Infinite Prandtl Number Limit of Rayleigh-Bénard Convection

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Abstract

We rigorously justify the infinite Prandtl number model of convection as the limit of the Boussinesq approximation to the Rayleigh-Bénard convection as the Prandtl number approaches infinity. This is a singular limit problem involving an initial layer.

1 Introduction

In physical situation, these are many scenario that fluid phenomena involves heat transfer. What we shall consider here is the Rayleigh-Bénard setting of a horizontal layer of fluids confined by two parallel planes a distance h apart and heated at the bottom plane at temperature T_2 and cooled at the top plane at temperature T_1 ($T_2 > T_1$). Hot fluid at the bottom then rises while cool fluid on top sinks by gravity force. If the relative change of density is small with respect to a background density, we may ignore density variation in the system except a buoyancy force proportional to the local temperature in the momentum balance and we arrive at the so-called Boussinesq approximation of the Rayleigh-Bénard convection. The dynamic model consists of the heat advection-diffusion of the temperature coupled with the incompressible Navier-Stokes equations via a buoyancy force proportional to the temperature (Tritton 1988). Taking into account the effect of rotation, normalizing the background density to 1, we arrive at the so-called **Boussinesq system** in a rotating frame:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + 2\Omega\mathbf{k}\times\mathbf{u} = \nu\Delta\mathbf{u} + g\alpha\mathbf{k}T, \quad \nabla\cdot\mathbf{u} = 0, \tag{1.1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\Delta T, \tag{1.2}$$

$$\mathbf{u}|_{z=0,h} = 0, \tag{1.3}$$

$$T|_{z=0} = T_2, \quad T|_{z=h} = T_1, \tag{1.4}$$

where $\mathbf{u}$ is the velocity field of the fluid , p is the pressure, Ω is the rotation rate, $\mathbf{k}$ is the unit upward vector, ν is the kinematic viscosity, α is the thermal expansion coefficient, T is the temperature field of the fluid, and

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κ is the thermal diffusive coefficient. We also impose periodic boundary condition in the horizontal directions for simplicity.

This set of equations is closely related to the rotating Boussinesq equations in geophysical fluid dynamics, with the temperature field replaced by the negative density, (See Gill 1982, Pedlosky 1987 and Salmon 1998) and suitable changes of boundary conditions appropriate to the atmosphere or ocean. However the physics is very different. For instance the buoyancy force plays a much more important role in Rayleigh-Bénard convection than in geophysical problems.

This set of equations is much more complex than the Navier-Stokes equations. For one thing, the dynamic similarity of the Navier-Stokes flows has only one parameter, namely the Reynolds number. On the other hand, for the Boussinesq approximation of Rayleigh-Bénard convection, the dynamic similarity requires two parameters, namely the Grashoff number ($Gr = g\alpha(T_2 - T_1)h^3/\nu^2$) and the Prandtl number ($Pr = \nu/\kappa$) (Tritton 1988) in the absence of rotation.

Since we are interested in convection, i.e., motion of the fluid induced by buoyancy, the standard/natural non-dimensional form of the system is achieved by using the units of the layer depth h as the typical length scale, the thermal diffusion time h^2/κ as the typical time, the ratio of typical length over typical time, i.e., κ/h as typical velocity, the temperature on a scale where the top plane is kept at 0 and the bottom plane kept at 1. The **non-dimensional form of the Boussinesq equations** then take the form

$$\frac{1}{Pr}(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p + \frac{1}{Ek}\mathbf{k}\times\mathbf{u} = \Delta\mathbf{u} + Ra\,\mathbf{k}T, \quad \nabla\cdot\mathbf{u} = 0, \tag{1.5}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \Delta T, \tag{1.6}$$

$$\mathbf{u}|_{z=0,1} = 0, \tag{1.7}$$

$$T|_{z=0} = 1, \quad T|_{z=1} = 0. \tag{1.8}$$

The parameters of the system are thus absorbed into the geometry of the domain plus three adimensional numbers: the Rayleigh number

$$Ra = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa} \tag{1.9}$$

measuring the ratio of overall buoyancy force to the damping coefficients; the Ekman number

$$Ek = \frac{\nu}{2\Omega h^2} \tag{1.10}$$

measuring the relative importance of viscosity over rotation; and the Prandtl number

$$Pr = \nu/\kappa \tag{1.11}$$

measuring the relative importance of kinematic viscosity over thermal diffusivity.

As we stated earlier, this is a very complicated system and hence simplifications are highly desirable. Physically simpler situations are achieved if we consider large Ekman number or large Prandtl number limit. In the large Ekman number case which is equivalent to strong rotation, the physics is simpler due to the so-called Taylor-Proudman type phenomena where the fluid flow is basically horizontal and hence inhibits heat convection. Rapidly rotating fluids have been the subject of recent intensive study in the mathematical community (see for instance the lecture notes by Majda (2002) and the references therein) and we will refrain from this subject except pointing out that this problem is more difficult due to the presence of solid boundary (see Masmoudi 2000 and the reference therein).

Another physically simpler case is when the Prandtl number is much bigger than one which is the case for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure (Bodenschatz, Pesch and Ahlers 2000, Busse 1989, Chandrasekhar 1961, Grossmann and Lohse 2000, Tritton 1988). This means that the viscous time scale of the fluid (h^2/ν) is much shorter than the thermal diffusive time scale of h^2/κ. Since we have normalized the time to the thermal diffusive time scale, we expect that the velocity field has settled into some "equilibrium" state due to the long time viscosity effect. Hence we expect that the velocity field to be "slaved" by the temperature field. Moreover, since the typical velocity is set to κ/h, the Reynolds number is expected to be small. Thus we anticipate "creeping" flow and hence the nonlinear advection term is negligible. Therefore the velocity field should be linearly "slaved" by the temperature field. Indeed, formally setting the Prandtl number equal to infinity in the non-dimensional form of the dynamic equations (1.5-1.8) we arrive at the so-called **infinite Prandtl number convection system** of the form

$$\frac{1}{Ek}\mathbf{k} \times \mathbf{u}^0 + \nabla p^0 = \Delta \mathbf{u}^0 + Ra\,\mathbf{k}T^0, \quad \nabla \cdot \mathbf{u}^0 = 0, \tag{1.12}$$

$$\frac{\partial T^0}{\partial t} + \mathbf{u}^0 \cdot \nabla T^0 = \Delta T^0, \tag{1.13}$$

$$\mathbf{u}^0|_{z=0,1} = 0, \tag{1.14}$$

$$T^0|_{z=0} = 1, \quad T^0|_{z=1} = 0. \tag{1.15}$$

The fact that the velocity field is linearly "slaved" by the temperature field has been exploited in several recent very interesting works on rigorous estimates on the rate of heat convection in this infinite Prandtl number setting (see Doering and Constantin 2001, Constantin and Doering 1999, Constantin-Hallstrom-Poutkaradze 2001 and the references therein, as well as the work of Busse 1989).

A natural question to ask then is if such an approximation is valid, i.e.

$$(\mathbf{u}, T) \to (\mathbf{u}^0, T^0) \quad \text{as } Pr \to \infty? \tag{1.16}$$

$$((\mathbf{u}_0, T_0) \to (\mathbf{u}_0^0, T_0^0))$$

In order to understand the problem, we consider a special case with the following type of initial data and ansatz for solutions

$$\begin{aligned} T_0 &= 1 - z, \\ \mathbf{u}_0 &= (u_{01}(z), 0, 0), \\ \mathbf{u} &= (u_1(z,t), 0, 0). \end{aligned}$$

The Boussinesq equations then reduce to

$$\begin{aligned} \frac{1}{Pr}\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial z^2}, \\ u_2 \equiv u_3 &\equiv 0, \\ T(t) &\equiv 1 - z \end{aligned}$$

whose solutions can be derived explicitly as

$$u_1(t,z) = e^{Pr \cdot t \frac{\partial^2}{\partial z^2}} u_{01}(z).$$

On the other hand, the infinite Prandtl number model reduces to an ODE with solutions given by

$$\begin{aligned} T^0 &\equiv 1 - z, \\ \mathbf{u}^0 &\equiv 0. \end{aligned}$$

We then observe that the limit as the Prandtl number approaches infinity is a *singular* one involving an initial layer.

The study of such a singular perturbation problems suffers another setback: the possible singularity of the solutions to the Boussinesq system. Such a system embodies the Navier-Stokes system and hence the regularity of its solution in finite time in this three dimensional setting is not known (see for instance Constantin and Foias 1988, Temam 2000, Majda and Bertozzi 2001). Thus the convergence that we can expect must be in some weak averaged form. As in most singular perturbation problems, we benefit from the fact that the limit system, i.e., the infinite Prandtl number model, has much more regular behavior as we can expect from the linear slavery of the velocity field by the temperature field. Nevertheless, the dynamics of the limit system is still highly nontrivial which is evident from numerical simulations and from analytical results such as a bifurcation analysis to be presented somewhere else.

The rest of the article is organized as follows. In the next section we derive the effective dynamics of the Boussinesq system at large Prandtl number utilizing two time scale approach. We then show that the effective dynamics is nothing but the infinite Prandtl number dynamics plus initial layer and lower order terms. In the fourth section we rigorously justify the infinite Prandtl number limit on a finite time interval. In the last section we provide concluding remarks and comments on the results proved here and other related topics.

2 Derivation of the Effective Dynamics

In this section we derive the simplified effective dynamics for the Boussinesq system at large Prandtl number. We then show in the next section that the effective dynamics is directly related to the infinite Prandtl number model modulo an initial layer and lower order terms.

In order to derive the effective dynamics for the Boussinesq system at large Prandtl number, we recognize that the large Prandtl number problem is really a problem involving two time scales, namely, the fast viscous time scale of $\frac{1}{Pr}$ ($\frac{h^2}{\nu}$ before non-dimensionalization) and the slow thermal diffusive time scale of 1 ($\frac{h^2}{\kappa}$ before non-dimensionalization). This suggests that we should take a two time scale approach (see for instance Holmes 1995, Majda 2002 among others) and introduce the fast time scale

$$\tau = Pr \cdot t = \frac{t}{\varepsilon}, \quad \text{with } \varepsilon = \frac{1}{Pr}, \tag{2.1}$$

and replace the time derivative with

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau} \tag{2.2}$$

and postulate the following formal asymptotic expansion

$$\mathbf{u} = \mathbf{u}^{(0)}(t,\tau) + \varepsilon \mathbf{u}^{(1)}(t,\tau) + h.o.t., \tag{2.3}$$

$$T = T^{(0)}(t,\tau) + \varepsilon T^{(1)}(t,\tau) + h.o.t. \tag{2.4}$$

where $h.o.t$ represents higher order terms in ε.

We also impose the customary sub-linear growth condition

$$\lim_{\tau\to\infty} \frac{(\mathbf{u}^{(1)}(t,\tau), T^{(1)}(t,\tau))}{\tau} = 0 \tag{2.5}$$

in order to ensure the validity of the formal asymptotic expansion for large values of the fast variable τ.

Inserting the formal asymptotic expansion into the Boussinesq equations (1.5-1.6), collecting the leading orders terms, we have

$$\frac{\partial \mathbf{u}^{(0)}}{\partial \tau} + \frac{1}{Ek}\mathbf{k}\times\mathbf{u}^{(0)} + \nabla p^{(0)} = \Delta\mathbf{u}^{(0)} + Ra\mathbf{k}T^{(0)}, \quad \nabla\cdot\mathbf{u}^{(0)} = 0, \tag{2.6}$$

$$\frac{\partial T^{(0)}}{\partial \tau} = 0. \tag{2.7}$$

The solutions can be represented as

$$\begin{aligned}\mathbf{u}^{(0)}(t,\tau) &= e^{-A\tau}\mathbf{u}^{(0)}(t,0) - Ra\, e^{-A\tau}A^{-1}(P(\mathbf{k}T^{(0)}(t))) \\ &\quad + Ra\, A^{-1}(P(\mathbf{k}T^{(0)}(t))),\end{aligned} \tag{2.8}$$

$$T^{(0)}(t,\tau) = T^{(0)}(t) \tag{2.9}$$

where P is the Leray-Hopf projector (see for instance Constantin and Foias 1988, Temam 2000 among others) and A is an elliptic operator defined as

$$A\mathbf{u} = \mathbf{f}, \tag{2.10}$$

if and only if the following holds

$$\frac{1}{Ek}\mathbf{k}\times\mathbf{u} + \nabla p = \Delta\mathbf{u} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \tag{2.11}$$

$$\mathbf{u}|_{z=0,1} = 0. \tag{2.12}$$

The next order dynamics is governed by

$$\frac{\partial \mathbf{u}^{(1)}}{\partial \tau} + A\mathbf{u}^{(1)} = -\frac{\partial \mathbf{u}^{(0)}}{\partial t} - P((\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}) + Ra\, P(\mathbf{k}T^{(1)}), \tag{2.13}$$

$$\frac{\partial T^{(1)}}{\partial \tau} = \Delta T^{(0)} - \frac{\partial T^{(0)}}{\partial t} - \mathbf{u}^{(0)}\cdot\nabla T^{(0)}. \tag{2.14}$$

We impose the sub-linear growth condition and we obtain, for the temperature field,

$$0 = \Delta T^{(0)} - \frac{\partial T^{(0)}}{\partial t} - \mathbf{u}^{(0)} \cdot \nabla T^{(0)} \tag{2.15}$$

which is basically the infinite Prandtl number model.

On the other hand, the sub-linear growth condition does not yield anything on the dynamics of $\mathbf{u}^{(0)}$ since the equation for $\mathbf{u}^{(1)}$ is dissipative. This seems to be a problem. Nevertheless, we observe that $\mathbf{u}^{(0)}$ has three terms, one term linearly slaved by the leading order temperature field $T^{(0)}$ and another two terms that decay exponentially in time (initial layer type). This means that no dynamics on $\mathbf{u}^{(0)}$ is necessary except the ones that we already have. Moreover, we may modify the initial layer terms in such a way so that the initial data are fixed. This only introduces lower order error. Thus we propose the following **effective dynamics**

$$\frac{\partial T^{(0)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla T^{(0)} = \Delta T^{(0)}, \tag{2.16}$$

$$\begin{aligned} \mathbf{u}^{(0)}(t) = {} & Ra\, A^{-1}(P(\mathbf{k}T^{(0)})) \\ & + e^{-A\tau}\mathbf{u}_0 - Ra\, e^{-A\tau} A^{-1}(P(\mathbf{k}T_0)), \end{aligned} \tag{2.17}$$

$$T^{(0)}|_{z=0} = 1, \quad T^{(0)}|_{z=1} = 0. \tag{2.18}$$

This is to be compared with the infinite Prandtl number model casted in a form with the operator A

$$\frac{\partial T^0}{\partial t} + \mathbf{u}^0 \cdot \nabla T^0 = \Delta T^0, \tag{2.19}$$

$$\mathbf{u}^0 = Ra\, A^{-1}(P(\mathbf{k}T^0)), \tag{2.20}$$

$$T^0|_{z=0} = 1, \quad T^0|_{z=1} = 0. \tag{2.21}$$

3 Effective Dynamics and the Infinite Prandtl Number Model

It is easy to see that the effective dynamics is closely related to the infinite Prandtl number model. Indeed, it is exactly the infinite Prandtl number dynamics if we neglect the initial layer corrections in the velocity field. Here we verify that the solutions to the effective dynamics and the solutions of the infinite Prandtl number model dynamics remain close on any fixed time interval modulo an initial layer. The proof here relies on the regularity of solutions of the infinite Prandtl number model which we shall briefly address later.

Let us denote the difference in temperature field between the effective dynamics and the infinite Prandtl number dynamics as w, i.e.,

$$w = T^{(0)} - T^{0}. \tag{3.1}$$

We then notice, after utilizing the effective dynamics (2.16-2.18) and the infinite Prandtl number dynamics (2.19-2.21)

$$\mathbf{u}^{(0)} - \mathbf{u}^{0} = RaA^{-1}(P(\mathbf{k}w)) + e^{-A\tau}\mathbf{u}_0 - Ra\, e^{-A\tau}A^{-1}(P(\mathbf{k}T_0)), \tag{3.2}$$

$$\Delta w = \frac{\partial w}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla w + (\mathbf{u}^{(0)} - \mathbf{u}^{0}) \cdot \nabla T^{0}, \tag{3.3}$$

$$w\Big|_{z=0,1} = 0 \tag{3.4}$$

together with zero initial data.

We proceed with usual energy method. For this purpose we first observe the following coercive property of the operator A which can be derived easily by multiplying (2.11) by $\mathbf{u}$, integrating over the domain and applying Cauchy-Schwarz and Poincaré inequality

$$\|A\mathbf{u}\|_{L^2} \geq \|\nabla \mathbf{u}\|_{L^2} \geq \|\mathbf{u}\|_{L^2}. \tag{3.5}$$

It is also easy to check that the semigroup e^{-At} has the following property

$$\|e^{-At}\mathbf{u}\|_{L^2} \leq e^{-t}\|\mathbf{u}\|_{L^2}. \tag{3.6}$$

Next we notice that the nonlinear terms can be estimated as

$$\begin{aligned} |\int_\Omega RaA^{-1}(P(\mathbf{k}w)) \cdot \nabla T^0\, w| &\leq Ra|A^{-1}(P(\mathbf{k}w))|_{L^2}|\nabla T^0|_{L^\infty}|w|_{L^2} \\ &\leq c_1 Ra|w|^2_{L^2} \end{aligned}$$

and

$$\begin{aligned} |\int_\Omega e^{-A\tau}\mathbf{u}_0 \cdot \nabla T^0\, w| &\leq |e^{-A\tau}\mathbf{u}_0|_{L^2}|\nabla T^0|_{L^\infty}|w|_{L^2} \\ &\leq c_2|w|_{L^2}e^{-\tau} \end{aligned}$$

and similarly,

$$|\int_\Omega Ra\, e^{-A\tau}A^{-1}(P(\mathbf{k}T_0)) \cdot \nabla T^0\, w| \leq c_3 Ra|w|_{L^2}e^{-\tau}.$$

Here and elsewhere, the c_js represent generic constants independent of the Prandtl number or Rayleigh number, but depend on the domain and the Ekman number as well as the initial data.

Combining these estimates together with usual energy method on (3.3) we have

$$\frac{1}{2}\frac{d}{dt}|w|_{L^2}^2 + |\nabla w|_{L^2}^2 \le c_1 Ra|w|_{L^2}^2 + c_4 Ra|w|_{L^2} e^{-\tau} \tag{3.7}$$

which further implies

$$\|w\|_{L^\infty(0,t;L^2)} \le \frac{c_4 \varepsilon Ra}{1 + c_1 \varepsilon Ra} e^{c_1 Ra\, t}. \tag{3.8}$$

This proves that the temperature fields of the effective dynamics and that of the infinite Prandtl number model remain close which further implies the closeness of the velocity fields modulo an initial layer according to (3.2).

In short we have the following result

THEOREM 1 *For any fixed time interval* $[0, t]$ *and any given initial data* $(\mathbf{u}_0, T_0)$*, the following hold*

$$\begin{aligned} &\|\mathbf{u}^{(0)} - \mathbf{u}^0 - e^{-A\tau}\mathbf{u}_0 - Ra\, e^{-A\tau} A^{-1}(P(\mathbf{k}T_0))\|_{L^\infty(0,t;H^2)} \\ &\quad \le c_5 \varepsilon Ra^2 e^{c_1 Ra\, t}, \end{aligned} \tag{3.9}$$

$$\|T^{(0)} - T^0\|_{L^\infty(0,t;L^2)} \le c_5 \varepsilon Ra\, e^{c_1 Ra\, t}. \tag{3.10}$$

where $(\mathbf{u}^{(0)}, T^{(0)})$ *is the solution of the effective dynamics (2.16-2.18) and* $(\mathbf{u}^0, T^0)$ *is the solution of the infinite Prandtl number dynamics (2.19-2.21).*

Estimates in higher order Sobolev spaces for the temperature field are possible. However, these estimates in higher order Sobolev spaces are not really useful since the difference between the effective dynamics and the Boussinesq system has to be measured in L^2 due to the regularity constraint on the Boussinesq system.

Alternatively, we may derive the effective dynamics utilizing the so-called corrector approach (see Lions 1973, Vishik and Lyusternik 1957). The end result is very much similar with an initial layer.

Here we have used the regularity of the infinite Prandtl number model which can be derived easily using classical methods as those available from the books of Lions (1969), Temam (2000) among others. A similar model, the so called Rayleigh-Bénard convection in porous media with zero Darcy-Prandtl number was study by Fabrie (1982), Ly and Titi (1999) for the question of regularity among others. Their method can be transplanted to our case. The center piece is a maximum principle for the temperature field. A proof without invoking the maximum principle is also possible.

4 The Justification of Effective Dynamics

We now proceed to justify the effective dynamics on any given finite time interval.

As was mentioned earlier, the possible loss of regularity of the Boussinesq system causes some difficulty in the rigorous justification which is partially compensated by the regularity of the effective dynamics.

We follow the usual energy method approach which is natural here due to the known regularity of the Boussinesq system. We consider the difference of the solution $(\mathbf{u}, T)$ to the Boussinesq system (1.5-1.8) and the solutions $(\mathbf{u}^{(0)}, T^{(0)})$ of the effective dynamics (2.16-2.18). Denoting

$$w = T - T^{(0)}, \tag{4.1}$$

$$\mathbf{v} = \mathbf{u} - \mathbf{u}^{(0)}, \tag{4.2}$$

we see that, after utilizing equations (1.5-1.8) and (2.16-2.18),

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w + \mathbf{v} \cdot \nabla T^{(0)} = \Delta w, \tag{4.3}$$

$$w|_{z=0,1} = 0, \tag{4.4}$$

$$w|_{t=0} = 0. \tag{4.5}$$

This leads to the following energy inequality

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \le \|T^{(0)}\|_{L^\infty}\|\nabla w\|_{L^2}\|\mathbf{v}\|_{L^2}.$$

Applying the classical Cauchy-Schwarz inequality we further deduce

$$\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \le \|T^{(0)}\|_{L^\infty}^2\|\mathbf{v}\|_{L^2}^2 \tag{4.6}$$

which implies that

$$\|w\|_{L^\infty(0,t;L^2)} \le c_6\|\mathbf{v}\|_{L^2(0,t;L^2)}, \tag{4.7}$$

$$\|w\|_{L^2(0,t;H^1)} \le c_6\|\mathbf{v}\|_{L^2(0,t;L^2)}. \tag{4.8}$$

This motivates us to study the difference of velocity fields, namely $\mathbf{v}$. For this purpose we look for equations satisfied by $\mathbf{v}$ casted in an appropriate form. Indeed, we have

$$\varepsilon\{\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}^{(0)}\} + \frac{1}{Ek}\mathbf{k}\times\mathbf{v} + \nabla p = \Delta\mathbf{v} + Ra\,\mathbf{k}w + \mathbf{f}, \tag{4.9}$$

$$\nabla\cdot\mathbf{v} = 0,$$

$$\mathbf{v}\Big|_{t=0} = 0, \tag{4.10}$$

$$\mathbf{v}\Big|_{z=0,1} = 0 \tag{4.11}$$

with the extra forcing term $\mathbf{f}$ given by

$$\begin{aligned}\mathbf{f} = &-\varepsilon\frac{\partial\mathbf{u}^{(0)}}{\partial t} - \varepsilon(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)} \\ &-\frac{1}{Ek}\mathbf{k}\times\mathbf{u}^{(0)} + \Delta\mathbf{u}^{(0)} + Ra\,\mathbf{k}T^{(0)}\,.\end{aligned} \tag{4.12}$$

This extra forcing term has to be small if the effective dynamics is to be valid. The smallness of this term is guaranteed by the explicit form of the effective dynamics as we shall see below. As a matter of fact, we have, thanks to the explicit form of the velocity field in the effective dynamics given in (2.17) and the definition of the dissipative operator A given in (2.11-2.12),

$$\begin{aligned}\mathbf{f} &= -\varepsilon Ra\,A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t})) + Ae^{-A\tau}\mathbf{u}_0 - Ra\,e^{-A\tau}(P(\mathbf{k}T_0)) \\ &\quad -\varepsilon(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)} - A\mathbf{u}^{(0)} + Ra\,\mathbf{k}T^{(0)} + \nabla q \\ &= -\varepsilon Ra\,A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t})) - \varepsilon(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)} + \nabla q\,.\end{aligned} \tag{4.13}$$

Applying energy estimates we deduce

$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}|\mathbf{v}|_{L^2}^2 + |\nabla\mathbf{v}|_{L^2}^2 &\le Ra|w|_{L^2}|\mathbf{v}|_{L^2} + \varepsilon\{|\mathbf{v}|_{L^4}|\mathbf{u}^{(0)}|_{L^4}|\nabla\mathbf{v}|_{L^2} \\ &\quad +|Ra\,A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))|_{L^2}|\mathbf{v}|_{L^2} + |(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}|_{H^{-1}}|\nabla\mathbf{v}|_{L^2}\} \\ &\le Ra|w|_{L^2}|\mathbf{v}|_{L^2} + \varepsilon\{c_7|\mathbf{u}^{(0)}|_{H^1}|\nabla\mathbf{v}|_{L^2}^2 \\ &\quad +|Ra\,A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))|_{L^2}|\mathbf{v}|_{L^2} + |(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}|_{H^{-1}}|\nabla\mathbf{v}|_{L^2}\}\end{aligned}$$

Thus we have

$$\frac{d}{dt}|\mathbf{v}|_{L^2}^2 + \frac{1}{2\varepsilon}|\nabla \mathbf{v}|_{L^2} \leq \frac{2Ra^2}{\varepsilon}|w|_{L^2}^2 + 2\varepsilon\{|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))|_{L^2}^2 + |(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}|_{H^{-1}}^2\} \tag{4.14}$$

provided that

$$c_7\varepsilon|\mathbf{u}^{(0)}|_{H^1} \leq \frac{1}{4} \tag{4.15}$$

which is satisfied for large enough Prandtl number since, thanks to the explicit formula for the velocity field of the effective dynamics (2.17) and the maximum principle on the temperature field as well as the dissipative nature of the operator A

$$\begin{aligned}|\mathbf{u}^{(0)}|_{H^1} &\leq Ra|A^{-1}(P(\mathbf{k}T^{(0)}))|_{H^1} + |e^{-A\tau}\mathbf{u}_0|_{H^1} + Ra|e^{-A\tau}A^{-1}(P(\mathbf{k}T^{(0)}))|_{H^1} \\ &\leq c_8(Ra|T_0|_{L^\infty} + |\mathbf{u}_0|_{H^1})\end{aligned} \tag{4.16}$$

Thus we have,

$$\frac{d}{dt}(e^{\frac{t}{2\varepsilon}}|\mathbf{v}|_{L^2}^2) \leq \frac{2Ra^2}{\varepsilon}e^{\frac{t}{2\varepsilon}}|w|_{L^2}^2 + 2\varepsilon e^{\frac{t}{2\varepsilon}}\{|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))|_{L^2}^2 + |(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}|_{H^{-1}}^2\}$$

Integrating in time and utilizing equation (4.7) we have

$$\begin{aligned}|\mathbf{v}|(t)_{L^2}^2 &\leq 4Ra^2\|w\|_{L^\infty(0,t;L^2)}^2 + 4\varepsilon^2\{\|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))\|_{L^\infty(0,t:L^2)}^2 \\ &\quad + \|(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}\|_{L^\infty(0,t:H^{-1})}^2\} \\ &\leq c_9Ra^2\|\mathbf{v}\|_{L^2(0,t;L^2)}^2 + 4\varepsilon^2\{\|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))\|_{L^\infty(0,t:L^2)}^2 \\ &\quad + \|(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}\|_{L^\infty(0,t:H^{-1})}^2\}\end{aligned} \tag{4.17}$$

This implies, thanks to Gronwall inequality

$$\begin{aligned}&\|\mathbf{v}\|_{L^2(0,t;L^2)}^2 \\ &\leq \frac{c_{10}\varepsilon^2 e^{c_9Ra^2t}}{Ra^2}\{\|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))\|_{L^\infty(0,t:L^2)}^2 + \|(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}\|_{L^\infty(0,t:H^{-1})}^2\}\end{aligned}$$

which further implies, thanks to (4.17),

$$(4.18) \qquad \|\mathbf{v}\|^2_{L^\infty(0,t;L^2)}$$
$$\leq c_{11}\varepsilon^2 e^{c_9 Ra^2 t}\{\|Ra\, A^{-1}(P(\mathbf{k}\frac{\partial T^{(0)}}{\partial t}))\|^2_{L^\infty(0,t:L^2)} + \|(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}\|^2_{L^\infty(0,t:H^{-1})}\}$$

This leads to, provided we have adequate uniform in Pr bound on the solutions to the effective dynamics which we shall derive later, the convergence of the velocity field which further leads to the convergence of the temperature field by (4.7-4.8).

It remains to prove appropriate bounds on the solutions of the effective dynamics.

Here we only give a sketch although all the following arguments can be made rigorous.

It is easy to see that there is a maximum principle for the temperature field, i.e.,

$$(4.19) \qquad \|T^{(0)}\|_{L^\infty(0,t;L^\infty)} \leq \|T_0\|_{L^\infty}.$$

This immediately implies, utilizing the formula for the velocity field in the effective dynamics (2.17) and the coercive property of the operator A (see (3.5))

$$(4.20) \qquad \|\mathbf{u}^{(0)}\|_{H^1} \leq c_{12} Ra\,.$$

This further implies

$$(4.21) \qquad \begin{aligned} \|(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)}\|_{H^{-1}} &\leq \|\mathbf{u}^{(0)}\|^2_{L^4} \\ &\leq c_{13}\|\mathbf{u}^{(0)}\|^2_{H^1} \\ &\leq c_{14} Ra^2\,. \end{aligned}$$

Since we need estimate on the time derivative, and since the velocity field has an initial layer, the natural way to estimate the time derivative is to estimate the gradient of the temperature field and use the temperature equation. For this purpose, we consider the effective dynamics in the perturbative variable $(\theta^{(0)}, \mathbf{u}^{(0)})$, i.e., perturbation away from the pure conduction state $(1-z, 0)$ with

$$(4.22) \qquad \theta^{(0)} = T^{(0)} - (1-z).$$

The effective dynamics in the $(\theta^{(0)}, \mathbf{u}^{(0)})$ variable takes the form

$$(4.23) \qquad \Delta\theta^{(0)} = \frac{\partial\theta^{(0)}}{\partial t} + \mathbf{u}^{(0)}\cdot\nabla\theta^{(0)} - u_3^{(0)},$$

$$\mathbf{u}^{(0)}(t) = Ra\,A^{-1}(P(\mathbf{k}\theta^{(0)})) + e^{-A\tau}\mathbf{u}_0 - Ra\,e^{-A\tau}A^{-1}(P(\mathbf{k}T_0)), \tag{4.24}$$

$$\theta^{(0)}\Big|_{z=0,1} = 0. \tag{4.25}$$

One advantage of such a decomposition is that the fluctuation part $\theta^{(0)}$ satisfies the homogeneous Dirichlet boundary condition. The velocity equation remains unchanged after we replace the temperature field with the deviation (fluctuation) from the pure conduction state since the contribution from the pure conduction state can be absorbed into the pressure field.

Next, we multiply the temperature equation (4.23) by $\Delta\theta^{(0)}$ and integrate over the domain in order to derive higher order estimates. For the nonlinear term we have

$$\begin{aligned} |\int_\Omega \mathbf{u}^{(0)}\cdot\nabla\theta^{(0)}\Delta\theta^{(0)}| &= |\int_\Omega \nabla(\mathbf{u}^{(0)}\cdot\nabla\theta^{(0)})\cdot\nabla\theta^{(0)}| \\ &= |\int_\Omega \sum_{i=1}^{3}(\nabla u_i^{(0)}\cdot\nabla\theta^{(0)}\frac{\partial\theta^{(0)}}{\partial x_i} + u_i^{(0)}\nabla\frac{\partial\theta^{(0)}}{\partial x_i}\cdot\nabla\theta^{(0)})| \\ &= |\int_\Omega \sum_{i=1}^{3}\nabla u_i^{(0)}\cdot\frac{\partial\nabla\theta^{(0)}}{\partial x_i}\theta^{(0)}| \\ &\le c_{15}\|\mathbf{u}^{(0)}\|_{H^1}\|\Delta\theta^{(0)}\|_{L^2}\|\theta^{(0)}\|_{L^\infty} \\ &\le c_{16}Ra\|\Delta\theta^{(0)}\|_{L^2} \end{aligned}$$

Utilizing this in the energy estimate on $\theta^{(0)}$ we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\theta^{(0)}\|_{L^2}^2 + \|\Delta\theta^{(0)}\|_{L^2}^2 &\le c_{16}Ra\|\Delta\theta^{(0)}\|_{L^2} + \|u_3^{(0)}\|_{L^2}\|\Delta\theta^{(0)}\|_{L^2} \\ &\le \frac{1}{2}\|\Delta\theta^{(0)}\|_{L^2}^2 + c_{17}Ra^2 \end{aligned}$$

This leads to

$$\begin{aligned} \|\nabla\theta^{(0)}\|_{L^\infty(0,T^*;L^2)} &\le c_{18}Ra \\ \|\Delta\theta^{(0)}\|_{L^2(0,T^*;L^2)} &\le c_{18}Ra \end{aligned}$$

which further implies

$$\|\nabla T^{(0)}\|_{L^\infty(0,T^*;L^2)} \le c_{19}Ra \tag{4.26}$$

$$\|\Delta T^{(0)}\|_{L^2(0,T^*;L^2)} \le c_{19}Ra \tag{4.27}$$

When this is combined with the temperature equation (2.16) we deduce

$$
\begin{aligned}
(4.28)\qquad & \|\frac{\partial T^{(0)}}{\partial t}\|_{L^\infty(0,T^*;H^{-1})} \\
& \le \|T^{(0)}\|_{L^\infty(0,T^*;H^1)} + \|u_3^0\|_{L^\infty(0,T^*;L^2)}\|T^{(0)}\|_{L^\infty(0,T^*;L^\infty)} \\
& \le c_{20} Ra
\end{aligned}
$$

Combing (4.18), (4.21), (4.28), and (4.7-4.8) we have the following result

Theorem 2 *For any fixed interval* $[0,t]$ *and any given initial data* $(\mathbf{u}_0, T_0)$*, the following hold*

$$
\|\mathbf{u} - \mathbf{u}^{(0)}\|_{L^\infty(0,t;L^2)} \le c_{21}\varepsilon Ra^2 e^{c_9 Ra^2 t/2}, \tag{4.29}
$$

$$
\|T - T^{(0)}\|_{L^\infty(0,t;L^2)} \le c_{21}\varepsilon Ra^2 e^{c_9 Ra^2 t/2}. \tag{4.30}
$$

which further implies, when combined with (46-47),

$$
\begin{aligned}
(4.31)\qquad & \|\mathbf{u} - \mathbf{u}^0 - e^{-A\tau}\mathbf{u}_0 - Ra\, e^{-A\tau}A^{-1}(P(\mathbf{k}T_0))\|_{L^\infty(0,t;L^2)} \\
& \le c_{22}\varepsilon Ra^2 e^{c_{23} Ra^2 t},
\end{aligned}
$$

$$
\|T - T^0\|_{L^\infty(0,t;L^2)} \le c_{22}\varepsilon Ra^2 e^{c_{23} Ra^2 t}. \tag{4.32}
$$

where $(\mathbf{u}^{(0)}, T^{(0)})$ *is the solution of the effective dynamics (2.16-2.18) and* $(\mathbf{u}^0, T^0)$ *is the solution of the infinite Prandtl number dynamics (2.17-2.21).*

Notice the convergence rate of order ε is optimal which can be justified via a systematic asymptotic expansion of the Boussinesq system in the small parameter $\varepsilon = \frac{1}{Pr}$.

5 Comments and Remarks

So far we have rigorously justified the effective dynamics (2.16-2.18) which in turn justifies the infinite Prandtl number limit modulo an initial layer on any fixed finite time interval with the other parameters such as the Rayleigh number Ra fixed (see Theorem 2). A physically important question is about the asymptotic behavior as the Prandtl number Pr and the Rayleigh number Ra simultaneously approach infinity. It is easy to see that our convergence rate has an exponential dependence on the Rayleigh number Ra (see (4.29-4.30)). This implies, with the convergence result that we proved here, we can allow the Rayleigh number Ra approach infinity simultaneously with a rate of lower order than $\sqrt{\ln Pr}$. This estimate can be improved if we consider higher order expansion with higher order correctors to $\mathbf{u}$ and T. However this could only improve the result by some algebraic factors.

Encouraged by the finite time convergence, we naturally inquire if the solutions of the Boussinesq system and solutions of the infinite Prandtl number model remain close on a large time interval for large Prandtl number. In general we should not expect long time proximity of each individual orbit. Such a long time orbital stability result shouldn't be expected for such complex systems where turbulent/chaotic behavior abound. Instead, the statistical properties for such systems are much more important and physically relevant and hence it is natural to ask if the statistical properties (in terms of invariant measures) as well as global attractors (if they exist) remain close.

The first obstacle in studying long time behavior is the well-posedness of the Boussinesq system global in time. This is closely related to the well-known problem related to the 3D Navier-Stokes equations(Constantin and Foias 1988, Temam 2000, Majda and Bertozzi 2001 among others). Fortunately, in the regime of large Prandtl number, we are able to prove the eventual regularity for suitably defined weak solutions to the Boussinesq system which exists for all time. The suitable weak solutions are defined as Leray-Hopf type weak solution plus suitable energy inequality which ensures certain maximum principle type estimates. We are then able to show that the Boussinesq system possesses a global attractor which attracts all suitable weak solutions for sufficiently large Prandtl number (over the Rayleigh number). Furthermore, the global attractors of the Boussinesq system converge to the global attractor of the limit infinite Prandtl number model in some appropriate sense (see Wang 2003). The set of invariant measures/ stationary statistical solutions for the Boussinesq system also converge to the set of invariant measures/ stationary statistical solutions to the infinite Prandtl number model. This further validates the infinite Prandtl number model.

A fundamental quantity in Rayleigh-Bénard convection is the total heat transport in the vertical direction. This is expressed in terms of the Nusselt number in a non-dimensional fashion as

$$Nu = 1 + \frac{1}{L_x L_y} \int_0^1 \int_0^{L_y} \int_0^{L_x} u_3 \, T. \tag{5.1}$$

It is apparent that the Nusselt number for the Boussinesq approximation of the Rayleigh-Bénard convection converges to the Nusselt number

(Nu^0) for the infinite Prandtl number for any fixed positive time due to the convergence of temperature field (4.32) and the convergence of the velocity field with initial layer (4.31). Again the convergence rate has an undesirable exponential dependence on the Rayleigh number. A physically more interesting quantity for turbulent flow is the time averaged heat transport in the vertical direction.

$$\overline{Nu} = 1 + \lim_{t\to\infty} \frac{1}{t}\int_0^t Nu. \tag{5.2}$$

The limit here may need to be replaced by upper limit if necessary. The convergence established in section 4 has no implication on the time averaged heat transport since the convergence there is for fixed time. The time averaged Nusselt number is actual a statistical property of the system since we can prove, following arguments similar to those for the Navier-Stokes equations (see for instance Foias, Manley, Rosa and Temam 2001) that this averaged Nusselt number corresponds to average with respect to some appropriate invariant measure (stationary statistical solutions). However the convergence of the set of invariant measures (stationary statistical solutions) that we mentioned above carry no information on the convergence of time averaged Nusselt numbers since we are not sure if the limit of invariant measures induced by time average still correspond to time average. Nevertheless, we can consider upper bounds on the Nusselt number. It seems that one is able to derive upper bound on the Nusselt number for the Boussinesq system which is consistent with the infinite Prandtl number system in the sense that the new upper bound is the sum of a physically relevant and known best upper bound for the infinite Prandtl number model of the form $Ra^{1/3}(\ln Ra)^{2/3}$ (Constantin-Doering 1999) plus a correction term which vanishes as the Prandtl number approaches infinity.

As we remarked in section 3, the dynamics of the infinite Prandtl number convection model is very regular, and the associated dynamical system possesses a compact global attractor. It is easy to check that the linearized solution operator contracts high modes. This further implies the finite dimensionality of the global attractor (see for instance Constantin and Foias (1988), Doering and Gibbon (1996), or Temam (1997)). See Ly-Titi 1999 for a study on a related convection model in porous media with zero Darcy-Prandtl number. Furthermore, one can check that the associated dynamical system possesses a finite dimensional exponential attractor which attracts all orbits at an exponential rate (see Eden, Foias, Nicolaenko and Temam (1994) for more on exponential attractor).

We do not dwell into the details here since we do not have a physical prediction for the degrees of freedom for this infinite Prandtl number model and hence we cannot make the connection to the dimension of attractors. An interesting question to ask is if the infinite Prandtl number model possesses a finite dimensional inertial manifold (See Temam (1997) or Constantin, Foias, Nicolaenko and Temam (1988) for more on inertial manifolds). This is not trivial since the spectral gap condition is not satisfied for this dynamical system which prevents us from applying the Foias-Sell-Temam theory in a straightforward fashion.

Within the simplified model of infinite Prandtl number, it is still interesting to consider the asymptotic behavior for large Rayleigh number. Heuristically, this corresponds to the situation of simultaneous large Prandtl and Rayleigh number but with the Prandtl number approaching infinity at a much faster rate. Such a limit is highly non-trivial. Indeed, for the special case of no rotation ($Ek = \infty$), we may rewrite the infinite Prandtl number model as

$$\frac{\partial T}{\partial t} + RaA^{-1}(\mathbf{k}T) \cdot \nabla T = \Delta T, \tag{5.3}$$

where A represents the Stokes operator. We see that on a short/fast time scale of $\tau = Ra\, t$, the problem of large Rayleigh number is equivalent to the problem of vanishing viscosity for this non-local advection diffusion problem with boundary. The large Rayleigh number asymptotics can be viewed as the long time (on the time scale of $Ra\, \tau$ for the fast time scale $\tau = t\, Ra$) asymptotics for the temperature field. Thus the leading order terms satisfies the following nonlinear non-local advection equation

$$\frac{\partial T^0}{\partial \tau} + A^{-1}(\mathbf{k}T^0) \cdot \nabla T^0 = 0. \tag{5.4}$$

This is quite different from the recent works on singular limits of PDEs which heavily rely on the linearity of the leading order (singular term) (see Schochet (1994), Majda (2002) and the references therein). In fact, most (if not all) singular limit problem relies either on the linearity of the leading order or ODE dynamics of the leading order. The study (even on a formal level) of singular problem with leading order nonlinear and not satisfying ODE may be extremely difficult and remains a major challenge.

A somewhat simpler problem is to consider pattern formation as the Rayleigh number various. It is easy to see that for small enough Rayleigh number, the global attractor is a single point consisting of the pure conduction solution. It is expected that we have spatially periodic solutions, corresponding to Bénard cells, as the Rayleigh number crosses a threshold value utilizing techniques of Rabinowitz 1968 and Yudovich 1967.

As a final remark, the method that we used here for dealing with two time scale problems related to dissipative system can be generalized to many other systems as long as the limit system is regular enough and the original system satisfies certain uniform estimates in terms of the singular parameter such as the L^2 estimates in the velocity field for the Boussinesq equations.

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Bibliography

[1] E. Bodenschatz, W. Pesch and G. Ahlers, *Recent developments in Rayleigh-Bnard convection*. Annual review of fluid mechanics, Vol. 32, 709–778, 2000

[2] F.H. Busse, *Fundamentals of thermal convection*. In *Mantle Convection: Plate Tectonics and Global Dynamics*, ed WR Peltier, pp. 23-95. Montreux: Gordon and Breach, 1989.

[3] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*. Oxford, Clarendon Press, 1961.

[4] P. Constantin and C. R. Doering, *Infinite Prandtl number convection*, J. Stat. Phys. 94, no. 1-2, 159–172, 1999.

[5] P. Constantin and C. Foias, *Navier-Stokes Equations,* Chicago University Press, 1988.

[6] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Integral and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer-Verlag, New York, 1988.

[7] P. Constantin, C. Hallstrom, V. Poutkaradze, *Logarithmic bounds for infinite Prandtl number rotating convection*, J. Math. Phys. 42, no. 2, 773–783, 2001.

[8] C. R. Doering and P. Constantin, *On upper bounds for infinite Prandtl number convection with or without rotation*, J. Math. Phys. 42, no. 2, 784–795, 2001.

[9] C.R. Doering and J. Gibbons, *Applied Analysis of the Navier-Stokes Equations*, Cambridge University Press, Cambridge, UK, 1996.

[10] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Masson, Paris and Wiley, New York,1994.

[11] P.Fabrie, *Existence, unicité et comportement asymptoticque de la solutions d'un problem de convection en mileu poreux*, Comptes Rendus de l'Academie des Sciences-Série I, 295, 423-425, 1982.

[12] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes equations and turbulence*. Encyclopedia of Mathematics and its Applications, 83. Cambridge University Press, Cambridge, 2001.

[13] A.V. Getling, *Rayleigh-Bénard convection. Structures and dynamics.* Advanced Series in Nonlinear Dynamics, 11. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[14] A. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, San Diego, California, 1982.

[15] S. Grossmann and D. Lohse, *Scaling in thermal convection: a unifying theory*, J. Fluid Mech., vol. 407, pp.27-56, 2000.

[16] M.H. Holmes, *Introduction to Perturbation Methods*, Springer-Verlag, New York, 1995.

[17] J.L. Lions, *Quelques Methodes de Resolution des Provlémes aux Limites non Linéaires*, Dunod, Paris, 1969.

[18] J.L. Lions, *Perturbations singulières dans les problèmes aux limites et en controle optimal*, Lecture Notes in Math., vol **323**, Springer-Verlag, New York, 1973.

[19] H.V. Ly and E.S. Titi, *Global Gevrey regularity for the Bénard convection in porous medium with zero Darcy-Prandtl number*, Jour. Nonlinear Sci., 9, 333-362, 1999.

[20] A.J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Vol. 9 of Courant Lecture Notes, American Mathematical Society, Rhode Island, 2003.

[21] A.J. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, England, 2001.

[22] N. Masmoudi, *Ekman layers of rotating fluids: The case of general initial data*, Comm. Pure Appl. Math., vol. 53, issue 4, pp. 432-483, 2000.

[23] J. Pedlosky, *Geophysical Fluid Dynamics*, 2nd Edition, Springer-Verlag, New York, 1987.

[24] P.H. Rabinowitz, *Existence and non-uniqueness of regular solutions of the Bénard problem.* Arch. Rational Mech. Anal., 29, 32–57, 1968.

[25] R. Salmon, *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, 1998.

[26] S. Schochet, *Fast singular limits of hyperbolic PDEs*, J. Diff. Eqns, 114(2), pp. 476-512, 1994.

[27] E. Siggia, *High Rayleigh number convection.* Annual review of fluid mechanics, Vol. 26, 137–168, Annual Reviews, Palo Alto, CA, 1994.

[28] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd Edition* , Springer-Verlag, New York, 1997.

[29] R. Temam, *Navier-Stokes Equations*, AMS Chelsea, Providence, Rhode Island, 2000.

[30] D.J. Tritton, *Physical Fluid Dynamics*, Oxford Science Publishing, Oxford, England, 1988.

[31] M. I. Vishik and L. A. Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter*, Uspekki Mat. Nauk, vol **12**, 3-122, 1957.

[32] X. Wang, *Large Prandtl Number Limit of the Boussinesq System of Rayleigh-Bénard Convection*, submitted to Applied Mathematics Letters, 2003.

[33] V.I. Yudovich, *An example of the loss of stability and the generation of a secondary flow of a fluid in a closed container*. Mat. Sb. (N.S.), 74, (116), 565–579, 1967.