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INFINITE TENSOR PRODUCTS IN FOURIER ALGEBRAS

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This paper is a continuation of the author's article [8], and the main purpose is to improve Theorem 4 in [8]. The reader is required to read [8] before proceeding to the present one.

Let G be a locally compact abelian group with dual $\hat{G}$. For a sequence $(E_j)_1^\infty$ of (non-empty) compact subsets of G, we write $E = \prod_{j=1}^\infty E_j$. We say that $\sum_{j=1}^\infty E_j$ converges if $\sum_{j=1}^\infty x_j$ converges for every $x = (x_j)_1^\infty \in E$. If this is the case, we define

$$\tilde{E} = \sum_{j=1}^\infty E_j = \left\{\sum_{j=1}^\infty x_j \colon (x_j)_1^\infty \in E\right\} .$$

Any set $\tilde{E}$ obtained in this way is called a *multi-symmetric* set. We also define a map $p_E \colon E \to \tilde{E}$ by setting

$$p_E(x) = \sum_{j=1}^\infty x_j \quad (x = (x_j)_1^\infty \in E) .$$

Notice that if $\sum_1^\infty E_j$ is a convergent series of compact sets then so is $\sum_n^\infty E_j$ for every natural number $n \in N$, and that to each neighborhood V of $0 \in G$ there corresponds an $N \in N$ such that

$$n \geqq N \Rightarrow \sum_{j=n}^\infty E_j \subset V .$$

In fact, suppose this is false for some compact neighborhood V. Then for each $p \in N$ there exists an arbitrarily large $M_p \in N$ such that

$$(1) \qquad x_{jp} \in E_j (j \geqq M_p) \quad \text{and} \quad \sum_{j=M_p}^\infty x_{jp} \notin V$$

for some choice of (x_{jp}). Suppose that such an M_p and a sequence (x_{jp}) have been chosen for some $p \in N$. Since V is compact, there is an $N_p \in N$, with $N_p > M_p$, such that

$$(2) \qquad \sum_{j=M_p}^n x_{jp} \notin V \quad (n \geqq N_p) .$$

Then we choose $M_{p+1} > N_p$ so that (1) with p replaced by $p + 1$ is satisfied for some sequence $(x_{j(p+1)})$. If we set $x_j = x_{jp}$ for $M_p \leqq j < M_{p+1}$, $p = 1, 2, \cdots$, then (2) and our choice of M_p show that the series $\sum_j x_j$ does not converge, which contradicts the convergence of $\sum_j E_j$.

Thus we conclude that for any convergent series $\sum_j E_j$ of compact sets the map p_E is continuous and therefore $\tilde{E} = p_E(E)$ is compact.

THEOREM 1. *Let $(F_j)_1^\infty$ be a sequence of non-empty finite subsets of the real line $\boldsymbol{R}$. Then every locally compact abelian I-group G contains a convergent series $\tilde{E} = \sum_1^\infty E_j$ of compact subsets satisfying the following three conditions*:

(a) *the map p_E induces an isometric isomorphism P_E of the restriction algebra $A(\tilde{E})$ onto the S-tensor product $A_E = \odot_1^\infty A(E_j)$ by $P_E f = f \circ p_E$. Moreover, $A(E_j)$ is isometrically isomorphic to $A(F_j)$ for each $j = 1, 2, \cdots$.*

(b) *$\tilde{E}$ is an S-set.*

(c) *$\tilde{E}$ is a Dirichlet set, that is,*

$$\liminf_{\hat{G} \ni \chi \to \infty} \|\chi - 1\|_{C(\tilde{E})} = 0 .$$

To prove this, we need two lemmas.

LEMMA 1.1. *Let G be a locally compact abelian I-group, and $F \subset \boldsymbol{R}$ and $E_0 \subset G$ finite sets. Then every neighborhood V of O_G contains a finite set E such that $Gp(E) \cap Gp(E_0) = \{O_G\}$ and $A(E) = A(F)$ algebraically and isomorphically.*

PROOF. Since F is finite, there exists a rationally independent finite set $\{v_1, \cdots, v_M\}$ in $\boldsymbol{R}$ such that

$$F \subset Gp(\{v_1, \cdots, v_M\}) .$$

Take a finite set $\tilde{F} \subset \boldsymbol{Z}^M$ so that

$$F = \left\{\sum_1^M n_j v_j \colon n = (n_j)_1^M \in \tilde{F}\right\} .$$

Let V be an arbitrary neighborhood of O_G. Since G is an I-group and E_0 is a finite subset thereof, we can find a finite set $\{x_1, \cdots, x_M\}$ in G, which is independent (over the ring $\boldsymbol{Z}$ of integers), so that

$$E = \left\{\sum_1^M n_j x_j \colon n \in \tilde{F}\right\} \subset V$$

and $Gp(E) \cap Gp(E_0) = \{O_G\}$.

Define a map $p\colon Gp(\{x_j\}_1^M) \to Gp(\{v_j\}_1^M)$ by setting

$$p\left(\sum_1^M n_j x_j\right) = \sum_1^M n_j v_j \quad (n \in \boldsymbol{Z}^M) .$$

Then p is an onto isomorphism and $p(E) = F$. Therefore it is easy to prove that

$$\|f \circ p\|_{A(E)} = \|f\|_{A(F)} \quad (f \in A(F)) ,$$

which completes the proof.

LEMMA 1.2. *Let E be a finite set in a locally compact abelian group G, and $\varepsilon > 0$. Then there exists a compact neighborhood V of O_G such that:*

(i) *The sets $x + V$, $x \in E$, are disjoint.*

(ii) *For each $\gamma \in \hat{G}_d$, G_d being the group G with the discrete topology, let $f_\gamma \in A(E + V)$ be defined by*

$$f_\gamma(x + v) = \gamma(x) \quad (x \in E, v \in V) .$$

Then $\|f\|_{A(E+V)} < 1 + \varepsilon$.

PROOF. Let $\eta > 0$ be given. Since E is finite, there exists a finite subset Γ of $\hat{G}$ such that $\{\chi|_E: \chi \in \Gamma\}$ is η-dense in $\{\gamma|_E: \gamma \in \hat{G}_d\} \subset C(E)$.

Take a compact neighborhood W of O_G so that

(1) $$x, y \in E \text{ and } x \neq y \Rightarrow (x + W) \cap (y + W) = \emptyset ,$$

(2) $$\chi \in \Gamma \Rightarrow \operatorname{diam}[\chi(W)] < \eta .$$

Next choose a $g \in A(G)$ so that

(3) $$\|g\|_{A(G)} < 2 , \quad \operatorname{supp} g \subset W , \text{ and}$$

(4) $$g = 1 \text{ on some compact neighborhood } V \text{ of } O_G .$$

Then $V \subset W$, and (i) holds.

Let $\gamma \in \hat{G}_d$ be given. By the choice of Γ, there exists a $\chi = \chi_\gamma \in \Gamma$ such that $|\gamma - \chi| < \eta$ on E. We can write

$$f_\gamma = \sum_{x \in E} \gamma(x) g_x = \sum_{x \in E} \{\gamma(x) - \chi(x)\} g_x + \sum_{x \in E} \{\chi(x) - \chi\} g_x + \chi \quad \text{on} \quad E + V ,$$

where $g_x(y) = g(y - x)$. It follows that

$$\begin{aligned} \|f_\gamma\|_{A(E+V)} &\leqq \sum_{x \in E} |\gamma(x) - \chi(x)| \cdot \|g_x\|_{A(G)} + \sum_{x \in E} \|\{\chi(x) - \chi\} g_x\|_{A(G)} + 1 \\ &\leqq 2\eta \operatorname{Card} E + \sum_{x \in E} \|\chi(x) - \chi\|_{A(x+W)} \|g_x\|_{A(G)} + 1 \\ &\leqq 2(\eta + M\eta) \operatorname{Card} E + 1 , \end{aligned}$$

where M is an absolute constant (cf. Lemma 1 in [8]). Therefore (ii) holds if $\eta > 0$ is sufficiently small.

PROOF OF THEOREM 1. Let G be any locally compact abelian group, and H a closed subgroup thereof. As is well-known, H is an S-set (see Theorem 2.7.5 in [4]), and if a closed subset E of H is an S-set (or a Dirichlet set) in H, then so is E in G. Moreover, the restriction algebra of $A(G)$ to H is isometrically isomorphic to the Fourier algebra $A(H)$ on H (Theorems 2.7.2 and 2.7.4 in [4]), and every I-group contains a metrizable closed I-group (Theorem 2.5.5 in [4]). Consequently, to prove Theorem 1, we may and will assume that G is a metric I-group with translation-invariant metric d.

Let $(\hat{K}_n)_1^\infty$ be an increasing sequence of compact subsets of $\hat{G}$ such that every compact subset of $\hat{G}$ is contained in some $\hat{K}_n$. We shall now inductively construct a sequence $(V_n)_1^\infty$ of compact neighborhoods of O_G, a sequence $(E_n)_1^\infty$ of finite subsets of G, and a sequence $(\chi_n)_1^\infty$ of characters in $\hat{G}$ which satisfy the following conditions:

(1) $$A(E_n) = A(F_n) \text{ algebraically and isometrically} .$$

(2) $$\chi_n \in \hat{G}\backslash\hat{K}_n \quad \text{and} \quad |\chi_n - 1| < n^{-1} \text{ on } E_1 + \cdots + E_n + V_{n+1} .$$

(3) $$O_G \in E_n \quad \text{and} \quad E_n + V_{n+1} \subset \operatorname{int} V_n .$$

(4) $$\text{The sets } x + V_{n+1},\ x \in E_1 + \cdots + E_n, \text{ are disjoint} .$$

(5) $$\|f_\gamma^n\|_{A(E_1+\cdots+E_n+V_{n+1})} < 1 + n^{-1} \quad (\gamma \in \hat{G}_d) ,$$

where f_γ^n is defined by

$$f_\gamma^n(x_1 + \cdots + x_n + V_{n+1}) = \gamma(x_1 + \cdots + x_n) \quad \forall (x_j \in E_j)_1^n .$$

For $n = 1$, we first take any compact neighborhood V_1 of O_G with $\operatorname{diam} V_1 < 1/2$. By Lemma 1.1, $\operatorname{int} V_1$ contains a finite set E_1 which contains O_G and satisfies (1) for $n = 1$. Since E_1 is finite, there is a $\chi_1 \in \hat{G}\backslash\hat{K}_1$ such that $|\chi_1 - 1| < 1$ on E_1.

Let $n \in \boldsymbol{N}$, and suppose that V_k, E_k, and χ_k have been chosen for all $k \leqq n$ so that

$$|\chi_n - 1| < n^{-1} \quad \text{on} \quad \sum_1^n E_k , \quad \text{and} \quad E_n \subset \operatorname{int} V_n .$$

Then we can take a compact neighborhood W_n of O_G so that

(2)′ $$|\chi_n - 1| < n^{-1} \quad \text{on} \quad \sum_1^n E_k + W_n ,$$

(3)′ $$E_n + W_n \subset V_n .$$

By Lemma 1.2, W_n contains a compact neighborhood V_{n+1} of O_G which satisfies (4) and (5). Clearly (2) and (3) hold. We can also demand that

(6) $$\operatorname{diam} V_{n+1} < 2^{-n-1} .$$

By Lemma 1.1, $\operatorname{int} V_{n+1}$ contains a finite set E_{n+1} with $O_G \in E_{n+1}$ which satisfies (1) with n replaced by $n+1$ and

$$(7) \qquad Gp(E_1 \cup \cdots \cup E_n) \cap Gp(E_{n+1}) = \{O_G\} .$$

Finally choose a $\chi_{n+1} \in \hat{G} \backslash \hat{K}_{n+1}$ so that

$$|\chi_{n+1} - 1| < (n+1)^{-1} \quad \text{on} \quad \sum_1^{n+1} E_k .$$

This completes the induction.

By (3) and (6), $\tilde{E} = \sum_1^\infty E_j$ converges. We now want to prove that $\tilde{E}$ has the required properties. Notice that (3) assures that

$$(8) \qquad \sum_{j=n}^{\infty} E_j \subset \operatorname{int} V_n \quad (n = 1, 2, \cdots) .$$

PROOF OF (a). We must prove that P_E is an isometric (onto) isomorphism.

Let $M \in N$ and $\gamma_1, \cdots, \gamma_M \in \hat{G}$ be given. Define $f \in A(\sum_1^M E_j + V_{M+1})$ by setting

$$(9) \qquad f(x_1 + \cdots + x_M + V_{M+1}) = \prod_{j=1}^{M} \gamma_j(x_j) \quad \forall (x_j \in E_j)_1^M ,$$

which is well-defined by (4) and (7). Then we claim that

$$(9.1) \qquad \|f\|_{A(\sum_1^M E_j + V_{M+1})} < 1 + M^{-1} , \quad \text{and}$$

$$(9.2) \qquad P_E f = \gamma_1 \odot \gamma_2 \odot \cdots \odot \gamma_M .$$

Indeed, $Gp(E_1 \cup \cdots \cup E_M)$ is the direct sum of $Gp(E_1), \cdots, Gp(E_M)$ by (7). Therefore

$$\chi(y_1 + \cdots + y_M) = \prod_{j=1}^{M} \gamma_j(y_j) \quad \forall (y_j \in Gp(E_j))_1^M$$

is a character of $Gp(E_1 \cup \cdots \cup E_M)$, and therefore it can be extended to a character of G_d. But then $f = f_\chi^M$, and so (5) yields (9.1). Also, for every $x = (x_j)_1^\infty \in E = \prod_1^\infty E_j$, we have by (8) and (9)

$$\begin{aligned} (P_E f)(x) &= f(x_1 + x_2 + \cdots + x_M + \cdots) \\ &= f(x_1 + x_2 + \cdots + x_M + V_{M+1}) \\ &= \prod_1^M \gamma_j(x_j) = (\gamma_1 \odot \cdots \odot \gamma_M)(x) , \end{aligned}$$

which establishes (9.2).

We now prove that the function f defined by (9) also satisfies

$$(9.3) \qquad \|f\|_{A(\tilde{E})} = 1 .$$

In fact, take any natural number $N > M$, and put $\gamma_j = 1$ for all j with $M < j \leqq N$. If we define $g \in A(E_1 + \cdots + E_N + V_{N+1})$ by the right-hand side of (9) with M replaced by N, then $f = g$ on the domain of g, and so

$$\|f\|_{A(\tilde{E})} \leqq \|g\|_{A(\Sigma_1^N E_j + V_{N+1})} < 1 + N^{-1}$$

by (9.1). Since N may be arbitrarily large, this establishes $\|f\|_{A(\tilde{E})} \leqq 1$ and hence (9.3).

Notice now that the absolute convex hull of elements of the form

$$\gamma_1 \circledast \gamma_2 \circledast \cdots \circledast \gamma_M \qquad (\gamma_j \in \hat{G},\ M \in N)$$

is dense in the unit ball of the Banach algebra A_E (see the proof of Theorem 3 in [8]). It follows from (9.2), (9.3), and Lemma 3 in [8] that P_E is an isometric isomorphism. This establishes part (a).

PROOF OF (b). For each $M \in N$, we define a homomorphism L_M from $A(\tilde{E})$ into $A(\sum_1^M E_j + V_{M+1})$ by setting

(10) $$(L_M f)(x_1 + \cdots + x_M + V_{M+1}) = f(x_1 + \cdots + x_M)$$

for $f \in A(\tilde{E})$ and $x_j \in E_j$, $1 \leqq j \leqq M$. Notice then

(10.1) $$\|L_M f\|_{A(\Sigma_1^M E_j + V_{M+1})} \leqq (1 + M^{-1})\|f\|_{A(\tilde{E})}$$

for all $f \in A(\tilde{E})$. In fact, since $\tilde{E}$ is compact, it suffices to prove this for $f = \gamma|_{\tilde{E}}$ with $\gamma \in \hat{G}$ (cf. Lemma 2 in [8]). But then (10.1) is a special case of (9.1). We now claim

(10.2) $$\lim_{M\to\infty} \|L_M\gamma - \gamma\|_{A(\Sigma_1^M E_j + V_{M+1})} = 0 \quad (\gamma \in \hat{G}) .$$

To see this, fix any $\gamma \in \hat{G}$. By (6) and the definition of L_M, we have

(10.3) $$\lim_{M\to 0} \|L_M\gamma - \gamma\|_{C(\Sigma_1^M E_j + V_{M+1})} = 0 .$$

On the other hand, (10.1) yields

(10.4) $$\|(L_M\gamma)^n\|_A = \|L_M(\gamma^n)\|_A \leqq 1 + M^{-1} \quad (n = 0, \pm 1, \pm 2, \cdots) .$$

Thus (10.2) follows from (10.3), (10.4), and Lemma 1 in [8].

Notice now that (8) implies

(11) $$\tilde{E} \subset \sum_{j=1}^{M} E_j + \operatorname{int} V_{M+1} \quad (M = 1, 2, \cdots) ,$$

and so $PM(\tilde{E}) \subset A(\sum_1^M E_j + V_{M+1})'$. To complete the proof of (b), take any $S \in PM(\tilde{E})$. Then, the definition of L_M shows

$$\operatorname{supp}(L_M^* S) \subset \sum_{j=1}^{M} E_j \subset \tilde{E} .$$

Since each E_j is a finite set, this implies that L_M^*S is a finitely supported measure in $M(\tilde{E})$ for each $M = 1, 2, \cdots$. Also, we have

$$||L_M^*S||_{PM} \leqq (1 + M^{-1})||S||_{PM} \quad (M = 1, 2, \cdots)$$

by (10.1); and (10.2) and (11) assure that for all $\gamma \in \hat{G}$

$$\begin{aligned} |(L_M^*S)^\wedge(\gamma^{-1}) - \hat{S}(\gamma^{-1})| &= |\langle \gamma, L_M^*S\rangle - \langle \gamma, S\rangle| \\ &= |\langle L_M\gamma - \gamma, S\rangle| \\ &\leqq ||L_M\gamma - \gamma||_{A(\Sigma_1^M E_j + V_{M+1})}||S||_{PM} = o(1) . \end{aligned}$$

It follows from Lemma 2 in [8] that the sequence $(L_M^*S)_1^\infty$ of measures in $M(\tilde{E})$ converges to S in the weak-* topology of $PM(G)$. Since this is true for every $S \in PM(\tilde{E})$, we conclude $\tilde{E}$ is an S-set (actually a strong S-set).

PROOF OF (c) follows from (2) and (11).

REMARKS. (a) If F is a compact Dirichlet set in G, then we have

(c)′ $$\limsup_{\chi \to \infty} |\hat{S}(\chi)| = ||S||_{PM} \quad (S \in PM(F)) .$$

To see this, take any $S \in PM(F)$. Let $\varepsilon > 0$, $\gamma \in \hat{G}$ and a compact subset $\hat{K}$ of $\hat{G}$ be given. Since F is a Dirichlet set, there exists a $\chi = \chi_\varepsilon \in \hat{G}\backslash\gamma^{-1}\hat{K}$ such that $|\chi - 1| < \varepsilon$ on F. But then $|\gamma\chi - \gamma| = |\chi - 1| < \varepsilon$ on some compact neighborhood V of F by the continuity of χ. Thus $||\gamma\chi - \gamma||_{A(V)} \leqq M\varepsilon$ by Lemma 1 in [8], where M is an absolute constant. Since $S \in PM(F) \subset A(V)'$, it follows that

$$\begin{aligned} \sup\{|\hat{S}(\alpha)| : \alpha \in \hat{G}\backslash\hat{K}\} &\geqq |\hat{S}(\gamma\chi)| \\ &\geqq |\hat{S}(\gamma)| - |\hat{S}(\gamma) - \hat{S}(\gamma\chi)| \geqq |\hat{S}(\gamma)| - M\varepsilon||S||_{PM} . \end{aligned}$$

Since $\gamma \in \hat{G}$ and $\varepsilon > 0$ are arbitrary, this shows

$$\sup\{|\hat{S}(\alpha)| : \alpha \in \hat{G}\backslash\hat{K}\} = \sup\{|\hat{S}(\gamma)| : \gamma \in \hat{G}\} = ||S||_{PM} ,$$

which establishes (c)′.

(b) In Theorem 1, we can replace $\boldsymbol{R}$ by any torsion-free group.

(c) The technique in the proof of Theorem 1 can be used to improve Example 4 in [8] as follows. Let $(E_j)_1^\infty$ be a sequence of finite subset of $\boldsymbol{R}^N$, N being a fixed natural number. Then there exists a sequence $(t_j)_1^\infty$ of positive real numbers which satisfies the following conditions. (i) The series $\tilde{K} = \sum_1^\infty t_jE_j$ converges; (ii) $A(\tilde{K})$ is isometrically isomorphic to $A_E = \odot_1^\infty A(E_j)$; (iii) $\tilde{K}$ is an S-set and a Dirichlet set.

THEOREM 2 (cf. Theorem 4 in [8]). *Every locally compact I-group G contains a multi-symmetric set $\tilde{K} = \sum_1^\infty K_j$, each K_j being a compact* *perfect Kronecker set in G, which satisfies the following conditions:*

(i) *The natural map* P_K: $A(\tilde{K}) \to S(K) = \odot_1^\infty C(K_j)$ *induced by* p_K: $K = \prod_1^\infty K_j \to \tilde{K}$ *is an isometric isomorphism.*

(ii) $\tilde{K}$ *is an S-set and a Dirichlet set.*

PROOF. Without loss of generality, we may assume that G has a translation-invariant metric d compatible with its topology. Then Theorem 1 and its proof show that there exists a countable subset $\{r_{jk}: j, k \in \boldsymbol{N}\}$ of G which is independent over $\boldsymbol{Z}$ and has the following properties:

$$d(0, r_{jk}) < 2^{-j-k} \quad (j, k = 1, 2, \cdots) . \tag{1}$$

$$\tilde{E} = \sum_{jk} E_{jk} \text{ satisfies the conclusions of Theorem 1} . \tag{2}$$

Here $E_{jk} = \{0, r_{jk}\}$ for all j and k.

Put $E = \prod_{jk} E_{jk}$, $\tilde{E}_j = \sum_k E_{jk}$, $E' = \prod_j \tilde{E}_j$, and define a map

$$q = p_{E'}: E' \to \tilde{E} = \sum_{jk} E_{jk} = \sum_j \tilde{E}_j$$

in the natural way. Then, by part (a) of Theorem 1, the natural map Q induced by q is an isometric isomorphism of $A(\tilde{E})$ onto

$$A_{E'} = \bigodot_1^\infty A(\tilde{E}_j) \cong \bigodot_j [\bigodot_k A(E_{jk})] \cong \bigodot_{jk} A(E_{jk}) .$$

(Notice that p_E is a homeomorphism from E onto $\tilde{E}$ since P_E is an isomorphism.)

We now claim that each $\tilde{E}_j$ contains a perfect Kronecker set. In fact, since $\{r_{jk}\}_k$ is independent over $\boldsymbol{Z}$, $\tilde{E}_j$ has the following property: for any natural number n, any $x_1, \cdots, x_n \in \tilde{E}_j$, and any $\varepsilon > 0$, there exist distinct $y_1, \cdots, y_n \in \tilde{E}_j$ such that $d(x_l, y_l) < \varepsilon$ for all l and $\{y_l\}_l$ is independent over $\boldsymbol{Z}$. This property assures that $\tilde{E}_j$ contains a perfect Kronecker set (cf. 5.2.3 and 5.2.4 in [4]).

We now choose and fix a perfect Kronecker set K_j in E_j for each $j = 1, 2, \cdots$, and first prove that $K_1 \times \cdots \times K_N$ is an S-set for the algebra $\odot_1^N A(\tilde{E}_j)$. In fact, every Kronecker set is an S-set (see [11], [5], and [7]). Since $A(G^N)$ is the N-fold projective tensor product of $A(G)$, it follows that $K_1 \times \cdots \times K_N$ is an S-set in G^N (see Theorem 1.5.1 in [12] and Theorem 2.2 in [6]). Since

$$\bigodot_1^N A(\tilde{E}_j) = A(\tilde{E}_1 \times \cdots \times \tilde{E}_N)$$

algebraically and isometrically, this assures that $K_1 \times \cdots \times K_N$ is an S-set for the algebra $\odot_1^N A(\tilde{E}_j)$.

Next we prove that $K = \prod_1^\infty K_j$ is an S-set for the algebra $A_{E'}$. To do this, choose and fix any point $y = (y_j)_1^\infty \in K$, and define a sequence of homomorphisms

$$J_N: A_{E'} \to \bigodot_1^N A(\tilde{E}_j) \subset A_{E'}$$

by setting

$$(J_N f)(x_1, \cdots, x_N) = f(x_1, \cdots, x_N, y_{N+1}, y_{N+2}, \cdots)$$

for $f \in A_{E'}$ and $x_j \in \tilde{E}_j$, $1 \leqq j \leqq N = 1, 2, \cdots$. Then we have

$$\lim_{N\to 0} \|J_N f - f\|_{A_{E'}} = 0 \quad (f \in A_{E'}) \tag{3}$$

(cf. [8: p. 283]). If $f \in A_{E'}$ vanishes on K, then each $J_N f$ vanishes on $K_1 \times \cdots \times K_N$. Since each $K_1 \times \cdots \times K_N$ is an S-set, it follows that

$$\begin{aligned} J_N f &\in \mathrm{cl}\left\{g \in \bigodot_1^N A(\tilde{E}_j)\colon \operatorname{supp} g \cap (K_1 \times \cdots \times K_N) = \varnothing\right\} \\ &\subset \mathrm{cl}\left\{h \in \bigodot_1^\infty A(\tilde{E}_j)\colon \operatorname{supp} h \cap K = \varnothing\right\} \end{aligned}$$

for all N, which combined with (3) implies that K is an S-set for $A_{E'}$.

Finally $\tilde{K} = \sum_1^\infty K_j = q(K)$ is an S-set for $A(\tilde{E})$ since $Q\colon A(\tilde{E}) \to A_{E'}$ is an isomorphism. Therefore $\tilde{K}$ is an S-set for $A(G)$ since so is $\tilde{E}$ by part (b) of Theorem 1. That $\tilde{K}$ is a Dirichlet set follows from part (c) of Theorem 1. Also we have

$$\begin{aligned} A(\tilde{K}) &= A(\tilde{E})|_{\tilde{K}} = A_{E'}|_K \\ &= \bigodot_1^\infty A(\tilde{E}_j)|_{K_j} = \bigodot_1^\infty C(K_j) = S(K) \end{aligned}$$

with natural identification, which completes the proof.

It is an interesting problem to find an explicit example of a multi-symmetric set $\tilde{E} = \sum_1^\infty E_j$ for which we have $A(\tilde{E}) = \bigodot_1^\infty A(E_j)$ algebraically and topologically. If G is an infinite product of compact groups, then this is very easy (Theorem 3 in [8]). Since every non-discrete non I-group contains such a group as a closed subgroup, it is reasonable to consider the problem only for I-groups. However, to obtain an explicit example of a set of a certain type, we much *know* the group under consideration. Consequently we will consider the above problem only for $G =$ the group of a-adic integers and for $G = \boldsymbol{R}^N$. Of course, then the problem will turn out trivial for any groups which contain, as a closed subgroup, one of the following groups: an infinite product of non-trivial compact groups; the group of a-adic integers for some a; $\boldsymbol{R}^N$ or $\boldsymbol{T}^N$ for some natural number N.

Let $a = (a_0, a_1, a_2, \cdots)$ be a sequence of positive integers $\geqq 2$, and $\Delta(a)$ the compact group of the a-adic integers (cf. [1: (10.2)]). Topologically we will identity $\Delta(a)$ with the product space of all $\{0, 1, \cdots, a_n - 1\}$, $n = 0, 1, 2, \cdots$. Let u_n be the element of $\Delta(a)$ whose n-th coordinate is one and other coordinates are all zero. Thus we have

$$u_n = a_{n-1}u_{n-1} = a_{n-1}a_{n-2} \cdots a_0 u_0 \qquad (n = 1, 2, \cdots)$$

and each element $x \in \Delta(a)$ can be uniquely written in the form

$$x = (x_n)_0^\infty = \sum_{n=0}^\infty x_n u_n ,$$

where $x_n \in \{0, 1, \cdots, a_n - 1\}$ for all $n = 0, 1, 2, \cdots$. We also set

$$a(l, m) = a_l a_{l+1} \cdots a_m \qquad (l < m) .$$

THEOREM 3. *Let a be as above, and let $(n_1, n_2, \cdots)$ and $(k_1, k_2, \cdots)$ be two sequences of natural numbers such that*

$$n_j < n_{j+1} \text{ and } k_j < a_{n_j} \qquad (j = 1, 2, \cdots) .$$

If

$$(*) \qquad \sum_{j=1}^\infty jk_j/a(n_j, n_{j+1} - 1) < \infty ,$$

then $A(\widetilde{E})$ is topologically isomorphic to $A_E = \odot_1^\infty A(E_j)$, where

$$E_j = \{\tau u_{n_j} \colon \tau = 0, 1, \cdots, k_j\} \text{ and } \widetilde{E} = \sum_{j=1}^\infty E_j .$$

PROOF. For each m, put

$$\Delta_m = \Delta(a, m) = \{(x_n)_0^\infty \in \Delta(a) \colon x_n = 0 \quad \text{for all} \quad n < m\} ,$$

which is an open-and-compact subgroup of $\Delta(a)$. Thus, if $l < m$, the coset $u_l + \Delta_m$ has order $a_l a_{l+1} \cdots a_{m-1} = a(l, m-1)$ as an element of the quotient group $\Delta(a)/\Delta_m$. Notice that the subgroup of $\boldsymbol{T} = \{z \colon |z| = 1\}$ consisting of p elements is η_p-dense in $\boldsymbol{T}$, where $\eta_p = |1 - \exp(\pi i/p)| = 2 \sin(\pi/2p)$. It follows that for each pair $l < m$ of non-negative integers and each character γ of $\Delta(a)$, there exists a character $\chi \in \Delta_m^\perp$ such that

$$(1) \qquad |\gamma(u_l) - \chi(u_l)| < \pi/a(l, m-1) ,$$

where $\Delta_m^\perp$ denotes the annihilator of Δ_m in $\widehat{\Delta(a)}$. Obviously (1) implies

$$(2) \qquad |\gamma(\tau u_l) - \chi(\tau u_l)| \leqq \tau\pi/a(l, m-1) \quad (\tau = 0, 1, 2, \cdots) .$$

If the sets E_j are defined as in the theorem, then $\widetilde{E} = \sum_1^\infty E_j$ converges, and

$$(3)\qquad \sum_{j=k}^{\infty} E_j \subset \varDelta_{n_k} \qquad (k = 1, 2, \cdots) .$$

Notice that (*) implies

$$(4)\qquad \sum_{j=N}^{\infty} \pi(j - N + 1)k_j/a(n_j, n_{j+1} - 1) \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty .$$

We apply the arguments in [8: pp. 294–295] with $\Gamma_j = \varDelta_{n_{j+1}}^{\perp}$ and $\varepsilon_j = \pi k_j/a(n_j, n_{j+1} - 1)$, and infer from (2), (3) and (4) that $A(\sum_N^{\infty} E_j)$ is topologically isomorphic to $\odot_N^{\infty} A(E_j)$ for all sufficiently large N. Since each E_j is a finite set and the natural map p_E associated with $(E_j)_1^{\infty}$ is injective, it follows that $A(\tilde{E})$ is topologically isomorphic to A_E. This completes the proof.

We now prove an analog of Theorem 3 for $G = \boldsymbol{Z}$. For each natural number $j \in \boldsymbol{N}$, let A_j be a semi-simple commutative Banach algebra with spectrum E_j. We identify A_j with a subalgebra of $C_0(E_j)$ in the usual way, and assume that A_j contains an idempotent ξ_j of norm one. If $f_1, \cdots, f_N$ are functions in $A_1, \cdots, A_N$, we define a function

$$\tilde{f} = f_1 \odot \cdots \odot f_N \odot \xi_{N+1} \odot \cdots$$

on the set

$$E_0 = \bigcup_{k=1}^{\infty} E_1 \times \cdots \times E_k \times \xi_{k+1}^{-1}(1) \times \cdots$$

by setting

$$\tilde{f}(x) = \left\{\prod_{j=1}^{N} f_j(x_j)\right\}\left\{\prod_{j=N+1}^{\infty} \xi_j(x_j)\right\} \qquad (x = (x_j)_1^{\infty} \in E_0) .$$

We denote by $S = S(A_1, A_2, \cdots)$ the algebra of all functions f on E_0 which have expansions of the form

$$f = \sum_{k=1}^{\infty} f_1^{(k)} \odot \cdots \odot f_{N_k}^{(k)} \odot \xi_{N_k+1} \odot \xi_{N_k+2} \odot \cdots ,$$

where $f_j^{(k)} \in A_j$, $N_k \in \boldsymbol{N}$, and

$$M = \sum_{k=1}^{\infty} \|f_1^{(k)}\|_{A_1} \cdots \|f_{N_k}^{(k)}\|_{A_{N_k}} < \infty .$$

For $f \in S$, the norm $\|f\|_S$ of f is defined to be the infimum of the numbers M taken over all expansions of f of the above form. We call S with norm $\|\cdot\|_S$ the S-tensor product of $A_1, A_2, \cdots$ relative to $\xi_1, \xi_2, \cdots$ (or, relative to $0_1, 0_2, \cdots$ if each $\xi_j^{-1}(1)$ is a singleton $\{0_j\}$). Therefore S is a semi-simple commutative Banach algebra. Notice that if $\xi_j = 1$ for all j, then S is the algebra $\odot_1^{\infty} A_j$ defined in [8].

THEOREM 4. *Let* $(a_1, a_2, \cdots)$ *and* $(k_1, k_2, \cdots)$ *be two sequences of natural numbers such that*

$$(*) \qquad k_j < a_j \ \forall j \quad and \quad \sum_{j=1}^{\infty} jk_j/a_j < \infty .$$

Let also $\tilde{E}_0$ *be the subset of* $\boldsymbol{Z}$ *consisting of all elements of the form*

$$\tau_1 + \tau_2 a_1 + \cdots + \tau_n a_1 a_2 \cdots a_{n-1} + \cdots ,$$

where $\tau_j \in \{0, 1, \cdots, k_j\}$ *for all* j *and* $\tau_j = 0$ *for all but except finitely many* j. *Then* $A(\tilde{E}_0)$ *is topologically isomorphic to the S-tensor product* S *of*

$$A_j = A(\{0, 1, \cdots, k_j\}) \qquad (j = 1, 2, \cdots)$$

relative to $0, 0, \cdots$.

PROOF. Let $a = (a_1, a_2, \cdots)$, and let $\varDelta(a)$ be the compact group of the a-adic integers. Put

$$E_j = \{\tau u_j : \tau = 0, 1, \cdots, k_j\} \qquad (j = 1, 2, \cdots) ,$$
$$E = \prod_{j=1}^{\infty} E_j = \sum_{j=1}^{\infty} E_j = \tilde{E} \subset \varDelta(a) .$$

Then the natural homomorphism P_E of $A(E)$ into $A_E = \odot_1^{\infty} A(E_j)$ is norm-decreasing by Lemma 3 in [8], and is actually an (onto) isomorphism by Theorem 3 and (*).

For each $N \in \boldsymbol{N}$, we define a norm-decreasing homomorphism $J_N: A_E \longrightarrow \odot_1^N A(E_j) \subset A_E$ by setting

$$(1) \qquad (J_N f)(x) = f(x_1, \cdots, x_N, 0, 0, \cdots) \qquad (x \in E) .$$

Notice that if we regard J_N as an operator on $A(E)$ then J_N has norm $\leqq \| P_E^{-1} \|$, and that

$$(2) \qquad \lim_{N \to \infty} \| J_N f - f \|_{A(E)} = 0 \qquad (f \in A(E)) .$$

(See [8: p. 283].)

Put

$$E_0 = \bigcup_{N=1}^{\infty} E_1 \times \cdots \times E_N \times \{0\} \times \{0\} \times \cdots ,$$

which is a dense subset of E. Let $B(E_0)$ be the restriction algebra of $B(\varDelta_d)$ to E_0. Here $\varDelta_d$ denotes the group $\varDelta(a)$ with the discrete topology, and $B(\varDelta_d)$ denotes the Banach algebra of Fourier-Stieltjes transforms of measures on $\widehat{\varDelta_d}$ = the Bohr compactification of $\widehat{\varDelta(a)}$. Let also $M_F(E_0)$ be the space of finitely supported measures on E_0. Then $\mu \in M_F(E_0)$ implies

$$\|\mu\|_{PM} = \sup\{|\hat{\mu}(\gamma)|: \gamma \in \widehat{\Delta(a)}\} = \sup\{|\hat{\mu}(\chi)|: \chi \in \widehat{\Delta_d}\},$$

since $\hat{\mu}$ is continuous on $\widehat{\Delta_d}$ and $\widehat{\Delta(a)}$ is dense in $\widehat{\Delta_d}$. The space $B(E_0)$ may be identified with the conjugate space of $M_F(E_0)$: $f \in B(E_0)$ if and only if

$$\|f\|_{B(E_0)} = \sup\left\{\left|\int_{E_0} f d\mu\right|: \mu \in M_F(E_0), \|\mu\|_{PM} \leqq 1\right\} < \infty .$$

Since E_0 is dense in E and $A(E) \subset C(E)$, we can and will identify $A(E)$ with its restriction to E_0. Then the embedding $A(E) \subset B(E_0)$ is a norm-decreasing homomorphism. We claim that $A(E)$ is indeed closed in $B(E_0)$. To see this, take any $f \in A(E)$. Then there exists a $\lambda \in M(\widehat{\Delta_d})$ such that $\hat{\lambda} = f$ on E_0 and $\|\lambda\|_M = \|f\|_{B(E_0)}$. Since E_0 is countable there exists a sequence $(f_n)_1^\infty$ in $A(\Delta(a))$ such that $\|f_n\|_{A(\Delta(a))} \leqq \|\lambda\|_M$ for all n and $f_n \to \hat{\lambda}$ on E_0 pointwise. Then we have

$$(3) \qquad \begin{cases} \|J_N f\|_{A(E)} \leqq \|J_N f - J_N f_n\|_{A(E)} + \|J_N f_n\|_{A(E)} \\ \qquad\qquad\quad \leqq \|J_N(f - f_n)\|_{A(E)} + \|J_N\| \cdot \|f\|_{B(E_0)} \end{cases}$$

for all $N, n = 1, 2, \cdots$. Notice that the range of J_N is finite-dimensional and $J_N f_n$ converges to $J_N f$ pointwise by (1), for each $N = 1, 2, \cdots$. Thus (3) yields

$$\|J_N f\|_{A(E)} \leqq \|J_N\| \cdot \|f\|_{B(E_0)} \leqq \|P_E^{-1}\| \cdot \|f\|_{B(E_0)} \quad (N = 1, 2, \cdots),$$

and hence

$$(4) \qquad \|f\|_{B(E_0)} \leqq \|f\|_{A(E)} \leqq \|P_E^{-1}\| \cdot \|f\|_{B(E_0)}$$

by (2). Since (4) holds for every $f \in A(E)$, we conclude that $A(E)$ is closed in $B(E_0)$.

We now prove that the S-tensor product S_E of the $A(E_j)$ relative to $0, 0, \cdots$ can be naturally identified with $A(E_0)$—the restriction algebra of $A(\Delta_d)$ to E_0. To do this, we introduce two maps

$$S_E \xrightarrow{K_N} \textstyle\bigodot_1^N A(E_j) \xrightarrow{L_N} S_E$$

for each N:

$$(K_N f)(x) = f(x_1, \cdots, x_N, 0, 0, \cdots) \quad (x \in E_1 \times \cdots \times E_N),$$
$$L_N f = f \odot \xi_{N+1} \odot \xi_{N+2} \odot \cdots .$$

It follows from the definition of S_E that K_N is norm-decreasing, that L_N is an isometry, and that the sequence $(L_N \circ K_N)_1^\infty$ converges to the identity operator on S_E in the strong operator topology. Take now any $f \in S_E$. Then, by the first inequality in (4), we have

(5) $\|K_N f\|_{B(E_0)} \leqq \|K_N f\|_{A(E)} \leqq \|P_E^{-1}\| \ \|K_N f\|_{A_E} \leqq \|P_E^{-1}\|\cdot\|f\|_{S_E}$

for all N. Here we regard $\bigodot_1^N A(E_j) \subset A_E = A(E)$ in the usual way. Since $K_N f \to f$ pointwise on E_0, (5) assures

(6) $$f \in B(E_0) \quad \text{and} \quad \|f\|_{B(E_0)} \leqq \|P_E^{-1}\|\cdot\|f\|_{S_E} .$$

To prove the converse inequality, choose a sequence $(f_n)_1^\infty$ in $A(E)$ so that $\|f_n\|_{A(E)} \leqq \|f\|_{B(E_0)}$ and $f_n \to f$ pointwise on E_0. Then we have

$$\begin{aligned}\|L_N J_N f_n\|_{S_E} &= \|J_N f_n\|_{A_E} \leqq \|f_n\|_{A_E} \\ &\leqq \|f_n\|_{A(E)} \leqq \|f\|_{B(E_0)} .\end{aligned}$$

But it is clear that $J_N f_n \to K_N f$ pointwise on E as $n \to \infty$ for each fixed N. Since $\bigodot_1^N A(E_j)$ is a finite-dimensional linear space, this implies

$$\|J_N f_n - K_N f\|_{A_E} \to 0 \quad \text{as} \quad n \to \infty \qquad (N = 1, 2, \cdots) .$$

Therefore we have

$$\|L_N K_N f\|_{S_E} = \lim_{n\to\infty} \|L_N J_N f_n\|_{S_E} \leqq \|f\|_{B(E_0)} \quad (N = 1, 2, \cdots) .$$

Since $L_N K_N$ converges to the identity operator, we have $\|f\|_{S_E} \leqq \|f\|_{B(E_0)}$, and hence

(7) $$\|f\|_{S_E} \leqq \|f\|_{B(E_0)} \leqq \|P_E^{-1}\| \ \|f\|_{S_E} \qquad (f \in S_E) .$$

Now it is easy to see that all the functions on E_0 with finite support are contained in $A(E_0) \cap S_E$ and are dense in both $A(E_0)$ and S_E. Therefore (7) assures $A(E_0) = S_E$.

Finally, there exists a unique group isomorphism $\phi: \boldsymbol{Z} \to Gp(E_0) \subset \Delta_d$ such that $\phi(1) = u_1$, and we have $\phi(\tilde{E}_0) = E_0$. The adjoint map ϕ^* induces an isometric isomorphism $\Phi: B(E_0) \to B(\tilde{E}_0)$ which maps $A(E_0)$ onto $A(\tilde{E}_0)$. The composite of the maps

$$A(\tilde{E}_0) \xrightarrow{\Phi^{-1}} A(E_0) \xrightarrow{id} S_E$$

is therefore a norm-decreasing topological isomorphism. Since $A(\{0, 1, \cdots, k_j\}) = A(E_j)$ algebraically and isometrically for all j, this completes the proof.

REMARK. The above proof shows that $B(\tilde{E}_0)$ contains a closed subalgebra which is topologically isomorphic to A_E.

We now fix a natural number N. For each $j = 1, 2, \cdots$, let $\{v_{kj}\}_{k=1}^N$ be an orthogonal basis in $\boldsymbol{R}^N$, and E_j a finite set such that

$$\{0\} \subsetneqq E_j \subset Gp(\{v_{1j}, \cdots, v_{Nj}\}) .$$

We put

$$R_j = \sup\{\|x\|: x \in E_j\},\ r_j = \inf\{\|v_{kj}\|: 1 \leqq k \leqq N\}\ ,$$

and assume that

(UTMS) $$\sum_{j=1}^{\infty} (R_{j+1}/r_j)^2 < \infty\ .$$

Under these conditions, we call $\tilde{E} = \sum_1^\infty E_j$ a UTMS set (ultra thin multi-symmetric set).

The following theorem is a generalization of the Meyer-Schneider theorem (cf. [3], [10], and [2: Chapter XIV]).

THEOREM 5. *Let $\tilde{E} = \sum_1^\infty E_j$ be a UTMS set in $\boldsymbol{R}^N$, and define a map $p_E: E = \prod_1^\infty E_j \to \tilde{E}$ as usual. Assume that p_E is one-to-one. Then we have:*

(a) *The map $P_E: A(\tilde{E}) \to A_E = \bigodot_1^\infty A(E_j)$ induced by p_E is a topological isomorphism.*

(b) *$\tilde{E}$ is an S-set.*

(c) *$\tilde{E}$ is a set of uniqueness, i.e., $PF(\tilde{E}) = \{0\}$.*

To prove this, we need several lemmas. Although the first two of these lemmas are well-known, we give a complete proof to make the paper self-contained.

For $\gamma = (\gamma_k)_1^N$ and $x = (x_k)_1^N \in \boldsymbol{R}^N$, write

$$\gamma(x) = e_\gamma(x) = e^{i\gamma x} = \exp\left[i(\gamma_1 x_1 + \cdots + \gamma_N x_N)\right]\ .$$

If u is a unit vector in $\boldsymbol{R}^N$ and $\phi \in C^1(\boldsymbol{R}^N)$, we define

$$(D_u\phi)(\gamma) = \sum_1^N u_j \frac{\partial\phi}{\partial\gamma_j}(\gamma) \qquad (\gamma \in \boldsymbol{R}^N)$$

which is the derivative of ϕ in the direction of u. We also write $S_l = \{x \in \boldsymbol{R}^N: \|x\| \leqq l\}$ for $l > 0$.

LEMMA 5.1. (Bernstein's inequality). *If $P \in PM(S_l)$, then we have*

$$\|D_u^k P\|_{C(\boldsymbol{R}^N)} \leqq l^k \|P\|_{PM} \qquad (k = 1, 2, \cdots)$$

for every unit vector u in $\boldsymbol{R}^N$.

PROOF. Let f_l be the $4l$-periodic odd function on $\boldsymbol{R}^1$ defined by

$$f_l(t) = \begin{cases} t & (0 \leqq t \leqq l) \\ 2l - t & (l \leqq t \leqq 2l)\ . \end{cases}$$

Then we have

(1) $$f_l(t) = l\sum_{n\neq 0}\left\{\left(\sin\frac{n\pi}{2}\right)\Big/\left(\frac{n\pi}{2}\right)\right\}^2(-i)^n \exp\left(i\, n\pi t/2l\right)\ ,$$

$$(2)\qquad \|f\|_{B(R)} = l \sum_{n\neq 0} \left\{\left(\sin \frac{n\pi}{2}\right) \Big/ \left(\frac{n\pi}{2}\right)\right\}^2 = l\ .$$

To prove (1), we identify $[-2l, 2l)$ with $\boldsymbol{T}$ in the usual way and compute the Fourier coefficients of $f_l(t-l)+l$. (2) follows from $\|f_l\|_{B(R)} = f_l(l) = l$.

Let now $P \in PM(S_l)$ be given. Since

$$\hat{P}(\gamma) = \langle e^{-i\gamma x}, P_x\rangle \qquad (\gamma \in \boldsymbol{R}^N)\ ,$$

we have $\hat{P} \in C^\infty(\boldsymbol{R}^N)$ and

$$(3)\qquad (D_u^k \hat{P})(\gamma) = \langle(-\mathrm{iux})^k e^{-i\gamma x}, P_x\rangle \quad (\gamma \in \boldsymbol{R}^N; k = 1, 2, \cdots)$$

for any unit vector u in $\boldsymbol{R}^N$. Notice that $|ux| \leqq \|x\|$ by Schwarz' inequality, and so

$$(4)\qquad f_l(ux) = ux \qquad (x \in S_l)\ .$$

Since S_l is an S-set [4: Theorem 7.5.4], we have by (2), (3), and (4)

$$\begin{aligned} |(D_u^k \hat{P})(\gamma)| &= |\langle f_l(ux)^k e^{-i\gamma x}, P_x\rangle| \\ &\leqq \|f_l(ux)^k e^{-i\gamma x}\|_{B(\boldsymbol{R}^N)} \cdot \|P\|_{PM} \\ &\leqq \{\|f_l\|_{B(\boldsymbol{R}^1)}\}^k \|P\|_{PM} = l^k \|P\|_{PM}\ . \end{aligned}$$

This completes the proof.

LEMMA 5.2. (Schneider's inequality [10]). *Let $P \in PM(S_l)$, $l > 0$, and $\eta > 0$ be given. Let also K be any η-dense subset of $\boldsymbol{R}^N$. Then we have*

$$\sup_{\gamma \in K} |\hat{P}(\gamma)| \geqq \{1 - 2^{-1}(l\eta)^2\} \|P\|_{PM}\ .$$

PROOF. We first prove this assuming $P \in PF(S_l)$, i.e., $\hat{P} \in C_0(\boldsymbol{R}^N)$. Then there exists a $\gamma_0 \in \boldsymbol{R}^N$ such that

$$|\hat{P}(\gamma_0)| = \|\hat{P}\|_{C(\boldsymbol{R}^N)} = \|P\|_{PM}\ .$$

Without loss of generality, we may assure $\hat{P}(\gamma_0) \geqq 0$. Choose any $\gamma_1 \in K$ so that $\|\gamma_0 - \gamma_1\| \leqq \eta$. Let u be the unit vector in the direction of $\gamma_1 - \gamma_0$. Thus

$$\gamma_1 = \gamma_0 + tu, \quad \text{where} \quad t = \|\gamma_1 - \gamma_0\| \leqq \eta\ .$$

By the Taylor formula, we then have

$$\begin{aligned} \mathrm{Re}\, \hat{P}(\gamma_1) &= \mathrm{Re}\, \hat{P}(\gamma_0 + tu) \\ &= \mathrm{Re}\left[\hat{P}(\gamma_0) + t(D_u\hat{P})(\gamma_0) + \frac{t^2}{2}(D_u^2\hat{P})(\gamma')\right] \\ &= \|P\|_{PM} + 0 + \frac{t^2}{2}\mathrm{Re}\,(D_u^2\hat{P})(\gamma') \end{aligned}$$

for some $\gamma' \in \boldsymbol{R}^N$. It follows from Bernstein's inequality that

$$\sup_{\gamma \in K} |\hat{P}(\gamma)| \geqq |\operatorname{Re} \hat{P}(\gamma_1)| \geqq (1 - 2^{-1}t^2l^2)\|P\|_{PM} \geqq (1 - 2^{-1}\eta^2 l^2)\|P\|_{PM} .$$

Let now $P \in PM(S_l)$ be arbitrary. Given $\varepsilon > 0$, take any probability measure $\mu_\varepsilon \in M(S_\varepsilon) \cap PF(S_\varepsilon)$. Then we have

$$P * \mu_\varepsilon \in PM(S_{l+\varepsilon}) \quad \text{and} \quad \widehat{P * \mu_\varepsilon} = \hat{P}\hat{\mu}_\varepsilon \in C_0(\boldsymbol{R}^N) .$$

It follows from the first case that

$$\sup_{\gamma \in K} |\hat{P}(\gamma)| \geqq \sup_{\gamma \in K} |\widehat{P * \mu_\varepsilon}(\gamma)| \geqq \{1 - 2^{-1}(\eta(l + \varepsilon))^2\}\|P * \mu_\varepsilon\|_{PM} .$$

Since $\lim_\varepsilon \hat{\mu}_\varepsilon(\gamma) = 1 \ \forall \gamma \in \boldsymbol{R}^N$, this yields the desired inequality.

LEMMA 5.3. *Let $\{v_k\}_1^N$ be an orthogonal basis in $\boldsymbol{R}^N$ and E any subset of $Gp(\{v_k\}_1^N)$. Then the set*

$$E^\perp = \{\gamma \in \boldsymbol{R}^N : e^{i\gamma x} = 1 \quad \forall x \in E\}$$

is η-dense in $\boldsymbol{R}^N$, where $\eta = \pi(\sum_1^N \|v_k\|^{-2})^{1/2}$.

PROOF. It suffices to note that $E^\perp$ contains

$$Gp(\{v_k\}_1^N)^\perp = \left\{\sum_{k=1}^N n_k 2\pi \|v_k\|^{-2} v_k : n \in \boldsymbol{Z}^N\right\} .$$

LEMMA 5.4. *Let E be a finite set in $\boldsymbol{R}^N$, and $0 < l < \infty$. Suppose that $E^\perp$ is η-dense in $\boldsymbol{R}^N$ for some $0 < \eta < 2^{1/2}/l$. Then*

$$\sup_{\gamma, \beta \in \boldsymbol{R}^N} \left|\sum_{x \in E} \hat{Q}_x(\gamma) e^{-i\beta x}\right| \leqq \left\{1 - \frac{(l\eta)^2}{2}\right\}^{-1} \left\|\sum_{x \in E} Q_x * \delta_x\right\|_{PM}$$

holds for every finite subset $\{Q_x : x \in E\}$ of $PM(S_l)$. Here δ_x is the unit mass at x.

PROOF. Let $\{Q_x : x \in E\} \subset PM(S_l)$ and $\beta \in \boldsymbol{R}^N$ be given. Then we have

$$(1) \qquad \left\|\sum_{x \in E} Q_x * \delta_x\right\|_{PM} = \sup_{\gamma \in \boldsymbol{R}^N} \left|\sum_{x \in E} \hat{Q}_x(\gamma) e^{-i\gamma x}\right| \geqq \sup_{\lambda \in E^\perp} \left|\sum_{x \in E} \hat{Q}_x(\lambda + \beta) e^{-i\beta x}\right| .$$

Let $Q \in PM(S_l)$ be the sum of all $e^{-i\beta x} Q_x$, $x \in E$. Since $E^\perp + \beta$ is η-dense in $\boldsymbol{R}^N$, it follows from Schneider's inequality that

$$\sup_{\lambda \in E^\perp} \left|\hat{Q}(\lambda + \beta)\right| \geqq \left\{1 - \frac{(l\eta)^2}{2}\right\} \|Q\|_{PM} ,$$

or, equivalently, that the last term in (1) is larger than or equal to

$$\left\{1 - \frac{(l\eta)^2}{2}\right\} \sup_{\gamma \in R^N} \left| \sum_{x \in E} \hat{Q}_x(\gamma) e^{-i\beta x} \right| .$$

Since $\beta \in R^N$ is arbitrary, this gives the desired inequality.

LEMMA 5.5. *Let* $\alpha = (2\pi N)^{-1}$, *and let*

$$l_j = \sum_{k=j+1}^{\infty} R_k \quad and \quad \eta_j = \pi\left(\sum_{k=1}^{N} \|v_{kj}\|^{-2}\right)^{1/2} .$$

To prove Theorem 5, *we can assume the following*:

(i) $r_j > 4\pi N l_j$ *and* $(1+\alpha) l_j \eta_j < 1$ $(j = 1, 2, \cdots)$.

(ii) *The sets* $\sum_1^n x_j + S_{\alpha r_n}$, $x_j \in E_j$ $(1 \leqq j \leqq n)$, *are disjoint for each* $n = 1, 2, \cdots$.

PROOF. We first prove that (i) implies (ii). Fix any $n \in N$, and take two distinct points $\sum_1^n x_j$ and $\sum_1^n y_j$ of $\sum_1^n E_j$. If $1 \leqq k \leqq n$ is the first number such that $x_k \neq y_k$, then we have

$$\left\| \sum_1^n x_j - \sum_1^n y_j \right\| \geqq r_k - 2l_k .$$

But (i) assures that $r_j - 2l_j > r_{j+1} - 2l_{j+1}$ for all j, and so

$$\left\| \sum_1^n x_j - \sum_1^n y_j \right\| \geqq r_n - 2l_n .$$

Moreover, we have

$$\begin{aligned}(r_n - 2l_n) - 2\alpha r_n &= (1 - 2\alpha) r_n - 2l_n \\ &> \{1 - 2\alpha - (2\pi N)^{-1}\} r_n > 0\end{aligned}$$

by (i) and the definition of α. Thus (i) implies (ii).

Take now any real a so large that

(1) $$a > 4\pi N \quad \text{and} \quad (1+\alpha)\pi N^{1/2}/a < 1 .$$

By (UTMS), there exists a natural number j_0 such that $r_j > (a+1) R_{j+1}$ for all $j > j_0$. Since $R_j \geqq r_j$, it follows that $j > j_0$ implies

(2) $$\begin{aligned} r_j &> aR_{j+1} + r_{j+1} > aR_{j+1} + aR_{j+2} + r_{j+2} \\ &\cdots > a \sum_{k=j+1}^{\infty} R_k = a l_j .\end{aligned}$$

Notice now that $\eta_j \leqq \pi N^{1/2}/r_j$. It follows from (1) and (2) that $j > j_0$ implies

$$(1+\alpha) l_j \eta_j < (1+\alpha) a^{-1} r_j \cdot \pi N^{1/2}/r_j < 1 .$$

In other words, (i) is the case for every $j > j_0$.

Let now $t_1, \cdots, t_{j_0}$ be any real positive numbers. Put $E'_j = E_j$ if $j > j_0$, $E'_j = t_j E_j$ if $j \leqq j_0$, and let $(r'_j)_1^\infty$, $(\eta'_j)_1^\infty$ and $(l'_j)_1^\infty$ be the numerical sequences corresponding to $(E'_j)_1^\infty$. We choose successively $t_{j_0}, \cdots, t_2, t_1$ so that the above three sequences satisfy (i).

Then both $\widetilde{E} = \sum_1^\infty E_j$ and $\widetilde{E}' = \sum_1^\infty E'_j$ are disjoint unions of the same number of translates of $\sum_{j>j_0} E_j$. Therefore it is trivial that if $\widetilde{E}'$ has the required properties in Theorem 5, then so does $\widetilde{E}$. This completes the proof.

LEMMA 5.6. *Suppose that the UTMS set $\widetilde{E}$ satisfies condition* (i) *in Lemma* 5.5. *Let $\{Q_{x_1 \cdots x_n};\ x_j \in E_j,\ 1 \leqq j \leqq n\}$ be a finite subset of $PM(S_{\alpha r_n})$, n being a natural number. Then we have*

$$\sup\left\{\left|\sum_{x_j \in E_j, 1 \leqq j \leqq n} \hat{Q}_{x_1 \cdots x_n}(\gamma) \exp\left(-i \sum_{j=1}^{n} \gamma_j x_j\right)\right| : \gamma,\ \gamma_j \in \boldsymbol{R}^N\right\}$$
$$\leqq (2/C_n) \sup\left\{\left|\sum_{x_j} \hat{Q}_{x_1 \cdots x_n}(\gamma) \exp\left(-i\gamma \sum_{j=1}^{n} x_j\right)\right| : \gamma \in \boldsymbol{R}^N\right\},$$

where $C_n = \prod_1^{n-1} \{1 - (\eta_j l_j)^2\}$.

PROOF. Write

$$s_n = \alpha r_n;\ s_{n-1} = s_n + R_n = \alpha r_n + R_n;\ \cdots;$$
$$s_1 = s_2 + R_2 = \alpha r_n + R_n + \cdots + R_2 .$$

Let $\gamma_1, \cdots, \gamma_n \in \boldsymbol{R}^N$ be fixed. In the expression

$$\phi(\gamma) = \sum_{x_n \in E_n} \left\{\sum_{\substack{x_j \in E_j \\ 1 \leqq j < n}} \hat{Q}_{x_1 \cdots x_n}(\gamma) \exp\left(-i \sum_{j=1}^{n-1} \gamma_j x_j\right)\right\} e^{-i\gamma_n x_n},$$

the functions of γ in the brackets are Fourier transforms of pseudo-measures in $PM(S_{s_n})$. Since $E_n^\perp$ is η_n-dense in $\boldsymbol{R}^N$ by Lemma 5.3, and since $\eta_n s_n \leqq \pi N^{1/2}\alpha < 2^{1/2}$, it follows from Lemma 5.4 that

$$\sup_\gamma |\phi(\gamma)| \leqq A_n^{-1} \sup_\gamma \left|\sum_{x_n \in E_n}\left\{\sum_{x_j \in E_j, 1 \leqq j < n}\right\} e^{-i\gamma x_n}\right|$$
$$= A_n^{-1} \sup_\gamma \left|\sum_{\substack{x_j \in E_j \\ 1 \leqq j < n}}\left\{\sum_{x_n \in E_n} (Q_{x_1 \cdots x_n} * \delta_{x_n})\hat{\ }(\gamma)\right\} \exp\left(-i \sum_{j=1}^{n-1} \gamma_j x_j\right)\right|,$$

where $A_n = 1 - (\eta_n s_n)^2/2$. Notice that

$$\operatorname{supp}\left\{\sum_{x_n \in E_n} (Q_{x_1 \cdots x_n} * \delta_{x_n})\right\} \subset S_{s_n} + S_{R_n} = S_{s_{n-1}}$$

for all $x_j \in E_j$, $1 \leqq j < n$. Therefore an inductive argument applies, and we have

$$\sup_{\gamma}\left|\sum_{x_j\in E_j,1\leqq j\leqq n}\hat{Q}_{x_1\cdots x_n}(\gamma)\exp\left(-i\sum_{j=1}^{n}\gamma_j x_j\right)\right|$$
$$\leqq (A_n\cdots A_2A_1)^{-1}\sup_{\gamma}\left|\sum_{x_j}(Q_{x_1\cdots x_n}*\delta_{x_n}*\cdots*\delta_{x_1})^\wedge(\gamma)\right|$$
$$\leqq 2C_n^{-1}\sup_{\gamma}\left|\sum_{x_j}\hat{Q}_{x_1\cdots x_n}(\gamma)\exp\left(-i\gamma\sum_{j=1}^{n}x_j\right)\right| .$$

Since $\gamma_1, \cdots, \gamma_n \in \boldsymbol{R}^N$ are arbitrary, this yields the required inequality.

PROOF OF THEOREM 5. We will assume the two additional conditions (i) and (ii) given in Lemma 5.5. Notice that then

$$C_0 = 2\lim_n C_n^{-1} = 2\prod_{j=1}^{\infty}\{1-(\eta_j l_j)^2\}^{-1} < \infty ,$$

since $\eta_j l_j \leqq (\pi N^{1/2}/r_j)\cdot(R_{j+1}+l_{j+1}) \leqq 2\pi N R_{j+1}/r_j$ and so $\sum_1^\infty (\eta_j l_j)^2 < \infty$ by condition (UTMS). Notice also that (i) implies

$$\sum_{j=n+1}^{\infty} E_j \subset S_{l_n} \subset S_{\alpha r_n} \qquad (n = 1, 2, \cdots) .$$

To prove part (a), take any $n \in \boldsymbol{N}$ and any n vectors $\gamma_1, \cdots, \gamma_n$ in $\boldsymbol{R}^N$. We define a function $f = f_{\gamma_1\cdots\gamma_n} \in A(\sum_1^n E_j + S_{\alpha r_n})$ by setting

$$(1) \qquad f\left(\sum_{j=1}^{n} x_j + S_{\alpha r_n}\right) = \exp\left(i\sum_{j=1}^{n}\gamma_j x_j\right) \qquad \forall (x_j \in E_j)_1^n ,$$

which is well-defined by (ii).

We then claim that

$$(1.1) \qquad \|f_{\gamma_1\cdots\gamma_n}\|_{A(\sum_1^n E_j+S_{\alpha r_n})} \leqq C_0 , \quad \text{and}$$

$$(1.2) \qquad P_E(f_{\gamma_1\cdots\gamma_n}) = e_{\gamma_1} \circledcirc \cdots \circledcirc e_{\gamma_n} .$$

In fact, (1.2) is trivial. To prove (1.1), take any $Q \in A(\sum_1^n E_j + S_{\alpha r_n})' = PM(\sum_1^n E_j + S_{\alpha r_n})$ (notice that $\sum_1^n E_j + S_{\alpha r_n}$ is a finite disjoint union of translates of the S-set $S_{\alpha r_n}$). Write

$$Q = \sum_{x_j\in E_j,1\leqq j\leqq n} Q_{x_1\cdots x_n} * \delta_{x_1+\cdots+x_n}$$

with $Q_{x_1\cdots x_n} \in PM(S_{\alpha r_n})$. Then we have

$$\langle f, Q\rangle = \sum_{x_j}\langle f, Q_{x_1\cdots x_n} * \delta_{x_1+\cdots+x_n}\rangle$$
$$= \sum_{x_j}\hat{Q}_{x_1\cdots x_n}(0)\exp\left(i\sum_{j=1}^{n}\gamma_j x_j\right) .$$

Therefore, by Lemma 5.6, we have

$$|\langle f, Q\rangle| \leqq C_0\|Q\|_{PM} \qquad \forall Q \in A\left(\sum_1^n E_j + S_{\alpha r_n}\right)' .$$

This, combined with the Hahn-Banach Theorem, yields (1.1).

It is now easy to see that P_E is a topological isomorphism of $A(\tilde{E})$ onto A_E and satisfies

$$\|P_E f\|_{A_E} \leqq \|f\|_{A(\tilde{E})} \leqq C_0 \|P_E f\|_{A_E} \qquad \forall f \in A(\tilde{E}) .$$

(cf. the proof of part (a) of Theorem 1).

To prove part (b), fix a natural number n, and define an algebra homomorphism

$$L_n\colon A(\tilde{E}) \to A\Big(\sum_1^n E_j + S_{\alpha r_n}\Big)$$

by setting

$$(2) \qquad (L_n f)\Big(\sum_1^n x_j + S_{\alpha r_n}\Big) = f\Big(\sum_1^n x_j\Big) \qquad \forall (x_j \in E_j)_1^n .$$

We then claim that

$$(2.1) \qquad \|L_n f\|_{A(\Sigma_1^n E_j + S_{\alpha r_n})} \leqq C_0 \|f\|_{A(\tilde{E})} \qquad \forall f \in A(\tilde{E}) .$$

In fact, it suffices to prove this for $f = e_\gamma$ with $\gamma \in \boldsymbol{R}^N$. But then $f = f_{\gamma_1 \cdots \gamma_n}$, where $\gamma_1 = \cdots = \gamma_n = \gamma$. Thus (2.1) is a special case of (1.1).

We next prove

$$(2.2) \qquad \|L_n e_\gamma - e_\gamma\|_{A(\Sigma_1^n E_j + S_{l_n})} \leqq M C_0 \|\gamma\| \cdot l_n$$

for every $\gamma \in \boldsymbol{R}^N$, where M is an absolute constant. Fix $\gamma \in \boldsymbol{R}^N$, and set $l = l_n$. We have by (2.1)

$$\begin{aligned} \|(L_n e_\gamma)^k\|_{A(\Sigma_1^n E_j + S_l)} &= \|L_n e_{k\gamma}\|_{A(\Sigma_1^n E_j + S_l)} \\ &\leqq C_0 \|e_{k\gamma}\|_{A(\tilde{E})} = C_0 \qquad (k = 0, \pm 1, \pm 2, \cdots) . \end{aligned}$$

On the other hand, (2) shows

$$|\arg [(\overline{L_n e_\gamma}) e_\gamma]| \leqq \|\gamma\| \cdot l \qquad \text{on } \textstyle\sum_1^n E_j + S_l .$$

Thus (2.2) follows from Lemma 1 in [8].

Notice now $\sum_{n+1}^\infty E_j \subset S_{l_n}$ and so

$$PM(\tilde{E}) \subset A\Big(\sum_1^n E_j + S_{l_n}\Big)' .$$

Given any $Q \in PM(\tilde{E})$, we prove

$$(2.3) \qquad L_n^* Q \in M\Big(\sum_1^n E_j\Big) \subset M(\tilde{E}) , \quad \text{and}$$

$$(2.4) \qquad |(L_n^* Q)\hat{}(\gamma) - \hat{Q}(\gamma)| \leqq M C_0 \|\gamma\| l_n \|Q\|_{PM} \qquad \forall \gamma \in \boldsymbol{R}^N .$$

The definition (2) of L_n shows supp $L_n^* Q$ is contained in the finite set

$\sum_1^n E_j$, and hence (2.3). If $\gamma \in \boldsymbol{R}^N$, we have by (2.2)

$$
\begin{aligned}
|(L_n^* Q)\hat{\ }(\gamma) - \hat{Q}(\gamma)| &= |\langle L_n e_{-\gamma} - e_{-\gamma}, Q\rangle| \\
&\leqq \|L_n e_{-\gamma} - e_{-\gamma}\|_{A(\Sigma_1^n E_j + S_{l_n})} \cdot \|Q\|_{PM} \\
&\leqq MC_0 \|\gamma\| \cdot l_n \cdot \|Q\|_{PM} ,
\end{aligned}
$$

which establishes (2.4).

We infer from (2.1), (2.3), and (2.4) that $M(\tilde{E})$ is weak-* dense in $PM(\tilde{E})$ and $\tilde{E}$ is therefore an S-set.

To prove part (c), let f be the characteristic function of the unit ball S_1 divided by its volume (hence $\|f\|_1 = \hat{f}(0) = 1$). Set $f_n(x) = (\alpha r_n)^{-N} f(\alpha r_n)^{-1} x)$ for $n = 1, 2, \cdots$, so that each f_n is supported by $S_{\alpha r_n}$ and has Fourier transform $\hat{f}_n(\gamma) = \hat{f}(\alpha r_n \gamma)$, $\gamma \in \boldsymbol{R}^N$. We can choose a positive real number B_0 so that $\gamma \in \boldsymbol{R}^N$ and $\|\gamma\| \geqq B_0$ imply $|\bar{f}(\alpha\gamma)| < (2C_0)^{-1}$. Notice then

$$
(3) \qquad \|\gamma\| \geqq B_0/r_n \Rightarrow |\hat{f}_n(\gamma)| < (2C_0)^{-1} \qquad (n = 1, 2, \cdots) .
$$

Given $n \geqq 1$, $\mu \in M(\sum_1^n E_j)$, and $\gamma_0 \in \boldsymbol{R}^N$, we now prove

$$
(3.1) \qquad \|\mu\|_{PM} \leqq C_0 \sup \{|\mu(\gamma)| : \gamma \in \boldsymbol{R}^N, \|\gamma - \gamma_0\| \leqq B_0/r_n\} .
$$

First notice that $\operatorname{supp}(f_n * \mu) \subset \sum_1^n E_j + S_{\alpha r_n}$. Regarding $L^1(\boldsymbol{R}^N)$ as a subspace of $PM(\boldsymbol{R}^N)$ in the usual way, we have for every $g \in A(\tilde{E})$

$$
\begin{aligned}
\langle g, L_n^*(f_n * \mu)\rangle &= \langle L_n g, f_n * \mu\rangle \\
&= \int_{\Sigma_1^n E_j + S_{\alpha r_n}} (L_n g)(x) \cdot (f_n * \mu)(x) dx \\
&= \sum \left\{ \int_{\Sigma_1^n x_j + S_{\alpha r_n}} g\Big(\sum_1^n x_j\Big) f_n\Big(x - \sum_1^n x_j\Big) dx \right\} \mu\Big(\Big\{\sum_1^n x_j\Big\}\Big) \\
&= \sum g\Big(\sum_1^n x_j\Big) \cdot \mu\Big(\Big\{\sum_1^n x_j\Big\}\Big) = \langle g, \mu\rangle
\end{aligned}
$$

where the sum $\sum$ is taken over all $x_j \in E_j$, $1 \leqq j \leqq n$. This shows $L_n^*(f_n * \mu) = \mu$. It follows from (2.1) and (3) that

$$
\begin{aligned}
\|\mu\|_{PM} &= \|L_n^*(f_n * \mu)\|_{PM} \leqq C_0 \|f_n * \mu\|_{PM} \\
&= C_0 \sup_\gamma |\hat{f}_n(\gamma) \hat{\mu}(\gamma)| \\
&\leqq C_0 \max \{\sup \{|\hat{\mu}(\gamma)| : \|\gamma\| \leqq B_0/r_n\}, \|\mu\|_{PM}/(2C_0)\}
\end{aligned}
$$

and so

$$
\|\mu\|_{PM} \leqq C_0 \sup \{|\hat{\mu}(\gamma)| : \|\gamma\| \leqq B_0/r_n\} .
$$

Replacing μ by $e_{-\gamma_0}\mu$, we thus have (3.1).

Take now any $Q \in PM(\tilde{E})$. By (2.3) and (2.4), we have $L_n^* Q \in M(\sum_1^n E_j)$

and

(3.2) $$|(L_n^*Q)^\wedge(\gamma)| \leqq |\hat{Q}(\gamma)| + MC_0\|\gamma\|l_n\|Q\|_{PM} \qquad (\gamma \in \boldsymbol{R}^N)$$

for all $n \geqq 1$. We apply (3.1) to $\mu = L_n^*Q$ and have

(3.3) $$C_0^{-1}\|L_n^*Q\|_{PM} \leqq \sup\{|(L_n^*Q)^\wedge(\gamma)|: \gamma \in \boldsymbol{R}^N, \|\gamma - \gamma_0\| \leqq B_0/r_n\}$$

for every $n \geqq 1$ and $\gamma_0 \in \boldsymbol{R}^N$. It follows from (3.2) and (3.3) that

(3.4) $$\begin{aligned} C_0^{-1}\|L_n^*Q\|_{PM} \leqq \sup\{|\hat{Q}(\gamma)|: \gamma \in \boldsymbol{R}^N, \|\gamma - \gamma_0\| \leqq B_0/r_n\} \\ + MC_0(\|\gamma_0\| + B_0/r_n)l_n\|Q\|_{PM}\,. \end{aligned}$$

Since $\gamma_0 \in \boldsymbol{R}^N$ is arbitrary, we can replace it by any vector γ_n with $\|\gamma_n\| = 2B_0/r_n$ for each n. Then (3.4) yields

$$\begin{aligned} C_0^{-1}\|L_n^*Q\|_{PM} \leqq \sup\{|\hat{Q}(\gamma)|: \gamma \in \boldsymbol{R}^N, \|\gamma\| \geqq B_0/r_n\} \\ + 3MC_0B_0(l_n/r_n)\|Q\|_{PM}\,, \end{aligned}$$

which shows

$$C_0^{-1}\|Q\|_{PM} \leqq \overline{\lim_{\gamma\to\infty}}\,|\hat{Q}(\gamma)|\,,$$

since $L_n^*Q \to Q$ in the weak-* topology of $PM(\tilde{E})$, $r_n \to 0$ and $l_n/r_n \to 0$ as $n \to \infty$.

This completes the proof of part (c) and Theorem 5 was established.

We now give four examples of "explicit" non S-sets in certain groups, although the first two of them are essentially contained in [8].

EXAMPLES OF NON S-SETS. Let U be the union of the two open intervals $(0, \pi^2/6 - 1)$ and $(1, \pi^2/6)$. Then the following sets, denoted by the same notation $\tilde{E}_a$, are non S-sets.

(1) Let G be the product group of any non-trivial compact abelian groups G_n, $n = 1, 2, \cdots$. Choose and fix a non-zero element $x_n \in G_n$ for each $n \geqq 1$. Put

$$\tilde{E}_a = \left\{(\varepsilon_n x_n)_1^\infty \in G: \varepsilon_n \in \{0, 1\}\ \forall n\,, \quad \text{and} \quad \sum_{n=1}^\infty n^{-2}\varepsilon_{2n-1}\varepsilon_{2n} = a\right\}$$

for $a \in U$.

(2) Let $G = \boldsymbol{T}$ or $\boldsymbol{R}$, and $p \geqq 3$ any natural number. Define

$$\tilde{E}_a = \left\{\sum_{n=1}^\infty \varepsilon_n p^{-n}: \varepsilon_n \in \{0, 1\}\ \forall n\,, \quad \text{and} \quad \sum_{n=1}^\infty n^{-2}\varepsilon_{2n-1}\varepsilon_{2n} = a\right\}$$

for $a \in U$.

(3) Let $G = \boldsymbol{R}^N$, and $(x_n)_1^\infty$ any sequence of non-zero vectors such that $\sum_1^\infty (\|x_{n+1}\|/\|x\|_n)^2 < 1/2$. For each $a \in U$, put

$$\tilde{E}_a = \left\{\sum_{n=1}^{\infty} \varepsilon_n x_n : \varepsilon_n \in \{0, 1\}\ \forall n\ , \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-2}\varepsilon_{2n-1}\varepsilon_{2n} = a\right\} .$$

(4) Let $a = (a_0, a_1, \cdots)$ be any sequence of natural numbers $\geqq 2$, $G = \Delta(a)$ the group of the a-adic integers, and $u_0, u_1, u_2, \cdots$ the elements of $\Delta(a)$ defined as before. Choose any increasing sequence $(n_j)_1^\infty$ of natural numbers so that $\sum_1^\infty j/a(n_j, n_{j+1} - 1) < \infty$, where $a(m, n) = a_m a_{m+1} \cdots a_n$ for $m < n$. Put

$$\tilde{E}_a = \left\{\sum_{j=1}^{\infty} \varepsilon_j u_{n_j} : \varepsilon_j \in \{0, 1\}\ \forall j\ , \quad \text{and} \quad \sum_{j=1}^{\infty} j^{-2}\varepsilon_{2j-1}\varepsilon_{2j} = a\right\}$$

for $a \in U$.

The proof that these sets are non S-sets mainly follows from Remark (a) in [8: p. 288]. We omit the details.

REMARKS. (a) The set $\tilde{E}$ given in Theorem 3 is an S-set. The proof is similar to that of part (a) of Theorem 1, although we need a more subtle argument.

(b) We can use Bernstein's and Schneider's inequalities to improve the estimate of $\eta(d)$ given in Lemma 1 of [8]. Let $0 < d < 2\sqrt{2}$, and $A(d)$ the restriction algebra of $A(\boldsymbol{T})$ to $[-d, d]$. Then we have

$$\eta(d) = \|e^{i\alpha x} - 1\|_{A(d)} \leqq |\alpha| d/(1 - 8^{-1}d^2) \qquad \forall \alpha \in \boldsymbol{R} .$$

In fact, fix any $\alpha > 0$. If $P \in PM([-d, d])$, then

$$\langle e^{i\alpha x} - 1, P_x\rangle = \left\langle \int_0^\alpha ix\, e^{itx} dt, P_x\right\rangle$$
$$= \int_0^\alpha \langle ix\, e^{itx}, P_x\rangle dt = -\int_0^\alpha \hat{P}'(-t) dt .$$

It follows from Bernstein's and Schneider's inequalities that

$$|\langle e^{i\alpha x} - 1, P_x\rangle| \leqq \alpha \|P'\|_{C(\boldsymbol{R})} \leqq \alpha d \|\hat{P}\|_{C(\boldsymbol{R})}$$
$$\leqq \alpha d(1 - 8^{-1}d^2)^{-1} \|\hat{P}\|_{C(\boldsymbol{Z})} .$$

This, combined with the Hahn-Banach Theorem, yields the desired inequality.

(c) Most of the results in this paper is part of the author's lecture notes [9].

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