq (1 + \epsilon)\binom{k}{j} 2^{-k}. \tag{3.30}$$

Let A_1, A_2, B, s_0, and C be as in Proposition 13. Let δ and j_0 be such that (3.16) holds and $\delta \leqq \delta_0$ of the above paragraph. Let k_0 be such that $k_0(1 - \delta) \geqq 2$. Set

$$\mu_2(x, A) = \sum_{k=k_0}^{\infty} \sum_{|k-2j| \leq \delta k} \binom{k}{j} 2^{-k} \cdot \int_{|x-z-jm_0| \leq \delta j} \psi^{(k-j)}(dz) \chi^{(j)}(x - z + A)$$

and

$$\mu_3(x, A) = \sum_{k=k_0}^{\infty} \sum_{|k-2j| \leq \delta k} \binom{k}{j} 2^{-k} \int_{|x-z-jm_0| \leq \delta j} \psi^{(k-j)}(dz)\chi^{(i+j)}(x - z + A).$$

$$\int_B \varphi(A - y)\, dy = \int_A \varphi(y - B)\, dy.$$

Then there is a measure μ_4 such that $\mu_4(R^d) < \infty$ and, for $x \in X$ and $y \in C$,

$$(3.31) \qquad (1 - \epsilon)\mu_3(x, y + is_0 + A_1) \leqq \mu(x + y + is_0 + A_1)$$
$$\leqq (1 + \epsilon)\mu_3(x, y + is_0 + A_1) + \mu_4(x + y + is_0 + A_1),$$

$$(3.32) \qquad \mu_2(x, A_2) \leqq \mu(x + A_2) \leqq \mu_2(x, A_2) + \mu_4(x + A_2),$$

and

$$(3.33) \qquad \mu_2(x, B) \leqq \mu(x + B) \leqq \mu_2(x, B) + \mu_4(x + B).$$

From Proposition 13 we have, for $x \in X$ and $y \in C$

$$(3.34) \qquad |\mu_3(x, y + is_0 + A_1) - \mu_2(x, A_2)| \leqq \epsilon\mu_2(x, B).$$

It follows from (3.31)–(3.34) that, for $x \in X$ and $y \in C$,

$$(3.35) \qquad |\mu(x + y + is_0 + A_1) - \mu(x + A_2)| \leqq \mu_4(x + y + is_0 + A_1)$$
$$+ \mu_4(x + A_2) + 2\epsilon(\mu(x + B) + \mu(x + y + is_0 + A_1)).$$

It also follows from (3.31)–(3.34) that, for $x \in X$ and $y \in C$,

$$(3.36) \qquad \mu(x + y + is_0 + A_1) \geqq \tfrac{1}{2}\mu(x + A_2) - \epsilon\mu(x + B) - \mu_4(x + A_2).$$

The remainder of the proof of Theorem 1 in the $(d - 1)$-dimensional case is similar to that in the d-dimensional case, using (3.35) and (3.36) instead of (3.24) and (3.27)

4. Proof of Theorems 2–6. Theorem 2 follows immediately from Theorems 2 and 4 of Stone [13].

In Theorems 3–6, $\kappa < \infty$; if $\kappa = 0$ the results follow trivially from Theorem 1. Thus throughout the rest of the paper it will be assumed that $0 < \kappa < \infty$.

Let $X^+ = \{x \in X \mid x \cdot v \geqq 0\}$.

Let S_n, $n \geqq 0$, be a random walk on X with $S_0 = 0$ and transition distribution φ. Let Z_n be the corresponding ladder random walk defined by $Z_n = S_{\tau_n}$, where $\tau_0 = 0$ and

$$\tau_n = \min \{k \mid S_k \cdot v > S_{\tau_{n-1}} \cdot v\}, \qquad n \geqq 1.$$

Then Z_n is also a random Walk on X. By definition $E(S_n \cdot v) = n\kappa^{-1}$.

Proposition 16. *X is the smallest closed subgroup of R^d containing the support of the distribution of Z_1.*

Proof. Let Y denote the smallest closed subgroup of R^d containing the support of the distribution of Z_1. There is an $s_0 \in S$ with $s_0 \cdot v > 0$. Clearly $s_0 \in Y$. Let $s \in S$. Then $s + ns_0 \in Y$ for some $n \geqq 0$. Therefore $s \in Y$. Consequently $Y = X$, as desired.

From the one-dimensional renewal theorem we obtain immediately

Proposition 17. $0 < E(Z_1 \cdot v) < \infty$.

Proposition 18. *If $x \in X$, $y \in x + X^+$, $z \in X$, and $z \notin x + X^+$, then $z \cdot v < y \cdot v$.*

Proof. Since $z - x \in X$ and $z - x \notin X^+$, it follows that $(z - x) \cdot v < 0$. Since $y - x \in X^+$, it follows that $(y - x) \cdot v \geqq 0$. Therefore

$$y \cdot v \geqq x \cdot v > z \cdot v,$$

as desired.

Proposition 19. *For $x \in X$ the processes S_n, $n \geqq 0$, and Z_n, $n \geqq 0$, hit $x + X^+$ first in the same place.*

Proof. We can assume $x \cdot v > 0$. If $S_{n_0} \in x + X^+$ and $S_k \notin x + X^+$ for $0 \leqq k < n_0$, then $S_k \cdot v < S_{n_0} \cdot v$ for $0 \leqq k < n_0$, by Proposition 18. Thus S_{n_0} is equal to one of the Z_n's, and in fact to the first one in $x + X^+$.

Let φ_L denote the distribution of Z_1 and let μ_L denote the corresponding renewal measure. Set $X^- = \{x \in X \mid x \cdot v < 0\}$.

Proposition 20. *For any Borel sets A and B in X*

$$\int_B \varphi(A - y)\, dy = \int_A \varphi(y - B)\, dy.$$

Proof. To obtain this special case of the duality of $\varphi(dz)$ and $\varphi(-dz)$, we note that

$$\begin{aligned}\int_B \varphi(A - y)\, dy &= \int 1_B(y)\, dy \int 1_A(y + z)\varphi(dz) \\ &= \int \varphi(dz) \int 1_B(y - z) 1_A(y)\, dy \\ &= \int_A \varphi(y - B)\, dy.\end{aligned}$$

Proposition 21. *If $A \in \mathfrak{A}$, then $\varphi(x + A)$, $x \in X$, is continuous on x except on a set of Haar measure zero.*

Proof. The function is continuous at x if $\varphi(x + \partial A) = 0$. By Proposition 20

$$\int_X \varphi(\partial A - y)\, dy = \int_{\partial A} \varphi(y - X)\, dy = 0,$$

since $|\partial A| = 0$ for $A \in \mathfrak{A}$. Thus $\varphi(x + \partial A) = 0$ except for x in a set of Haar measure zero.

For some purposes the hitting times

$$\sigma_B = \min [n \geqq 0 \mid X_n \in B]$$

are not convenient, so let

$$\tau_B = \min [n \geqq 1 \mid X_n \in B].$$

Set

$$G(x, A) = E_x \sum_1^\infty 1_A(X_n),$$

$$\tilde{G}(x, A) = \tilde{E}_x \sum_1^\infty 1_A(X_n),$$

$$G_B(x, A) = E_x \sum_1^{\tau_B} 1_A(X_n),$$

and

$$\tilde{G}_B(x, A) = \tilde{E}_x \sum_1^{\tau_B} 1_A(X_n).$$

Let $G_{L,B}(x, A)$ and $\tilde{G}_{L,B}(x, A)$ be defined similarly in terms of the random walk with transition distribution φ_L. Then

$$L_B(x) = P_x(\tau_B = \infty) = (1 - G_B(x, B))$$

and

$$\tilde{L}_B(x) = 1 - \tilde{G}_B(x, B).$$

Let $L_{L,B}(x)$ and $\tilde{L}_{L,B}(x)$ correspond to the random walk with transition distribution φ_L. We have the well-known dual relations

$$\int_C G_B(x, A)\, dx = \int_A \tilde{G}_B(x, C)\, dx$$

and

$$\int_C G_{L,B}(x, A)\, dx = \int_C \tilde{G}_{L,B}(x, C)\, dx,$$

valid for Borel sets A, B, and C. Somewhat more generally, we have

$$\int_C f(x) \int G_B(x, dy) g(y) = \int_A g(x) \int_C \tilde{G}_B(x, dy) f(y),$$

valid for Borel sets A, B, and C and non-negative Borel functions f and g.

Proposition 22. *Let A, B, C be Borel sets such that $A \subseteq C$, $B \subseteq C$, and*

$P_x(X_n \in C$ *for infinitely many values of* $n) = 0, x \in X$. *Then*

$$\int_A \tilde{L}_B(x)\,dx = \int_A \tilde{L}_C(x)\,dx + \int_{C\cap B^c} \tilde{L}_C(x) G_B(x, A).$$

Proof. For $x \in X$

$$\tilde{L}_B(x) - \tilde{L}_C(x) = \tilde{P}_x(\tau_B = \infty, \tau_C < \infty)$$

$$= \sum_{n=1}^{\infty} \tilde{P}_x$$

$$(\tau_B = \infty \text{ and last visit to } C \text{ takes place at time } n)$$

$$= \int_{C\cap B^c} \tilde{G}_B(x, dy)\tilde{L}_C(y).$$

Therefore

$$\int_A \tilde{L}_B(x)\,dx - \int_A \tilde{L}_C(x)\,dx = \int_A dx \int_{C\cap B^c} \tilde{G}_B(x, dy)\tilde{L}_C(y)$$

$$= \int_{C\cap B^c} \tilde{L}_C(x) G_B(x, A),$$

from which the proposition follows immediately.

Proposition 23. *Assume* $d = 1$, $A \in \mathfrak{A}$, B *a Borel set bounded from above. Then*

$$\lim_{y\to\infty} G_B(x, y + A) = \kappa L_B(x)\,|A|.$$

Proof. Let $x \in X$. Then

$$G_B(x, y + A) = G(x, y + A) - \int_B G_B(x, dz) G(z, y + A).$$

By the renewal theorem

$$\lim_{y\to\infty} G_B(x, y + A) = \kappa\,|A|\,(1 - G_B(x, B)) = \kappa L_B(x)\,|A|.$$

The next proposition was obtained by Port [7] through the use of several identities from fluctuation theory.

Proposition 24. (Port). *If* $d = 1$, *then*

$$\kappa_L \tilde{L}_{L,[0,\infty)}(x) = \kappa \tilde{L}_{[0,\infty)}(x), \qquad 0 \leqq x < \infty.$$

Proof. Observe first that if A *is a Borel subset of* $[0, \infty)$, *then*

$$G_{[0,\infty)}(x, A) = G_{L,[0,\infty)}(x, A), \qquad -\infty < x < 0.$$

Choose $A \in \mathfrak{A}$ and $C \in \mathfrak{A}$ such that $|C| > 0$, $A \subseteq [0, \infty)$, and $C \subseteq [-\infty, 0]$. Then

$$\int_{y+C} G_{[0,\infty)}(x, A)\,dx = \int_{y+C} G_{L,[0,\infty)}(x, A)\,dx, \qquad y < 0,$$

and hence

$$\int_A \tilde{G}_{[0,\infty)}(x, y + C)\, dx = \int_A \tilde{G}_{L,[0,\infty)}(x, y + C)\, dx, \qquad y < 0.$$

Letting $y \to -\infty$ and using Proposition 23, we get

$$\kappa\, |C| \int_A \tilde{L}_{[0,\infty)}(x)\, dx = \kappa_L\, |C| \int_A \tilde{L}_{L,[0,\infty)}(x)\, dx.$$

Therefore

$$\kappa \tilde{L}_{[0,\infty)}(x) = \kappa_L \tilde{L}_{L,[0,\infty)}(x)$$

for almost all $x \geqq 0$. But $\tilde{L}_{[0,\infty)}(x)$ and $L_{L,[0,\infty)}(x)$ are non-increasing in x and, in fact, right continuous. Proposition 23 now follows immediately.

Proposition 25. *In general*

$$\kappa \tilde{L}_{X^+}(x) = \kappa_L \tilde{L}_{L,X^+}(x), \qquad x \in X^+.$$

Proof. This result reduces immediately to the previous result.

Proposition 26. *If $\nu \in \mathfrak{N}_{v,\lambda}$ and $A \in \mathfrak{B}$, then*

$$(\nu * \mu)(x + A), \qquad x \in X,$$

is bounded.

Proof. We can write

$$(\nu * \mu)(x + A) = \int \nu(x - y + A)\mu(dy).$$

Now $\nu(x - y + A)$ is bounded by some finite number M and vanishes for $y \notin x + T$, for some $T \in \mathfrak{J}_v$. Thus

$$(\nu * \mu)(x + A) \leqq M\mu(x + T).$$

But $\mu(x + T)$ is bounded by the one-dimensional renewal theorem.

Proposition 27. *If $\nu \in \mathfrak{N}_{v,\lambda}$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, then*

$$\sum_n \sup_{x \in X} \int_{n+x+B} (\nu * \mu_L)(dy)\varphi_L(x - y + A) < \infty.$$

Proof. Let C be a compact set containing $A - B$. Then

$$\varphi_L(x - y + A) \leqq \varphi_L(C - n), \qquad y \in n + x + B.$$

By Proposition 26 applied to μ_L instead of μ there is an $M < \infty$ such that $(\nu * \mu_L)(n + x + B) \leqq M$, $n \in Z^d$ and $x \in X$. Then

$$\sum_n \sup_{x \in X} \int_{n+x+B} (\nu * \mu_L)(dy)\varphi_L(x - y + A) \leqq M \sum_n \varphi_L(C - n) < \infty.$$

Proposition 28. *If $\nu \in \mathfrak{N}_{v,\lambda}$, $A \in \mathfrak{A}^*$, and $n \in Z^d$, then*

$$\lim_{x\cdot v\to\infty} \int_{(n+x+\Delta)\cap(x+X^-)} (\nu * \mu_L)(dy)\varphi_L(x - y + A) = \lambda\kappa_L \int_{(n+\Delta)\cap X^-} dy\varphi_L(A - y).$$

Proof. Clearly $|\partial((n + \Delta \cap X^-)| = 0$. The function $\varphi_L(x + A)$, $x \in X$, is bounded and, by Proposition 21 applied to φ_L instead of φ, is continuous almost everywhere. Proposition 28 now follows from Theorem 1 applied to μ_L.

Proposition 29. *If $\nu \in \mathfrak{N}_{v,\lambda}$ and $A \in \mathfrak{A}^*$, then*

$$\lim_{x\cdot v\to\infty} \int_{x+X^-} (\nu * \mu_L)(dy)\varphi_L(x + A - y) = \lambda\kappa_L \int_{X^-} \varphi_L(A - y)\, dy.$$

Proof. This proposition follows immediately from Propositions 27 and 28 and the dominated convergence theorem.

Proposition 30. *If $\nu \in \mathfrak{N}_{v,\lambda}$ and $A \in \mathfrak{A}^*$, then*

$$\lim_{x\cdot v\to\infty} \int_{x+X^-} (\nu * \mu)(dy)\varphi_L(x + A - y) = \lambda\kappa \int_A \tilde{L}_{X^+}(x)\, dx.$$

Proof. Note first that $\varphi_L(x - X^-) = \tilde{L}_{L,X^+}(x)$. Thus by Propositions 20 and 25

$$\lambda\kappa_L \int_{X^-} \varphi_L(A - y)\, dy = \lambda\kappa_L \int_A \varphi_L(x - X^-)\, dx$$
$$= \lambda\kappa_L \int_A \tilde{L}_{L,X^+}(x) = \lambda\kappa \int_A \tilde{L}_{X^+}(x)\, dx,$$

and the desired result follows immediately from Proposition 29.

From Proposition 30 we obtain immediately as a special case of Theorem 3

Proposition 31. *If $\nu \in \mathfrak{N}_{v,\lambda}$ and $A \in \mathfrak{A}^*$, then*

$$\lim_{x\cdot v\to\infty} \int_X \nu(dy)E_y N_{x+X^+}(x + A) = \lambda\kappa \int \tilde{L}_{X^+}(y)\, dy.$$

Proposition 32. *If $\nu \in \mathfrak{N}_{v,\lambda}$, $Y \in \mathcal{Y}$, and $X^+ \in \mathfrak{D}_Y$, then*

$$\lim_{x\cdot v\to\infty} \int_X \nu(dy)E_y Y_x = \lambda\kappa \int_{X^+} \tilde{L}_{X^+}(y)E_y Y.$$

Proof. For $x \in X$ define the measure ν_x by

$$\nu_x(A) = \int_X \nu(dy)E_y N_{x+X^+}(x + A).$$

Then

$$\int_X \nu(dy)E_y Y_x = \int_X \nu_x(dy)E_y Y.$$

Let $A \varepsilon \mathfrak{A}$ be such that $A \subseteq X^+$ and $Y \leqq N_\phi(A)$. Then $E_y Y \leqq E_y N_\phi(A)$. Also

$$\int_X \nu_x(dy) E_y N_\phi(A) = (\nu * \mu)(x + A)$$

and hence by Theorem 1

$$\lim_{x \cdot v \to \infty} \int_X \nu_x(dy) E_y N_\phi(A) = \lambda\kappa \, |A|.$$

It follows from Proposition 22 that

$$\kappa\lambda \, |A| = \kappa\lambda \left[\int_A \tilde{L}_{X^+}(y) \, dy + \int_{X^+} \tilde{L}_{X^+}(y) G(y, A) \right] = \kappa\lambda \int_X \tilde{L}_{X^+}(y) E_y N_\phi(A).$$

Consequently

$$\lim_{x \cdot v \to \infty} \int_X \nu_x(dy) E_y N_\phi(A) = \kappa\lambda \int_X \tilde{L}_{X^+}(y) E_y N_\phi(A).$$

Therefore by Proposition 31

$$\lim_{c \to \infty} \limsup_{x \cdot v \to \infty} \int_{|y| \geqq c} \nu_x(dy) E_y N_\phi(A) = 0.$$

Therefore we have the main point of this paragraph, namely that

$$\lim_{c \to \infty} \limsup_{x \cdot v \to \infty} \int_{|y| \geqq c} \nu_x(dy) E_y Y = 0.$$

The conclusion of Proposition 32 now follows immediately from Proposition 31.

Proof of Theorem 3. Let $\nu \varepsilon \mathfrak{N}_{v,\lambda}$, $Y \varepsilon \mathcal{Y}$, and $D \varepsilon \mathfrak{D}_Y$. With no loss of generality, we can assume that $X^+ \varepsilon \mathfrak{D}_Y$ and $D \subseteq X^+$. Then

$$\int_{X^+} \tilde{L}_{X^+}(y) E_y Y \, dy = \int_D \tilde{L}_{X^+}(y) E_y Y \, dy + \int_{X^+ \cap D^c} \tilde{L}_{X^+}(y) \, dy \int_D G_D(y, dz) E_z Y$$

$$= \int_D \tilde{L}_{X^+}(y) E_y Y \, dy + \int_D E_y Y \, dy \int_{X^+ \cap D^c} \tilde{G}_D(y, dz) \tilde{L}_{X^+}(z).$$

From the proof of Proposition 22, we see that

$$\tilde{L}_D(y) = \tilde{L}_{X^+}(y) + \int_{X^+ \cap D^c} \tilde{G}_D(y, dz) \tilde{L}_{X^+}(z).$$

Therefore

$$\int_{X^+} \tilde{L}_{X^+}(y) E_y Y \, dy = \int_D \tilde{L}_D(y) E_y Y \, dy.$$

Theorem 3 now follows immediately from Proposition 32.

Proposition 33. *If $EN \varepsilon \mathfrak{N}_v$ and* (1.9) *holds, $A \varepsilon \mathfrak{A}^*$ and $n \varepsilon Z^d$, then*

$$\lim_{x\cdot v\to\infty} E \left| \int_{(n+x+\Delta)\cap(x+X^-)} (N * \mu_L)(dy)\varphi_L(x - y + A) - \lambda\kappa_L \int_{(n+\Delta)\cap X^-} dy\varphi_L(A - Y) \right| = 0.$$

Proof. This proposition follows directly from Theorem 2 and Proposition 28.

Proposition 34. If $EN \varepsilon \mathfrak{N}$, and (1.9) *holds and if $A \varepsilon \mathfrak{A}^*$, then*

$$\lim_{x\cdot v\to\infty} E \left| \int_X N(dy)E_v N_{x+X^+}(x + A) - \lambda\kappa \int_A \tilde{L}_{X^+}(y)\, dy \right| = 0.$$

Proof. The proof of this proposition is similar to that of Proposition 31, except that Proposition 33 is used instead of Proposition 28.

Proposition 35. If $EN \varepsilon \mathfrak{N}$, and (1.9) *holds, $Y \varepsilon \mathcal{Y}$ and $X^+ \varepsilon \mathfrak{D}_Y$, then*

$$\lim_{x\cdot v\to\infty} E \left| \int_X N(dy)E_v Y_x - \lambda\kappa \int_{X^+} \tilde{L}_{X^+}(y)E_v Y \right| = 0.$$

Proof. The proof is similar to that of Proposition 32, except that Proposition 34 is used in place of Proposition 31.

Proof of Theorem 4. Theorem 4 follows immediately from Proposition 35 and the formula

$$\int_{X^+} \tilde{L}_{X^+}(y)E_v Y\, dy = \int_D \tilde{L}_D(y)E_v Y\, dy$$

obtained in the proof of Theorem 3.

Proposition 36. Let $EN \varepsilon \mathfrak{N}$, and (1.9) *hold, $Y \varepsilon \mathcal{Y}_1$ and $D \varepsilon \mathfrak{D}_Y$.*

(*i*) *As $x\cdot v \to \infty$, Y_x has an asymptotic distribution which is Poisson with mean*

$$\lim_{x\cdot v\to\infty} EV_{Y_x} = \lambda\kappa \int_D \tilde{L}_D(y)P_v(Y = 1).$$

(*ii*) *If $Y_1, \cdots, Y_k$ are in Y_1 and the product of any two of them is identically zero, then*

$$V_{(Y_1)_x}, \cdots, V_{(Y_k)_x}$$

are asymptotically independent as $x\cdot v \to \infty$.

Proof. This proposition follows easily from Theorem 4 and well known elementary results about convergence to the Poisson distribution.

Proof of Theorem 5. Let $Y \varepsilon \mathcal{Y}$. It is easily seen that except for countably many values of a

$$P_x(Y = a) = 0$$

except for values of x lying in a set of measure zero. Choose $-\infty < a < b < \infty$

such that

$$P_x(Y = a) = P_x(Y = b) = 0$$

except for values of x lying in a set of measure zero. Let

$$Y' = 1 \quad \text{if} \quad a \leqq Y < b,$$

$$Y' = 0 \quad \text{if} \quad Y < a \quad \text{or} \quad Y \geqq b.$$

Then $Y' \varepsilon \mathcal{Y}_1$. Theorem 5 now follows easily from Proposition 36.

Proposition 37. Let $A \varepsilon \mathcal{A}^*$ *and* $B \varepsilon \mathfrak{D}_v$. *Then*

$$\int_{A\cup B} \tilde{L}_{A\cup B}(y) E_y N_B(A)\, dy = \int_A \tilde{L}_B(y)\, dy.$$

Proof. By Proposition 22

$$\int_{A\cup B} \tilde{L}_{A\cup B}(y) E_y N_B(A)\, dy = \int_A \tilde{L}_{A\cup B}(y)\, dy + \int_{A\cap B^c} \tilde{L}_{A\cup B}(y) G_B(y, A)$$
$$= \int_A \tilde{L}_B(y)\, dy,$$

as desired.

Proof of Theorem 6. Theorem 6 follows immediately from Theorem 5 and Proposition 37.

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